Esercizi su processi MA ed AR

funzioni di autocorrelazione

spettro

equazioni di Yule-Walker

Stima della funzione di autocorrelazione

Equazioni di Yule-Walker

$\{y(t)\}$ processo stocastico stazionario

$$E[y(t)] = 0$$

k	0	1	2	3	4	5	6	7
y	3,211	2,316	3,447	4,751	3,447	0,722	0,568	-0,712

(a) $\hat{\gamma}_y(0) = ?$ $\hat{\gamma}_y(1) = ?$ $\hat{\gamma}_y(2) = ?$

(b) MA(1) oppure AR(1)?

(c) $\hat{\sigma} = ?$

$$\hat{\gamma}(k) = \frac{1}{N-|k|} \sum_{i=0}^{N-|k|-1} y(i)\, y(i+k) \qquad |k| < N$$

$$\hat{\gamma}(0) = \frac{1}{8} \sum_{i=0}^{7} \left[y(i) \right]^2 \simeq 7{,}357$$

$$\hat{\gamma}(1) = \frac{1}{7} \sum_{i=0}^{6} y(i)\, y(i+1) \simeq 6{,}746$$

$$\hat{\gamma}(2) = \frac{1}{6} \sum_{i=0}^{5} y(i)\, y(i+2) \simeq 6{,}218$$

$$\hat{\gamma}(0) \simeq 7.357$$

$$\hat{\gamma}(1) \simeq 6.746$$

$$\hat{\gamma}(2) \simeq 6.218$$

$\hat{\gamma}(2) \neq 0$

NON può essere MA(1)

$\hat{\gamma}(2) \neq 0 \Rightarrow$ il processo è AR(1)

$\mathcal{M} \quad y(t) = a\, y(t-1) + \eta(t)$

$\eta(\cdot) \sim WN(0, \lambda^2)$

$\hat{a} = ?$

$\hat{\lambda}^2 = ?$

equazioni di Yule-Walker

$$\gamma(1) = a\,\gamma(0)$$

$$\gamma(2) = a\,\gamma(1)$$

$$\gamma(0) = a\,\gamma(1) + \lambda^2$$

$$\hat{\gamma}(\tau) \xrightarrow[N \to \infty]{} \gamma(\tau)$$

$$\hat{\gamma}(1) = \hat{a}\,\hat{\gamma}(0) \Longrightarrow \hat{a} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \approx 0{,}92$$

$$\hat{\gamma}(2) = \hat{a}\,\hat{\gamma}(1) \Longrightarrow \hat{a} = \frac{\hat{\gamma}(2)}{\hat{\gamma}(0)} \approx 0{,}92$$

$$\hat{\gamma}(0) = \hat{a}\,\hat{\gamma}(1) + \hat{\sigma}^2 \Longrightarrow \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{a}\,\hat{\gamma}(1) \approx 1{,}15$$

Stima dei parametri

di un modello MA(1)

Spettro del processo

Del processo stocastico stazionario $v(t)$ si conosce:

a) $E[v(t)] \equiv 0$

b) $\begin{cases} \gamma(0) = +5 \\ \gamma(\pm 1) = +2 \\ \gamma(\pm \tau) = 0 \quad |\tau| \geq 2 \end{cases}$

È un processo MA(1). Determinare il modello del processo

$$\mathcal{M}: \quad v(t) = C(z)\,\eta(t)$$

$$\eta \sim WN(0, \lambda^2)$$

?

$$C(z) = 1 + c_1 z^{-1}$$

$$c_1 = ?$$

MA(1) $\qquad C(z) = 1 + c_1 z^{-1}$

$$\gamma(0) = (1^2 + c_1^2)\,\lambda^2 = +5$$

$$\gamma(\pm 1) = c_0 c_1 \lambda^2 = c_1 \lambda^2 = +2$$

In definitiva:

$$\begin{cases} (1 + c_1)\,\lambda^2 = +5 \\ c_1 \lambda^2 = +2 \end{cases}$$

$$c_1 d^2 = 2 \Rightarrow d^2 = \frac{2}{c_1}$$

$$\Downarrow$$

$$(1 + c_1^2) d^2 = 5$$

$$(1 + c_1^2) \frac{2}{c_1} = 5 \quad \Big/ \cdot c_1 \left(\neq 0 \text{ perché? }\right)$$

$$2c_1^2 - 5c_1 + 2 = 0$$

$$\nearrow$$

$$c_1 \begin{cases} +2 \\ +\frac{1}{2} \end{cases} \Rightarrow \begin{matrix} \lambda^2 = +1 \\ \lambda^2 = 4 \end{matrix}$$

2 possibili modelli?

Quale scelgo?

$$\eta \longrightarrow \boxed{W(z)} \longrightarrow v$$

$\eta \sim WN$

$$W(z) = \frac{N(z)}{D(z)} \Longleftarrow$$

per il Teorema della fattorizzazione spettrale

$$\Longleftrightarrow$$

$$|p_i| < 1 \quad \forall i$$

$$|z_j| \le 1 \quad \forall j$$

$N(z), D(z)$ pol. monici

stesso grado

$$W(z) = 1 + c_1 z^{-1} = \frac{z + c_1}{z}$$

$N(z)$, $D(z)$ pol. monici ✓
stesso grado ✓

$|p_i| < 1 \quad p_1 = 0$ ✓

$|z_i| = |-c_1| \leq 1$ NB $c_1 = +\frac{1}{2}$

$c_1 = +\frac{1}{2}$ $\quad C(z) = 1 + \frac{1}{2} z^{-1}$ $\quad MA(1)$

$\lambda^2 = 4$ $\quad W(z) = \dfrac{z + 1/2}{z}$

$\eta \sim WN(0, 4)$

$\Gamma(\omega) = ?$ $\Rightarrow \Gamma(\omega) = |W(e^{i\omega})|^2 \cdot \lambda^2$

$$\Gamma(\omega) = \frac{\left| e^{j\omega} + \frac{1}{2} \right|^2}{\left| e^{j\omega} \right|^2} \cdot 4$$

$$= \frac{4 \cdot \left| \cos\omega + \frac{1}{2} + j\,\text{sen}\,\omega \right|^2}{1}$$

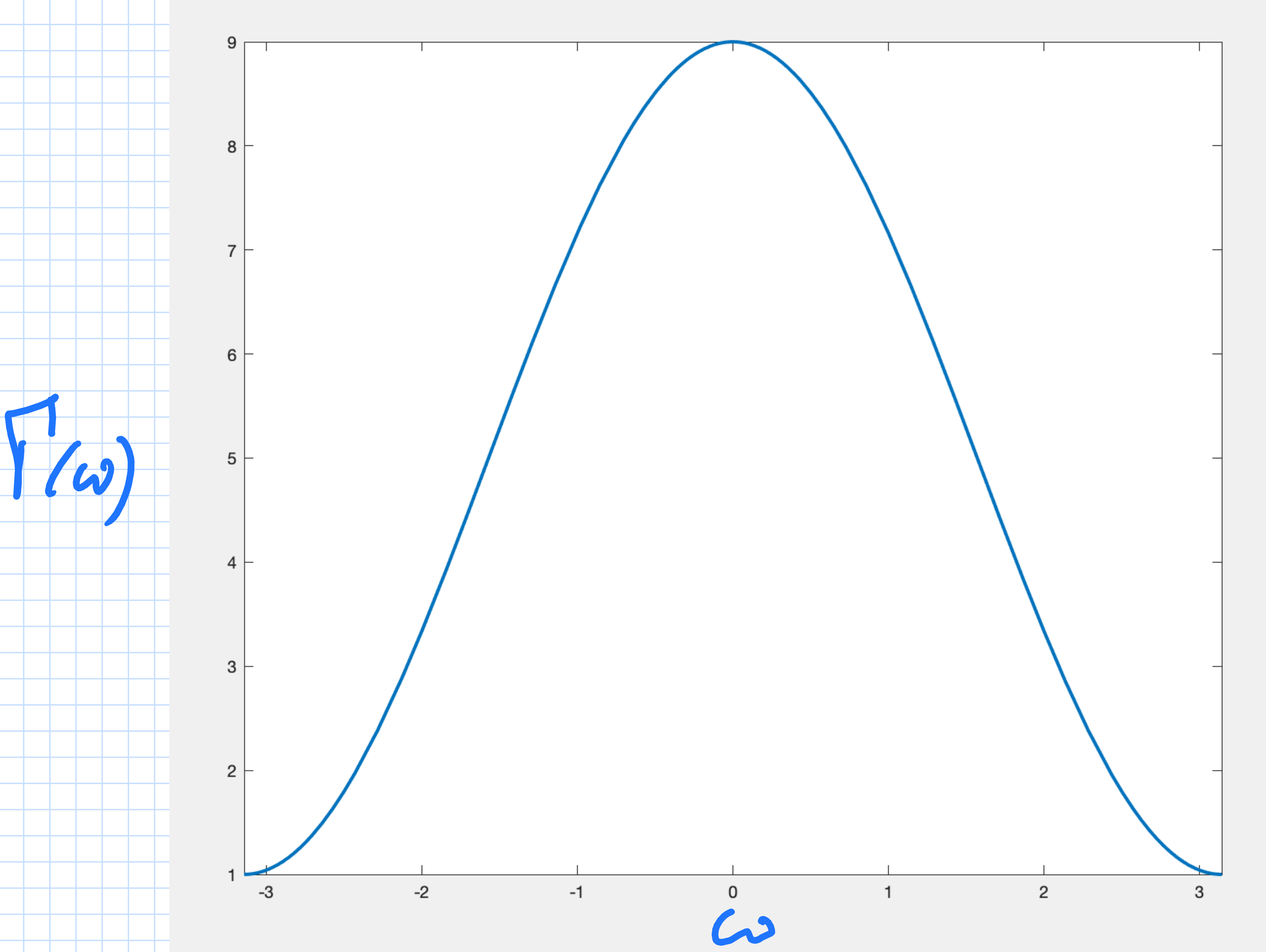
$$= 4 \cdot \left[\left(\cos\omega + \frac{1}{2} \right)^2 + \text{sen}^2\omega \right] = $$

$$\Gamma(\omega) = 4 \cdot \left[\cos^2\omega + \frac{1}{4} + 2 \cdot \frac{1}{2} \cos\omega + \operatorname{sen}^2\omega \right] =$$

$$1$$

$$= 4\left[1 + \frac{1}{4} + \cos\omega \right]$$

$$\Gamma(\omega) = 5 + 4\cos\omega$$



9
8
7
6
5
4
3
2
1
-3
-2
-1
0
1
2
3
$\Gamma(\omega)$
ω

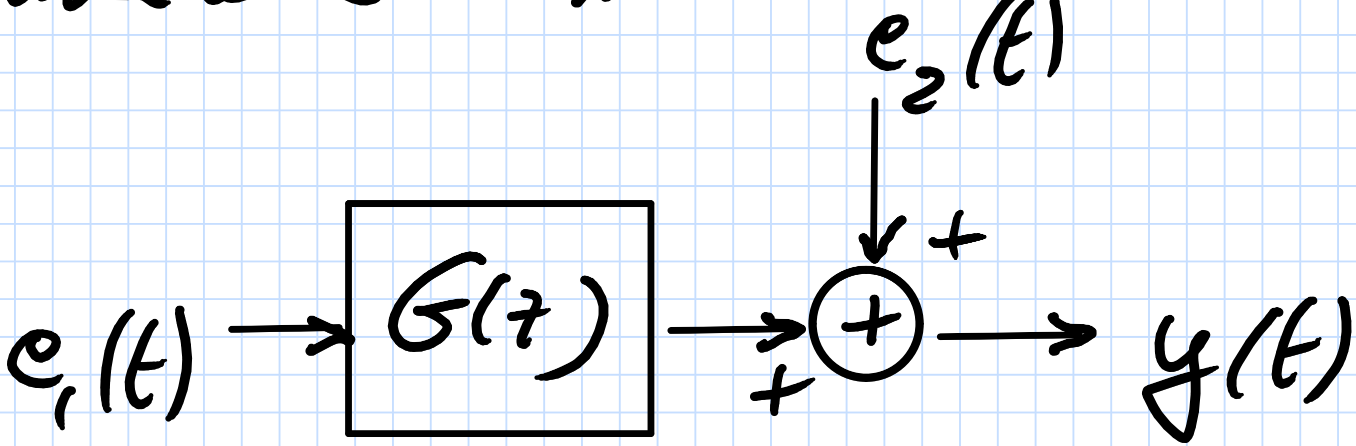
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[5+4\cos\omega\right]d\omega =$$

$$= \frac{1}{\cancel{2\pi}_{1}} \cdot 5 \cdot \cancel{2\pi}^{1} + \frac{4}{2\pi}\int_{-\pi}^{\pi}\cos\omega \, d\omega$$

$$= 5 + \frac{4}{2\pi}\,\operatorname{sen}\omega\Big|_{-\pi}^{\pi} = 5$$

Determinazione di modello, valore atteso e spettro su un processo stocastico stazionario

Dato il processo stocastico descritto dallo schema a blocchi



con

$$\begin{cases} G(z) = \dfrac{z}{z - \frac{1}{4}} \\ e_1(\cdot) \sim WN(0, 1) \end{cases}$$

Determinare valore atteso e spettro di $y(\cdot)$ nelle seguenti 3 condizioni

(a) $e_2(t) \equiv 0 \quad \forall t$

(b) $e_2(\cdot) \sim WN(0,1)$

indipendente da $e_1(\cdot)$

(c) $e_2(t) \equiv e_1(t) \quad \forall t$

Dallo schema a blocchi si ottiene

$$y(t) = \frac{1}{4}\, y(t-1) + e_1(t) + e_2(t)$$

caso a $\quad e_2 \equiv 0$

$\rightarrow$ l'equazione si riduce a

$$y_a(t) = \frac{1}{4}\, y_a(t-1) + e_1(t) \qquad \text{AR(1)}$$

In questo caso:

$$E\left[y_2(t)\right] = 0$$

$$\Gamma_2(\omega) = \frac{\left|e^{j\omega}\right|^2}{\left|e^{j\omega} - \frac{1}{4}\right|^2} \lambda^2 =$$

$$\Gamma_x(\omega) = \frac{1}{\left|\cos\omega - \frac{1}{4} + j\,\mathrm{sen}\,\omega\right|^2} =$$

$$= \frac{1}{\cos^2\omega + \frac{1}{16} - \frac{1}{2}\cos\omega + \mathrm{sen}^2\omega} =$$

$$= \frac{1}{\frac{17}{16} - \frac{1}{2}\cos\omega}$$

$$\Gamma_k(\omega) = \frac{16}{17 - 8\cos\omega}$$

$$\omega \in [-\pi; \pi]$$

Caso (I) È possibile scrivere:

$$y_b(t) = y_a(t) + e_2(t)$$

Ricordando che

$$\left.\begin{array}{l} E[y_a(t)] = 0 \\ E[e_2(t)] = 0 \end{array}\right\} \Rightarrow E[y_b(t)] = 0$$

$$\gamma_b(\pm\tau) = E\left[y_b(t)\cdot y_b(t\pm\tau)\right]$$

$$= E\left\{\left[y_a(t)+e_2(t)\right]\cdot\left[y_a(t\pm\tau)+e_2(t\pm\tau)\right]\right\}$$

$$\gamma_b(\pm\tau) = \underbrace{E\{y_a(t)\cdot y_a(t\pm\tau)\}}_{\gamma_a(\pm\tau)} + \underbrace{E\{e_2(t)\cdot e_2(t\pm\tau)\}}_{\gamma_{e_2}(\pm\tau)} + \cancel{E\{y_a(t)\cdot e_2(t\pm\tau)\}}^{0} + \cancel{E\{e_2(t)\cdot y_a(t\pm\tau)\}}^{0}$$

$$\gamma_{y_b}(t\tau) = \gamma_{y_a}(t\tau) + \gamma_{e_2}(t\tau)$$

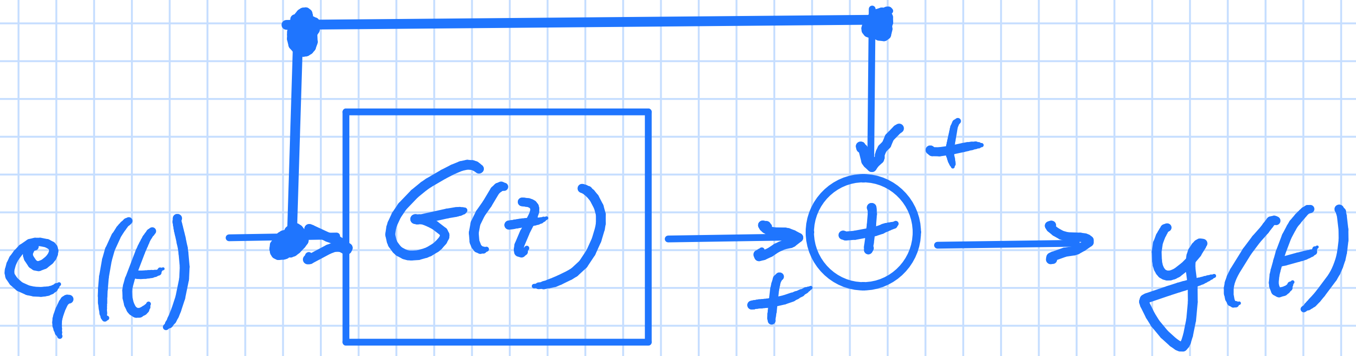
$$\Updownarrow$$

$$\Gamma_b(\omega) = \Gamma_a(\omega) + \Gamma_{e_2}(\omega)$$

$$\Gamma_b(\omega) = \frac{16}{17-8\cos\omega} + 1 = \frac{33-8\cos\omega}{17-8\cos\omega}$$

$$\mathrm{var}\left[y_b(t)\right] = ?$$

caso c) lo schema a blocchi diventa



$$W(z) = 1 + G(z) = 1 + \frac{z}{z - \frac{1}{4}}$$

$$= \frac{2 - \frac{1}{4} z^{-1}}{1 - \frac{1}{4} z^{-1}}$$

ARMA(1,1)