Esercizi su variabili aleatorie

Sistemi Dinamici

a.a. 2022/2023

Somma e differenza di 2 v.a.

Considariamo 2 v. aleatorie

X t.c. $E(X) = m_x$

$var(X) = \sigma_x^2$

Y t.c. $E(Y) = m_y$

$var(Y) = \sigma_y^2$

$Z = X + Y$

$E(Z) = ?$

$var(Z) = ?$

$W = X - Y$

$E(W)?$

$var(W)?$

$$Z = X + Y \qquad E(Z) = E(X+Y) = E(X) + E(Y)$$

ju lineeik

$$m_z = m_x + m_y$$

$$W = X - Y \qquad E(W) = E(X-Y) = E(X) - E(Y)$$

$$m_w = m_x - m_y$$

$$Z = X + Y$$

$$\sigma_z^2 = E\left[(z - \mu_z)^2\right]$$

$$\mu_z = \mu_x + \mu_y$$

$$= E\left[z^2 - 2\mu_z z + \mu_z^2\right]$$

$$z = X + Y$$

$$= E\left\{(X+Y)^2 - 2(\mu_x + \mu_y)(X+Y) + (\mu_x + \mu_y)^2\right\} = ?$$

$$\sigma_z^2 = E[X^2] + E[Y^2] + 2E[XY] + $$

$$-2(m_x + m_y)(m_x + m_y) + (m_x + m_y)^2$$

$$= E[X^2] + E[Y^2] - (m_x + m_y)^2 + $$

$$+ 2E[XY]$$

$$= \{E[X^2] - m_x^2\} + \{E[Y^2] - m_y^2\} + $$

$$+ 2\{E[XY] - m_x m_y\}$$

Ricordando che valgono:

$$\sigma_X^2 = E[X^2] - m_x^2$$

$$\sigma_Y^2 = E[Y^2] - m_y^2$$

$$\sigma_{XY} \triangleq E[(X - m_x)(Y - m_y)] =$$

$$= E[XY - m_y X - m_x Y + m_x m_y]$$

$$\sigma_{xy} \triangleq E\left[(X-m_x)(Y-m_y)\right] = E(XY) - m_x m_y$$

In definitiva:

$$\sigma_t^2 = \overbrace{\left\{E\left[X^2\right]-m_x^2\right\}}^{\sigma_x^2} + \overbrace{\left\{E\left[Y^2\right]-m_y^2\right\}}^{\sigma_y^2} + 2\underbrace{\left\{E\left[XY\right]-m_x m_y\right\}}_{\sigma_{xy}}$$

$$\sigma_t^2 = \sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}$$

Se x, y scorrelate $\rightarrow \sigma_{xy} = 0$

e quindi

$$\sigma_t^2 = \sigma_x^2 + \sigma_y^2$$

$$W = X - Y$$

$$\sigma_W^2 = E\left[(W - m_w)^2\right]$$

$$= E\left\{W^2 - 2m_w W + m_w^2\right\} \leftarrow m_w = m_x - m_y$$

$$= E\left\{X^2 + Y^2 - 2XY - 2(m_x - m_y)(X - Y) + \right.$$

$$\left. + m_x^2 + m_y^2 - 2m_x m_y\right\} \Rightarrow$$

$$\sigma_W^2 = E\{X^2\} + E\{Y^2\} - 2E(XY) - 2m_x^2 +$$

$$+ 2m_x m_y + 2m_x m_y - 2m_y^2 + m_x^2 + m_y^2 - 2m_x m_y$$

$$= \underbrace{\{E[X^2] - m_x^2\}}_{\sigma_x^2} + \overbrace{\{E[Y^2] - m_y^2\}}^{\sigma_y^2} +$$

$$-2\,\overbrace{\{E(XY) - m_x m_y\}}^{\sigma_{xy}}$$

$$\sigma_W^2 = \sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}$$

Se X, Y scorrelate $\rightarrow \sigma_{xy} = 0$

$$\sigma_W^2 = \sigma_x^2 + \sigma_y^2$$

Valore atteso e varianza di una v.a. $\sim \mathcal{U}[0,1]$

v.a. con distribuzione uniforme sull'intervallo $[0,1]$

$$v \sim \mathcal{U}[0,1] \Leftrightarrow f_v(q) = \begin{cases} 1 & q \in [0,1] \\ 0 & q \in \mathbb{R}/[0,1] \end{cases}$$

$E(v) = ?$ $\qquad \sigma_v^2 = ?$

Data $f_v(q)$, quale è $F_v(q)$?

$$F_v(q) = \int_{-\infty}^{q} f_v(q)\,dq =$$ ← devo distinguere i casi:

$q < 0$

$0 \le q \le 1$

$q > 1$

$q < 0$ $$F_v(q) = \int_{-\infty}^{q} f_v(q)\,dq = 0$$

$f_v \equiv 0$

$0 \leq q \leq 1$ $\quad f_v(q) = 1$ per $q \in [0, 1]$

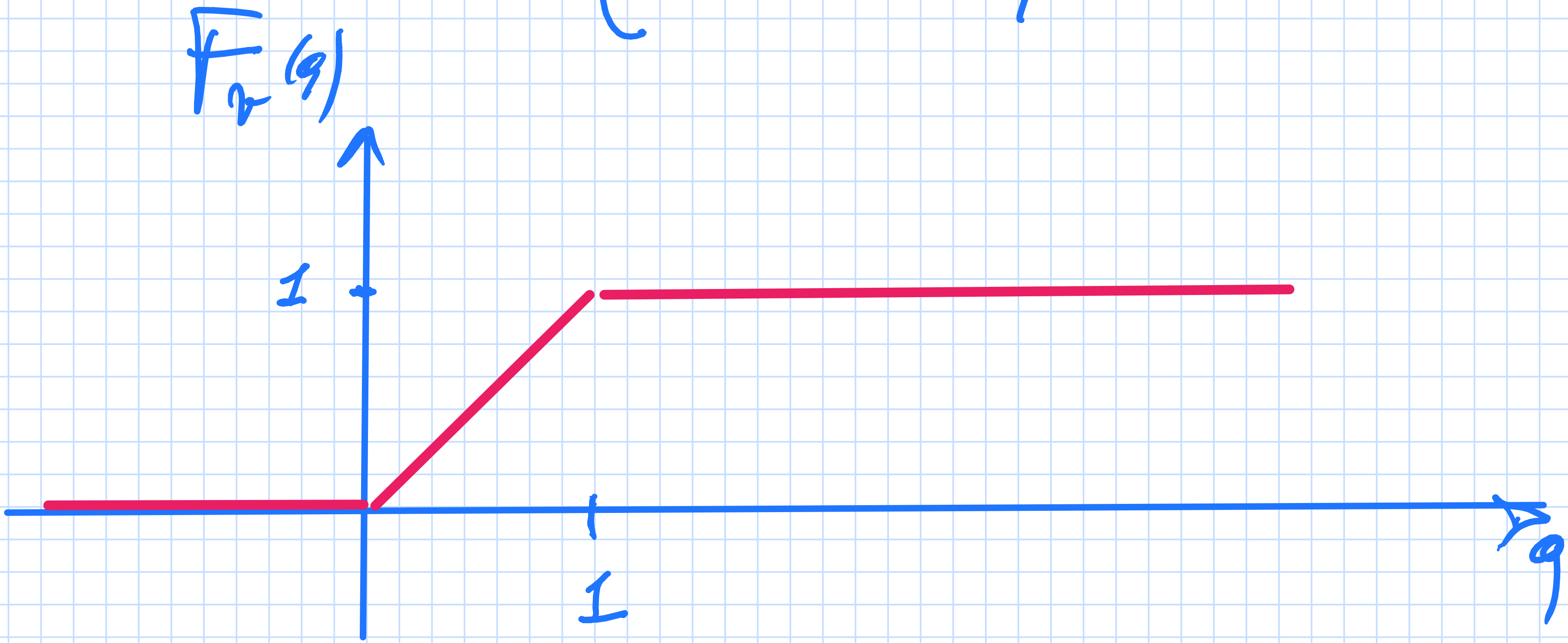
$$F_v(q) = \int_{-\infty}^{q} f_v(q)\, dq = \int_{0}^{q} 1\, dq = q$$

$q > 1$ $\quad f_v(q) = 0$

$$F_v(q) = \int_{-\infty}^{q} f_v(q)\, dq = \int_{0}^{1} 1 \cdot dq = 1$$

In definitiva:

$$F_v(q) = \begin{cases} 0 & q < 0 \\ q & 0 \le q \le 1 \\ 1 & q > 1 \end{cases}$$



$v \sim \mathcal{U}[0,1]$ $\quad E(v) = ?$

$$E(v) = \int_{-\infty}^{+\infty} q\, f_v(q)\, dq = \int_0^1 q\, dq = \frac{1}{2} q^2 \Big|_0^1 = \frac{1}{2}$$

$$\sigma_v^2 = \int_{-\infty}^{+\infty} \left[q - E(v) \right]^2 f_v(q)\, dq \longrightarrow$$

$$\sigma_v^2 = E\left\{\left[v - E(v)\right]^2\right\}$$

$$= E\left[v^2\right] - \left[E(v)\right]^2$$

$$E(v) = \frac{1}{2}$$

$$E(v^2) = \int_{-\infty}^{+\infty} q^2 f_v(q)\,dq = \int_0^1 q^2\,dq = \frac{1}{3} q^3 \Big|_0^1 = \frac{1}{3}$$

$$\sigma_v^2 = E\left[v^2\right] - \left[E(v)\right]^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Trasformazione lineare applicata a v.a.

Data v.a. X, con

$$\mu_X = E(X), \quad \sigma_X^2 = \mathrm{var}(X)$$

Sia

$$Y = aX + b \qquad \text{con } a, b \in \mathbb{R}$$

$E(Y)$? $\quad \mathrm{var}(Y) = ?$

Per linearità

$$E(Y) = E\left[aX + b\right] = aE(X) + b$$

quindi

$$\mu_y = a\,\mu_x + b$$

Per la varianza:

$$\sigma_y^2 = E\left[\left(Y - \mu_y\right)^2\right] =$$

sostituisco

$Y = aX + b$

$\mu_y = a\mu_x + b$

$$\sigma_y^2 = E\left\{\left[(aX+b) - (am_x+b)\right]^2\right\} =$$

$$= E\left\{\left[a(X-m_x) + \cancel{b} - \cancel{b}\right]^2\right\}$$

$$= E\left\{a^2 (X-m_x)^2\right\} = a^2 E\left[(X-m_x)^2\right]$$

$$\sigma_y^2 = a^2 E\left[(x - \mu_x)^2\right]$$

$$= a^2 \sigma_x^2$$

Esempio d'uso:

$$v \sim \mathcal{U}[0,1] \implies z = a + (b-a)v$$

$$z \sim \mathcal{U}[a,b]$$

$$E(z) = a + (b-a)\,m_v = a + \frac{b-a}{2} = \frac{b+a}{2}$$

$$\sigma_z^2 = (b-a)^2\sigma_v^2 = \frac{(b-a)^2}{12}$$

In Matlab: v. aleatorie con distribuzione uniforme in $[0,1]$

$\text{rand}(\ )$

$$z = a + (b-a) \cdot \text{rand}(\ )$$

$$z \sim \mathcal{U}[a,b]$$

Altro esempio d'uso delle trasf. lineari

$v \sim \mathcal{N}(0,1)$ v.a. gaussiana

$E(v) = 0$

$\sigma_v^2 = 1$

$X = \mu + \sigma v$ X è v.a. gaussiana

con $\mu, \sigma \in \mathbb{R}$

$E(X) = \mu$

$\sigma_X^2 = \sigma^2 \cdot 1$

In Matlab:

v.a. gaussiana con valor atteso $\emptyset$ e varianza 1:

randn( )

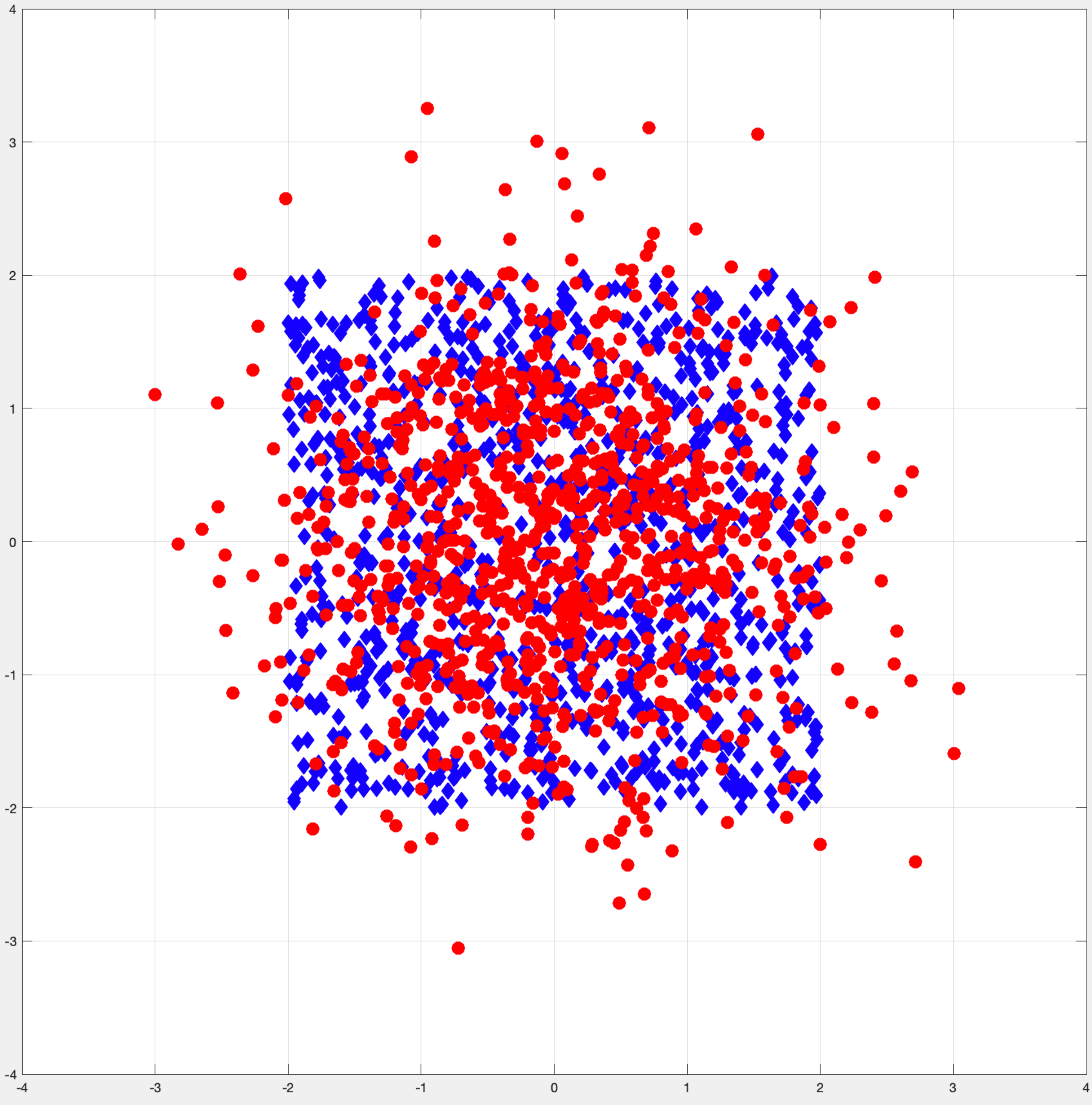
$x = \mu + \sigma \cdot$ randn( )

```matlab
N = 1000;
a = -2; b = +2;

v_unif = a + (b-a) * rand(N,2);
% N bidimensional random variables, with unif.
distribution in [-1 +1]

figure;
plot(v_unif(:,1), v_unif(:,2),...
    'd','MarkerEdgeColor','b',...
        'MarkerFaceColor','b',...
        'MarkerSize',10);
grid on
hold on;

v_gauss = randn(N,2);
% gaussian random varaible, with expected value 0 and
variance 1
plot(v_gauss(:,1), v_gauss(:,2),...
        'o','MarkerEdgeColor','r',...
            'MarkerFaceColor','r',...
            'MarkerSize',10);
axis square
```



4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4