$\rightarrow E[Z; A] = E[X; A]$ PER OGNI $A \in \mathcal{G}$

EQUIVALE A

$E[Z|A] = E[X|A]$ PER OGNI $A \in \mathcal{G}$
CON $P(A) > 0$

$E[E[X|\mathcal{G}]|A] = E[X|A]$

SE Y SEMPLICE, $\mathcal{G}$ - MISURABILE

$$Y = \sum c_i 1_{A_i} \qquad A_i \in \mathcal{G}$$

$$\underline{\underline{E[Z \cdot Y]}} = E\left[Z \sum c_i 1_{A_i}\right]$$

$$= E\left[\sum c_i Z 1_{A_i}\right]$$

$$= \sum c_i \; E[Z 1_{A_i}]$$

$$= \sum c_i \; E[Z ; A_i]$$

$$= \sum c_i \; E[X ; A_i] = \ldots = \underline{E[X \cdot Y]}$$

$$E[1_A \mid \mathcal{G}] = P(A \mid \mathcal{G})$$

$$\mathcal{G} = \sigma(Y_1, Y_2, \ldots, Y_n)$$

$$E[X \mid \mathcal{G}] = E[X \mid Y_1 \ldots, Y_n]$$ È UNA V.A. $\mathcal{G}$-MISURABILE CIOÈ $\sigma(Y_1 \ldots Y_n)$-MISURABILE

QUINDI ESISTE UNA FUNZIONE $h: \mathbb{R}^n \to \mathbb{R}$ MISURABILE TALE CHE

TALE CHE

$$E[X \mid Y_1 \ldots Y_n] = h(Y_1 \ldots Y_n)$$

PER CONVENZIONE, SI PONE

$$h(y_1, \ldots, y_n) = E[X \mid \underbrace{Y_1 = y_1, \ldots, Y_n = y_n}_{\text{PROB } 0\,?}]$$

$$E[X \mid \{\emptyset, \Omega\}] = E[X]$$

1) OGNI FUNZIONE COSTANTE È MISURABILE RISPETTO A $\{\emptyset, \Omega\}$

2) OK

3) $A \in \mathcal{G} = \{\emptyset, \Omega\}$

$$E[E[X]; A] = E[X; A]$$

$$\overbrace{E[E[X]; \emptyset]}^{=0} = \overbrace{E[X; \emptyset]}^{=0}$$

$$\underbrace{E[E[X]; \Omega]}_{E[X]} = \underbrace{E[X; \Omega]}_{E[X]}$$

$$E[E[X|\mathcal{G}]] = E[X]$$

BASTA PRENDERE IN (3), $A = \Omega \in \mathcal{G}$

$$E[\alpha X+\beta Y\mid \mathcal{G}] = \underbrace{\alpha E[X\mid \mathcal{G}] + \beta E[Y\mid \mathcal{G}]}_{Z}$$

DIMOSTRO CHE Z HA LE PROPRIETÀ 1, 2, 3 DELLA SPERANZA CONDIZIONATA $E[\alpha X+\beta Y\mid \mathcal{G}]$

1) Z È $\mathcal{G}$-MISURABILE

$$Z = \alpha \underbrace{E(X\mid \mathcal{G})}_{\mathcal{G}\text{-MIS.}} + \beta \underbrace{E(Y\mid \mathcal{G})}_{\mathcal{G}\text{-MIS.}} \Bigg\} \mathcal{G}\text{-MIS.}$$

2) $Z \in L^1$? (L^1 È UNO SPAZIO VETTORIALE)

3) $A \in \mathcal{G}$

$$E[Z;A] = E[(\alpha X+\beta Y);A] \qquad ?$$

$$E[Z;A] = E[(\alpha E(X|\mathcal{G}) + \beta E(Y|\mathcal{G}))\cdot 1_A] =$$

$$= \alpha \underbrace{E[E(X|\mathcal{G})1_A]}_{\underbrace{E[E(X|\mathcal{G});A]}_{E(X;A)}} + \beta \underbrace{E[E(Y|\mathcal{G})1_A]}_{E(Y;A)}$$

$$= E((\alpha X + \beta Y);A)$$

$E(XY|\mathcal{G}) = Y E(X|\mathcal{G})$ SE Y - $\mathcal{G}$ MIS. TALE CHE $XY \in L^1$

MOSTRO CHE $Y \cdot E(X|\mathcal{G})$ HA LE PROPRIETÀ 1, ~~2~~, 3 CHE DEFINISCONO $E(XY|\mathcal{G})$

1) $\underbrace{Y}_{\mathcal{G}\text{-MIS.}} \cdot \underbrace{E(X|\mathcal{G})}_{\mathcal{G}\text{ MIS.}}$ È $\mathcal{G}$-MIS.

3) MOSTRO CHE

$$E(Y\,E(X|\mathcal{G})\cdot V) = E(XY\cdot V)$$

PER OGNI V $\mathcal{G}$-MIS. (TALE CHE LE SPERANZE ESISTONO)

$$E(Y\,E(X|\mathcal{G})\cdot V) = E(E(X|\mathcal{G})\cdot \underbrace{YV}_{\mathcal{G}\text{ MISURABILE}}) =$$

$$= E(XYV)$$

Y $\mathcal{G}$ MIS, X INDIP. DA $\mathcal{G}$

$$E[g_1(X)\cdot g_2(Y) \mid \mathcal{G}] = g_2(Y)\, E[g_1(X) \cancel{\mid \mathcal{G}}]$$

$$= g_2(Y)\, E[g_1(X)]$$

$$E[g(X,Y) \mid \mathcal{G}] = E[g(X,y) \cancel{\mid \mathcal{G}}]_{y=Y}$$

$$\mathrm{COV}(X,Y) \quad \begin{array}{ll} E\left[\mathrm{COV}(X,Y \mid \mathcal{G})\right] & \text{WITHIN} \\ + & \\ \mathrm{COV}\left(E(X \mid \mathcal{G}),\, E(Y \mid \mathcal{G})\right) & \text{BETWEEN} \end{array}$$

$$\mathrm{VAR}(X) = E\left(\mathrm{VAR}(X \mid \mathcal{G})\right) + \mathrm{VAR}\left(E(X \mid \mathcal{G})\right)$$

$$\underbrace{E(X \mid N, V)}_{\text{LEGGE CONDIZIONATA: BINOMIALE}(N,\, V)} = N \cdot V$$

$$E[X \mid N] = E\big(E(X \mid N, V) \mid N\big)$$

[PROP. ITERATIVA

$$= E(N \cdot V \mid N)$$

$$= N \cdot E(V \mid \cancel{N}) = \frac{N}{2}$$