Quantum Foundations
4 – Nonlocality

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Einstein’s boxes

We are in ‘20s - ‘30s of the previous century and a central question for Einstein was the **role of the wave function**.

Specifically, when the wave function associated e.g. to an electron is in the superposition of two different locations, what does it entail for the electron?

Conception 1 [Incompleteness]: The electron, a point particle, is somewhere in space, and the wave function does not specify where. It only gives the probability of find it somewhere upon measurement. The actual position of the electron, which becomes manifest in measurements, is the missing piece of information and as such the wave function represents and incomplete description of physical phenomena. At best, it refers to an ensemble of electrons.

Conception 2 [Completeness]: The wave function describes everything there is to know about the individual electron. When it is spread out in space, there is no fact about the electron occupying a specific position in space. The appearance of the electron upon measurement is a genuinely stochastic event which is not triggered by anything pre-existing.

Einstein was against Completeness - essentially because he did not accept the intrinsic statistical character of the theory [QUOTE FROM SCHILPP] - and argued against it. In several of this arguments, he shows that it violated locality. One of these arguments is called **Einstein’s boxes**.

Consider first a **classical** – very trivial – situation: a ball in a box. The box is divided into two parts, which are separated from each other.

The ball is either on the left or on the right, but we do not know where until we open one of the two boxes. Moral:

- Perfect correlations: if the ball is on the left box, it is not in the right
- Nothing mysterious going on: the ball was from the very beginning in the box, where it is later found.

Now consider the **quantum version** of the classical situation. Consider a box with a particle in it. In an ideal situation, after some time its wave function is uniformly spread out all over the box. The box is divided into two halves, without revealing in which half the particle ends up; the two halves are brought far apart.

The shadow indicates the intensity of the wave function

A measurement of the particle's position is performed on the left half box and the particle, for example, is not found there. How do the two conceptions describe such a situation?

Conception 1 [Incompleteness]: the particle was all the time on the left side. The measurement simply revealed its pre-existing position. So nothing special happens in this case, it is like in classical physics.

Conception 2 [Completeness]: before the measurement there is no fact about the particle being on the left half or the right half of the box. The measurement on the left changes the state of things on both sides: after the measurement the particle is certainly not on the left, and is certainly on the right.

In this second case there is a **nonlocal** effect occurring. On this ground, Einstein rejects incompleteness.

Quote from Einstein: My way of thinking is now this: properly considered, one cannot [refute the completeness doctrine, i.e., Conception 2, i.e., the YES view] if one does not make use of a supplementary principle: the 'separation principle.' That is to say: 'the second box, along with everything having to do with its contents, is independent of what happens with regard to the first box (separated partial systems).' If one adheres to the separation principle, then one thereby excludes the [YES] point of view, and only the [NO] point of view remains, according to which the above state description is an incomplete description of reality, or of the real states.
[NORSEN]

Quote from Heisenberg: ...one other idealized experiment (due to Einstein) may be considered. We imagine a photon which is represented by a wave packet built up out of Maxwell waves. It will thus have a certain spatial extension and also a certain range of frequency. By reflection at a semi- transparent mirror, it is possible to decompose it into two parts, a reflected and a transmitted packet. There is then a definite probability for finding the photon either in one part or in the other part of the divided wave packet. After a sufficient time the two parts will be separated by any distance desired; now if an experiment yields the result that the photon is, say, in the reflected part of the packet, then the probability of finding the photon in the other part of the packet immediately becomes zero. The experiment at the position of the reflected packet thus exerts a kind of action (reduction of the wave packet) at the distant point occupied by the transmitted packet, and one sees that this action is propagated with a velocity greater than that of light [NORSEN].

However, it is also obvious that this kind of action can never be utilized for the transmission of signals so that it is not in conflict with the postulates of the theory of relativity [NORSEN]

The EPR Argument

Einstein was probably the first to recognize that Quantum Mechanics is potentially in tension with Special Relativity.

We present the EPR argument in the form considered by Bohm, which is the one used by Bell for developing his inequalities, and which is closer to experimental verification.

Consider to 1/2 spin particles generated by a common source in a singlet state and moving apart from each other in opposite directions. At the two ends, Alice and Bob perform spin measurements by using Stern-Gerlach devices.

The spin part of the wave function reads:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

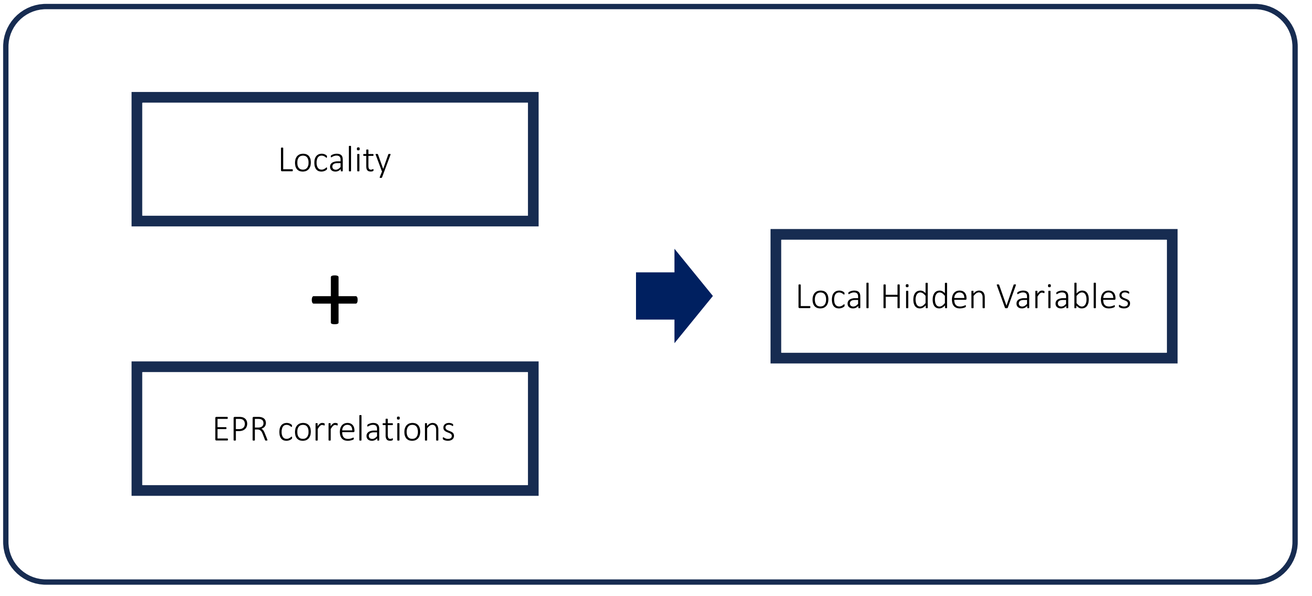
It is a fact that Alice and Bob, separately, will find with probability 1/2 the spin of their particle up, and with probability 1/2 the spin down, along any direction. However, every-time Alice and Bob measure the spin along the **same direction**, they will find always **opposite results**: one particle has spin up and the other spin down.
The situation is similar to the case of the boxes.

Conception 1 [Incompleteness]: The spin of the particle(s) is a pre-existing property and the measurement simply reveals that property. There is nothing mysterious in the experiment, it is like in classical physics.

Conception 2 [Completeness]: The wave function represents the most accurate description of the physical system, and the system's properties emerge at the time of measurement. Then, Einstein concludes, the theory is nonlocal: before Alice's measurement, Bon had 1/2 probability to find the spin up or down along a given direction; after Alice's measurement - no matter how far apart - Bob's outcome will be perfectly anti-correlated.

As before, Completeness leads to non-locality, which is refused by Einstein. The only other alternative is incompleteness, what we now call hidden variables.

The **EPR argument can be summarized as follows**:



Bell on the EPR Argument

From "On the Einstein-Podolski-Rosen paradox", Physics 1, 195 (1964). This is the first paper of the series about Bell's inequalities.

Let us **set up the framework and notation**, which we will use also for the later proof of Bell's theorem. Let us consider the EPR situation in the reformulation of Bohm: two spin 1/2 particles initially in a singlet state, and freely moving arbitrarily far away from each other in opposite directions.

Let **a** the direction along which the spin is measured by Alice (on the left) and **b** the direction along which the spin is measured by Bob (on the right).

Let A and B be the outcomes of the two measurements, which can be +1 (spin up) or -1 (spin down):

$A = A(\mathbf{a}|\psi) = \pm 1$, $B = B(\mathbf{b}|\psi) = \pm 1$

Let

$\mathbb{P}(A \mid \mathbf{a}, \psi)$ be the probability for Alice to obtain outcome A for a spin measurement along direction **a**, and similarly

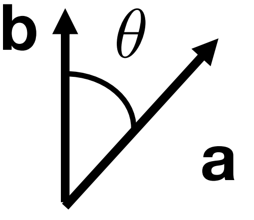
$\mathbb{P}(B \mid \mathbf{b}, \psi)$ the probability for Bob to obtain outcome B for a spin measurement along direction **b**, given that the initial two-spin state is ψ.

Let also

$\mathbb{P}(AB \mid \mathbf{a},\mathbf{b}, \psi)$ be the joint probability for Alice to obtain outcome A in a spin measurement along direction **a**, and for Bob to obtain outcome B in a spin measurement along direction **b** , given that the initial two-spin state is ψ.

The Quantum Mechanical description of the singlet state is the following. If Alice performs a measurement along direction **a**, with probability 1/2 she will find outcome +1 and with probability 1/2 out come -1. Suppose the outcome is +1. Then the state collapses to:

$$|\mathbf{a}\rangle|-\mathbf{a}\rangle$$



Now Bob performs a measurement along direction **b**.
It holds:

$$\begin{aligned}|\mathbf{a}\rangle &= \cos\frac{\theta}{2}|\mathbf{b}\rangle + \sin\frac{\theta}{2}|{-\mathbf{b}}\rangle \\ |-\mathbf{a}\rangle &= \sin\frac{\theta}{2}|\mathbf{b}\rangle - \cos\frac{\theta}{2}|{-\mathbf{b}}\rangle\end{aligned}$$

Therefore:

$$\begin{aligned}\mathbb{P}_{\mathrm{QM}}(++|\mathbf{a},\mathbf{b},\psi) &= \mathbb{P}_{\mathrm{QM}}(--|\mathbf{a},\mathbf{b},\psi) = \frac{1}{2}\sin^2\frac{\theta}{2} \\ \mathbb{P}_{\mathrm{QM}}(+-|\mathbf{a},\mathbf{b},\psi) &= \mathbb{P}_{\mathrm{QM}}(-+|\mathbf{a},\mathbf{b},\psi) = \frac{1}{2}\cos^2\frac{\theta}{2}\end{aligned}$$

These results can be summarized as:

$$\mathbb{P}_{\mathrm{QM}}(AB|\mathbf{a},\mathbf{b},\psi) = \frac{1}{4}(1 - AB\mathbf{a}\cdot\mathbf{b})$$

where A and B denote the measurement outcome (±1). The expectation value is

$$\begin{aligned}E_{\mathrm{QM}}(\mathbf{a},\mathbf{b}|\psi) &= \mathbb{P}_{\mathrm{QM}}(++|\mathbf{a},\mathbf{b},\psi) + \mathbb{P}_{\mathrm{QM}}(--|\mathbf{a},\mathbf{b},\psi) \\ &\quad -\mathbb{P}_{\mathrm{QM}}(+-|\mathbf{a},\mathbf{b},\psi) - \mathbb{P}_{\mathrm{QM}}(-+|\mathbf{a},\mathbf{b},\psi) \\ &= -\cos\theta = -\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

The question is: can we construct a hidden variable model which makes the outcomes deterministic and at the same time reproduces quantum probabilities for the singlet state?

Let λ represent a unit vector uniformly distributed on the entire sphere

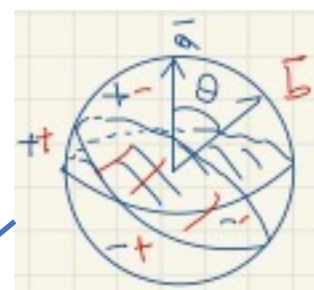
$$p_{\mathrm{HV}}(\lambda|\psi) = \frac{1}{4\pi}$$

Locality assumption. The outcome A of Alice's measurement is uniquely determined by the value of the hidden variable λ, the state vector ψ and the direction **a** of measurement, and nothing else; in particular, it does not depend on the direction of measurement **b** chosen by Bob. Similarly for Bob's outcome B:

$$\begin{aligned} A_{\mathrm{HV}} &= A_{\mathrm{HV}}(\mathbf{a}, \lambda, \psi) \\ B_{\mathrm{HV}} &= B_{\mathrm{HV}}(\mathbf{b}, \lambda, \psi) \end{aligned}$$

In particular we choose

$$\begin{aligned} A_{\mathrm{HV}} &= +\mathrm{sign}\, \mathbf{a} \cdot \lambda \\ B_{\mathrm{HV}} &= -\mathrm{sign}\, \mathbf{b} \cdot \lambda \end{aligned}$$



Then we have

$$\begin{aligned} \mathbb{P}_{\mathrm{HV}}(++|\mathbf{a},\mathbf{b},\psi) &= \mathbb{P}_{\mathrm{HV}}(--|\mathbf{a},\mathbf{b},\psi) = \frac{1}{4\pi}2\theta = \frac{\theta}{2\pi} \\ \mathbb{P}_{\mathrm{HV}}(+-|\mathbf{a},\mathbf{b},\psi) &= \mathbb{P}_{\mathrm{HV}}(-+|\mathbf{a},\mathbf{b},\psi) = \frac{1}{4\pi}(2\pi - 2\theta) = \frac{1}{2} - \frac{\theta}{2\pi} \end{aligned}$$

and

$$E_{\mathrm{HV}}(\mathbf{a},\mathbf{b}) = -1 + 2\frac{\theta}{\pi}$$

This model reproduces the cases considered by EPR:

$$\begin{aligned}
\mathbb{P}_{\rm HV}(++\,|\mathbf{a},\mathbf{a},\psi) &= \mathbb{P}_{\rm QM}(++\,|\mathbf{a},\mathbf{a},\psi) \;=\; 0 \\
\mathbb{P}_{\rm HV}(--\,|\mathbf{a},\mathbf{a},\psi) &= \mathbb{P}_{\rm QM}(--\,|\mathbf{a},\mathbf{a},\psi) \;=\; 0 \\
\mathbb{P}_{\rm HV}(+-\,|\mathbf{a},\mathbf{a},\psi) &= \mathbb{P}_{\rm QM}(+-\,|\mathbf{a},\mathbf{a},\psi) \;=\; 1/2 \\
\mathbb{P}_{\rm HV}(-+\,|\mathbf{a},\mathbf{a},\psi) &= \mathbb{P}_{\rm QM}(-+\,|\mathbf{a},\mathbf{a},\psi) \;=\; 1/2
\end{aligned}$$

Perfect anti-correlations are reproduced. However the quantum probabilities for general directions a and b are not reproduced by the hidden variable model. In particular:

$$E_{\rm HV}(\mathbf{a},\mathbf{b}) = -1 + 2\frac{\theta}{\pi} \;\neq\; E_{\rm QM}(\mathbf{a},\mathbf{b}|\psi) = -\cos\theta$$

With a **nonlocal model**, it would be easy to reproduce the quantum probabilities. It suffices for example to take

$$\begin{aligned}
A_{\rm HV} &= A_{\rm HV}(\mathbf{a},\mathbf{b},\lambda,\psi) = +\mathrm{sign}\ \mathbf{a}'\cdot\lambda \\
B_{\rm HV} &= B_{\rm HV}(\mathbf{b},\lambda,\psi) = -\mathrm{sign}\ \mathbf{b}\cdot\lambda
\end{aligned}$$

where **a’** is coplanar to **a** and **b** such that

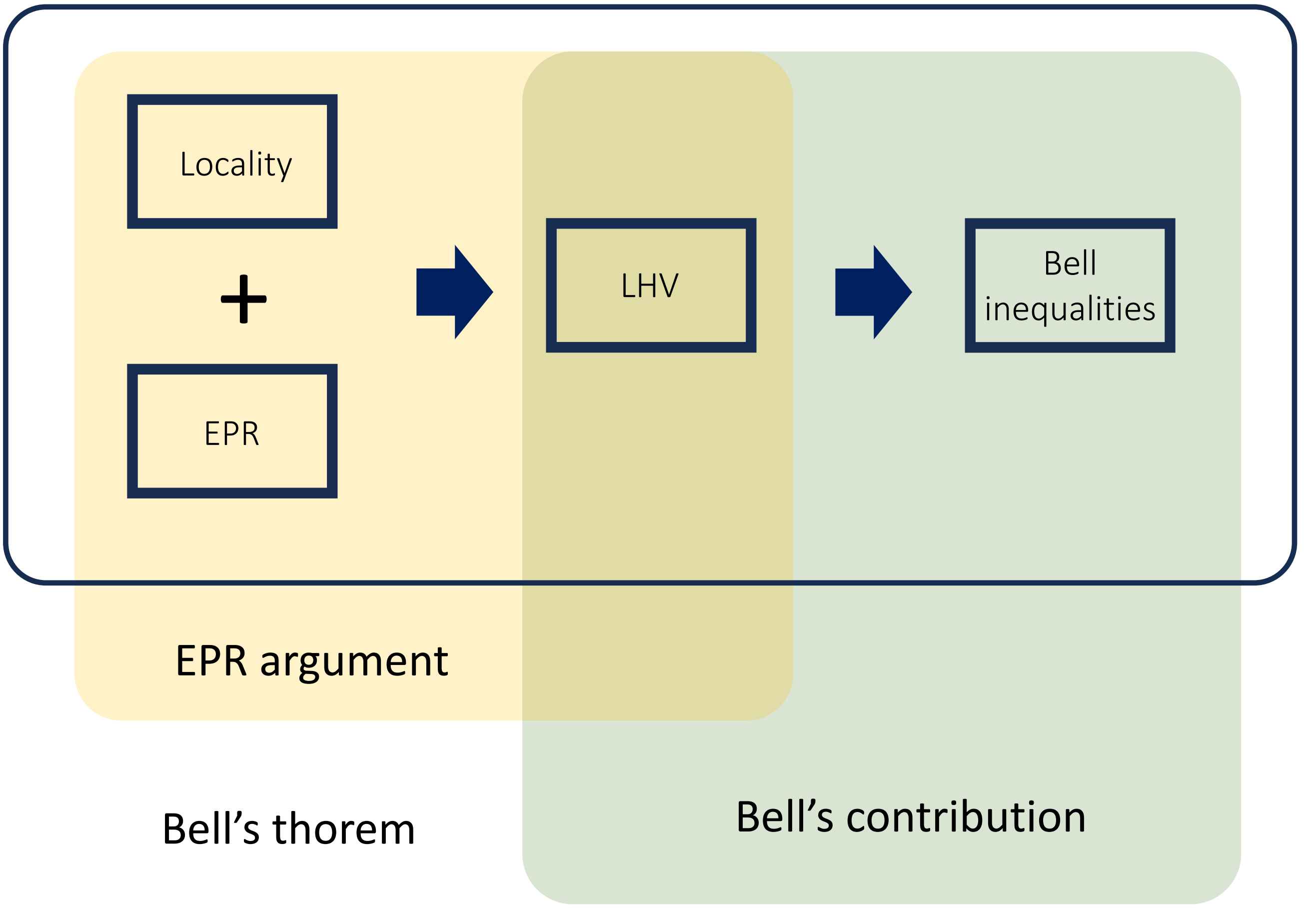
$$1 - 2\frac{\theta'}{\pi} = \cos\theta$$

Where θ’ is the angle between **a’** and **b**. Then, following the previous reasoning

$$E_{\rm HV}(\mathbf{a},\mathbf{b}) = -1 + 2\frac{\theta'}{\pi} = -\cos\theta = E_{\rm QM}(\mathbf{a},\mathbf{b})$$

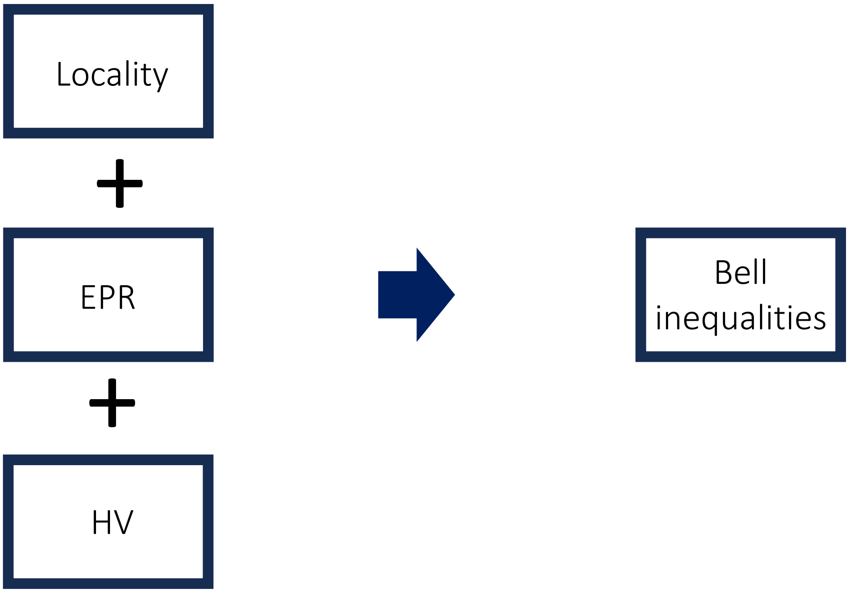
THE QUESTION is whether there exists a local hidden variable model capable of accounting for quantum predictions; Bell with his celebrated theorem gives a negative answer.

The logic of the 1964 Bell paper is the following



The experimental violation of Bell's inequalities implies that one of the two hypotheses is wrong. Since singlet's state correlations have been confirmed in experiments, **it is locality which must be abandoned**.

Over the years, it seems that Bell's theorem has been misunderstood as follows



So that the violation of Bell's inequalities, given that Singlet's correlations are verified, imply that **one can either give up locality or the existence of hidden variables**. This is summarized in the often cited supposed hypothesis of non-local realism underlying Bell's theorem. In this way one is allowed to give up hidden variables (after all, they are hidden), and maintain locality of homage to relativity.

This is a wrong way of presenting Bell's theorem. In the Introduction to the 1964 paper, Bell himself was clear that: "It is the requirement of locality [...] that creates the essential difficulty". And in the paper entitled "Bertlmann's socks and the nature of reality", footnote 10, he writes: "My own first paper on the subject (...) starts with a summary of the EPR argument *from locality to* deterministic hidden variables. But the commentaries have almost universally reported that it begins with deterministic hidden variables".

In the next section, we will present the modern version of Bell's inequalities, in the form provided in 1969 by Clauser-Holt-Horne-Shimony, without any reference to hidden variables. It is a form which is particularly suitable for experimental verification.

Note that the focus of Bell's theorem has shifted from determinism / hidden variables as in EPR to locality. The output of the EPR analysis did not go as Einstein would have hoped.

Bell's Theorem

With reference to the EPRB setup, we derive Bell's inequality in one of its simplest formulations, which is close to the original formulation.

Let

$$\mathbb{P}_{HV}(A(\mathbf{a}) = - B(\mathbf{b}) \mid \psi)$$

be the probability that, in a spin measurement performed by Alice along direction **a**, and a spin measurement performed by Bob along direction **b**, the outcomes are perfectly anti-correlated.

Note that the direction of measurements are different, and in this case there si no need for the outcomes to be antic-correlated. Note also the Bell is focussing on different directions, while the EPR argument considers only same directions, where the singlet's correlations are more manifest.

Now, considering three different directions **a**, **b** and **c** (we omit ψ):

$$\begin{aligned} \mathbb{P}_{\mathrm{HV}}(A(\mathbf{a}) = -B(\mathbf{b})) \;&+\; \mathbb{P}_{\mathrm{HV}}(A(\mathbf{b}) = -B(\mathbf{c})) \;+\; \mathbb{P}_{\mathrm{HV}}(A(\mathbf{c}) = -B(\mathbf{a})) \\ &= \mathbb{P}_{\mathrm{HV}}(A(\mathbf{a}) = A(\mathbf{b})) \;+\; \mathbb{P}_{\mathrm{HV}}(A(\mathbf{b}) = A(\mathbf{c})) \;+\; \mathbb{P}_{\mathrm{HV}}(A(\mathbf{c}) = A(\mathbf{a})) \\ &\geq \mathbb{P}_{\mathrm{HV}}(\{A(\mathbf{a}) = A(\mathbf{b})\} \cup \{A(\mathbf{b}) = A(\mathbf{c})\} \cup \{(\mathbf{c}) = A(\mathbf{a})\}) \\ &= 1 \end{aligned}$$

EPR anti-correlations

Subadditivity of measures

The last equality holds because, being the LHV (local hidden variables) dicotomic, at least one of the three cases holds true.

a	+	+	+	+	-	-	-	-
b	+	+	-	-	+	+	-	-
c	+	-	+	-	+	-	+	-

Space of the hivven variables λ, divided into 8 subsets

= subset where A(a) = a(b)

= subset where A(b) = a(c)

= subset where A(c) = a(a)

To summarize we have:

$$\mathbb{P}_{\text{HV}}(A(\mathbf{a}) = -B(\mathbf{b})) + \mathbb{P}_{\text{HV}}(A(\mathbf{b}) = -B(\mathbf{c})) + \mathbb{P}_{\text{HV}}(A(\mathbf{c}) = -B(\mathbf{a})) \geq 1$$

This is one version of Bell's inequality.

Quantum Mechanics contradicts this inequality; let us consider three directions of intermediate angles 120°. Then

$$\begin{aligned} \mathbb{P}_{\text{QM}}(A(\mathbf{a}) = -B(\mathbf{b})) &= \mathbb{P}_{\text{QM}}(+- |\mathbf{a}, \mathbf{b}, \psi) + \mathbb{P}_{\text{QM}}(-+ |\mathbf{a}, \mathbf{b}, \psi) \\ &= \cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta) \end{aligned}$$

where theta as usual is the angle formed by the two unit vectors. With our choice of directions we have:

$$\mathbb{P}_{\text{QM}}(A(\mathbf{a}) = -B(\mathbf{b})) = \frac{1}{4}$$

The same holds for the other two probabilities. Therefore:

$$\mathbb{P}_{\text{QM}}(A(\mathbf{a}) = -B(\mathbf{b})) + \mathbb{P}_{\text{QM}}(A(\mathbf{b}) = -B(\mathbf{c})) + \mathbb{P}_{\text{QM}}(A(\mathbf{c}) = -B(\mathbf{a})) = \frac{3}{4}$$

thus violating the inequality.

The moral is that there exists no local hidden variable model capable of reproducing the predictions of Quantum Mechanics (with respect to the singlet state). But since local hidden variables are a consequence of the locality assumption, given the EPR correlations, it follows that **there exists no local theory capable of reproducing the quantum mechanical predictions.**

The experimental violation of Bell's inequalities implies that no local theory is capable of describing the physics of entangled (singlet) states: Nature is nonlocal.

Bell’s Theorem - CHSH

Consider

$$\mathbb{P}(A, B|\mathbf{a}, \mathbf{b}, \lambda)$$

This is the probability that in a measurement of spin of the left particle along direction **a** the outcome is A, and in a measurement of spin of the right particle along direction **b** the outcome is B. (p can also be 0 or 1, if the theory is deterministic)

$\lambda$is the state of the two-particle system.

Classical mechanics: λ = positions and momenta of the particles
Quantum mechanics: λ = wave function
Bohmian mechanics: λ = wave function and positions of the particles

We are not committing to any specific theory.

As before

$\mathbb{P}(A|\mathbf{a}, \lambda)$ = same as before, but with no measurement B
$\mathbb{P}(B|\mathbf{b}, \lambda)$ = same as before, but with no measurement A

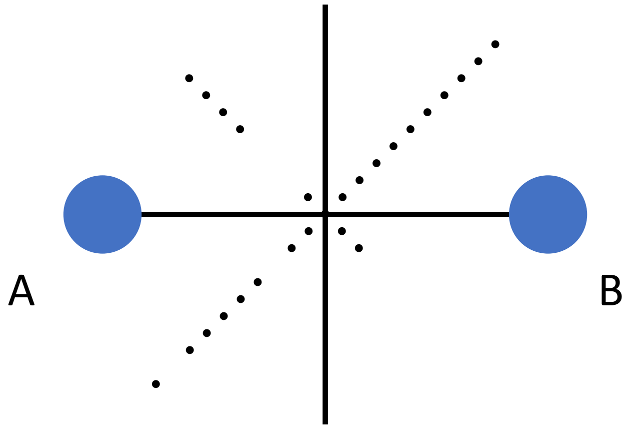
The definition of conditional probability implies:

$$\mathbb{P}(A, B|\mathbf{a}, \mathbf{b}, \lambda) = \mathbb{P}(A|B, \mathbf{a}, \mathbf{b}, \lambda) \cdot \mathbb{P}(B|\mathbf{a}, \mathbf{b}, \lambda)$$

Bell's definition of **locality**: When the two measurements are space-like separated from each other, what happens on one side cannot influence the other side.

$$\mathbb{P}(A|B,\mathbf{a},\mathbf{b},\lambda) = \mathbb{P}(A|\mathbf{a},\lambda)$$

$$\mathbb{P}(B|A,\mathbf{a},\mathbf{b},\lambda) = \mathbb{P}(B|\mathbf{b},\lambda)$$



Together with the rules of conditional probability, locality implies

$$\mathbb{P}(A,B|\mathbf{a},\mathbf{b},\lambda) = \mathbb{P}(A|\mathbf{a},\lambda)\cdot\mathbb{P}(B|\mathbf{b},\lambda)$$

Now we prove the theorem. Consider the expectation value, given λ

$$E_\lambda(\mathbf{a},\mathbf{b}) = \mathbb{P}(+,+|\mathbf{a},\mathbf{b},\lambda) + \mathbb{P}(-,-|\mathbf{a},\mathbf{b},\lambda) - \mathbb{P}(+,-|\mathbf{a},\mathbf{b},\lambda) - \mathbb{P}(-,+|\mathbf{a},\mathbf{b},\lambda)$$

which is the sum of agreements minus sum of disagreements. Then, using Bell's locality condition:

$$E_\lambda(\mathbf{a},\mathbf{b}) = [\mathbb{P}(+|\mathbf{a},\lambda) - \mathbb{P}(-|\mathbf{a},\lambda)][\mathbb{P}(+|\mathbf{b},\lambda) - \mathbb{P}(-|\mathbf{b},\lambda)]$$

and

$$E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d}) = \underbrace{[\mathbb{P}(+|\mathbf{a},\lambda) - \mathbb{P}(-|\mathbf{a},\lambda)]}_{= 1 - 2\mathbb{P}(-|\mathbf{a},\lambda) \in [-1,+1]}[(\mathbb{P}(+|\mathbf{b},\lambda) - \mathbb{P}(-|\mathbf{b},\lambda)) - (\mathbb{P}(+|\mathbf{d},\lambda) - \mathbb{P}(-|\mathbf{d},\lambda))]$$

Therefore

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| \leq |\underbrace{(\mathbb{P}(+|\mathbf{b},\lambda) - \mathbb{P}(-|\mathbf{b},\lambda))}_{\text{r}} - \underbrace{(\mathbb{P}(+|\mathbf{d},\lambda) - \mathbb{P}(-|\mathbf{d},\lambda))}_{\text{s}}|$$

$$|E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \leq |\underbrace{(\mathbb{P}(+|\mathbf{b},\lambda) - \mathbb{P}(-|\mathbf{b},\lambda))}_{\text{r}} + \underbrace{(\mathbb{P}(+|\mathbf{d},\lambda) - \mathbb{P}(-|\mathbf{d},\lambda))}_{\text{s}}|$$

So:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \leq |r-s| + |r+s|$$

Taking the square we have:

$$[|r-s| + |r+s|]^2 = 2r^2 + 2s^2 + 2|r^2 - s^2|$$

which is either equal to $4r^2$ or to $4s^2$; in either case, it is less than or equal to 4, since $r, s \in [-1,+1]$. So:

$$|r-s| + |r+s| \leq 2$$

So we end up with:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \leq 2$$

This is Bell's inequality, which is the direct consequence of Bell's locality condition alone

$$\mathbb{P}(A,B|\mathbf{a},\mathbf{b},\lambda) = \mathbb{P}(A|\mathbf{a},\lambda) \cdot \mathbb{P}(B|\mathbf{b},\lambda)$$

$$\Downarrow$$

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \leq 2$$

Now always we have full control of the state λ of the system. For example, in Bohmian Mechanics we can reasonably control the wave function ψ, but not the positions of all particles Therefore the above inequalities, as they are, are not always testable.

Solution: let us separate $\lambda = (\mu, \nu)$, where μ are controllable and ν are uncontrollable degrees of freedom. The physically measurable quantity is:

$$E_\mu(\mathbf{a}, \mathbf{b}) = \int E_{(\mu,\nu)}(\mathbf{a}, \mathbf{b}) \rho(\nu) d\nu$$

↑ Probability distribution; it reflects our ignorance

Given

$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq 2$$

We have

$$|E_\mu(\mathbf{a}, \mathbf{b}) - E_\mu(\mathbf{a}, \mathbf{d})| + |E_\mu(\mathbf{c}, \mathbf{b}) + E_\mu(\mathbf{c}, \mathbf{d})| \leq$$

$$\leq \int d\nu \rho(\nu) \left[|E_{(\mu,\nu)}(\mathbf{a}, \mathbf{b}) - E_{(\mu,\nu)}(\mathbf{a}, \mathbf{d})| + |E_{(\mu,\nu)}(\mathbf{c}, \mathbf{b}) + E_{(\mu,\nu)}(\mathbf{c}, \mathbf{d})| \right]$$

$$\leq 2 \int d\nu \rho(\nu) = 2$$

The inequality still holds.

Application to QM

As we have seen:

$$E_{\mathrm{QM}}(\mathbf{a},\mathbf{b}|\psi) = -\cos\theta_{\mathbf{a},\mathbf{b}}$$

Then:

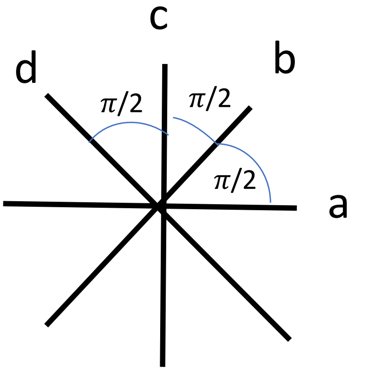
$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| =$$

$$= |\cos\theta_{\mathbf{a},\mathbf{b}} - \cos\theta_{\mathbf{a},\mathbf{d}}| + |\cos\theta_{\mathbf{c},\mathbf{b}} + \cos\theta_{\mathbf{c},\mathbf{d}}|$$

Let us choose the four angles as in the picture

$$= \left|\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right| + \left|\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right|$$

$$= 2\sqrt{2} \geq 2$$



The inequality is violated: QM is nonlocal.

The source of the nonlocality is in the collapse of the wave function. Since the wave function is real and since the collapse changes it at a distance, then there is a nonlocal effect.

Experiments showed that the inequalities are violated, therefore we conclude that Nature is nonlocal

Conspiracy theories

Bell's proof worked because we considered the distribution of the hidden variables independent from the settings a and b

$$\rho(\lambda) \neq \rho(\lambda, \mathbf{a}, \mathbf{b})$$

<u>But if it does depend on the settings</u>, it is easy to recover Quantum probabilities from LHV. Consider the same LHV proposed by Bell, with now

$$\rho(\lambda) = \rho(\lambda, \mathbf{a}, \mathbf{b}) = \frac{1+\cos\theta}{8(\pi-\theta)} \quad \text{if} \quad \mathrm{sign}(\lambda\cdot\mathbf{a}) = \mathrm{sign}(\lambda\cdot\mathbf{b})$$

$$= \frac{1-\cos\theta}{8\theta} \quad \text{if} \quad \mathrm{sign}(\lambda\cdot\mathbf{a}) \neq \mathrm{sign}(\lambda\cdot\mathbf{b})$$

where, again, theta is the angle between a and b.

$$\mathbb{P}_{\mathrm{HV}}(++|\mathbf{a},\mathbf{b},\psi) = \frac{1-\cos\theta}{8\theta}\cdot 2\theta = \frac{1-\cos\theta}{4} = \mathbb{P}_{\mathrm{QM}}(++|\mathbf{a},\mathbf{b},\psi)$$

and similarly for the other probabilities. We are capable of recovering quantum probabilities from a local model.

There are essentially two ways of satisfy the condition

$$\rho(\lambda) = \rho(\lambda, \mathbf{a}, \mathbf{b})$$

The above condition implies that the distribution of the hidden variables depend on the choice of the settings. The are basically two way for this to happen:

- Retro-causation: the choice of the settings in the future influence the distribution of the hidden variables in the past.
- Super-determinism: the settings were decided in the first place, together with the hidden variables.

The first situation implies that the future can influence the past; the second assumption denies free will in experiments: everything is pre-determined.

Both assumptions are demanding and given that they are put forward only to escape the conclusions of Bell’s theorem, it is much simpler to accept that Nature is nonlocal.

No signalling theorem

Quantum nonlocality cannot be used to send information faster than the speed of light. Actually measurements cannot send information at all

We have two systems A and B, which in general share an entangled state ρ_{AB}. They are apart from each other. Arbitrary measurements can be performed on each of them.

Alice performs a measurement of an observable Â with eigen-projectors P_n^A. The state at Bob's side changes to:

$$\rho_{AB} \to \rho'_{AB} = \sum_n Tr[(P_n^A \otimes I^B)\rho_{AB}] \frac{(P_n^A \otimes I^B)\rho_{AB}(P_n^A \otimes I^B)}{Tr[(P_n^A \otimes I^B)\rho_{AB}]} = \sum_n (P_n^A \otimes I^B)\rho_{AB}(P_n^A \otimes I^B)$$

Born rule

Von Neumann collapse

Then the average value of measurements Bob performs are given by:

$$\langle O^B \rangle' = \mathbf{Tr}[(I^A \otimes O^B)\rho'_{AB}] = \sum_n \mathbf{Tr}[(I^A \otimes O^B)(P_n^A \otimes I^B)\rho_{AB}(P_n^A \otimes I^B)]$$

Cyclicity of trace

$$= \sum_n \mathbf{Tr}[(I^A \otimes O^B)(P_n^A \otimes I^B)^2 \rho_{AB}]$$

Idempotent

$$= \sum_n \mathbf{Tr}[(I^A \otimes O^B)(P_n^A \otimes I^B)\rho_{AB}]$$

Linearity of trace + Projectors sum to 1

$$= \mathbf{Tr}[(I^A \otimes O^B)\rho_{AB}] = \langle O^B \rangle$$

The value Bob gets is the same before and after Alice’s measurement. Bob does not see any difference in the statistics of the outcomes of his measurements. There is no quantum operation (= unitary evolution or measurement) Alice can do, that allows her to send information to Bob.

If one looks at the reason why it is so, it ultimately rests on the fact that

$$\rho_{AB} \rightarrow \rho'_{AB} = \sum_n Tr[(P_n^A \otimes I^B)\rho_{AB}] \frac{(P_n^A \otimes I^B)\rho_{AB}(P_n^A \otimes I^B)}{Tr[(P_n^A \otimes I^B)\rho_{AB}]} = \sum_n (P_n^A \otimes I^B)\rho_{AB}(P_n^A \otimes I^B)$$

Born rule

Von Neumann collapse

In measurements, the Born rule and the von Neumann collapse are just the right recipes that avoid superluminal communication.

This calls for an explanation.