Svolgimento simulazione 10 gennaio 23

martedì 9 gennaio 2024 17:30

① $\begin{cases} x_1 + x_2 = a \\ 2x_1 + x_2 + x_3 = b \\ x_1 + x_3 = 1 \\ x_2 - x_3 = 2 \end{cases}$

Matrice completa $\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 2 & 1 & 1 & b \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$

$$\xrightarrow[\mathrm{III} - \mathrm{I}]{\mathrm{II} - 2\mathrm{I}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-2a \\ 0 & -1 & 1 & 1-a \\ 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow[\mathrm{IV} + \mathrm{II}]{\mathrm{III} - \mathrm{II}} \left(\begin{array}{ccc|cc} 1 & 1 & 0 & a & 3 \\ 0 & -1 & 1 & b-2a & -2 \\ 0 & 0 & 0 & 1+a-b = 0 & \\ 0 & 0 & 0 & 2+b-2a = 0 & \end{array}\right) \circledast$$

è compatibile $\Longleftrightarrow$ $\begin{cases} 1+a-b = 0 \\ 2+b-2a = 0 \end{cases}$

è un ulteriore sistema lineare nelle incognite a e b!

$\begin{cases} a - b = -1 \\ -2a + b = -2 \end{cases}$

Matrice completa $\underbrace{\left(\begin{array}{cc|c} 1 & -1 & -1 \\ -2 & 1 & -2 \end{array}\right)}$ $\xrightarrow[\mathrm{II} + 2\cdot\mathrm{I}]{} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & -1 & -4 \end{array}\right)$

è compatibile perchè $rg\, A = rg\,(A|b) = 2$, ed ha un'unica soluzione

$a = b - 1 = 3$

$\Uparrow$

$\underbrace{\begin{cases} a - b = -1 \\ -b = -4 \end{cases}}$ $\Rightarrow$ $b = 4$

Il SL è compatibile $\Longleftrightarrow$ $a = 3$ e $b = 4$

Generica soluzione in questo caso:

Sostituisco $a=3$ e $b=4$ alla matrice $\circledast$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + x_2 = 3 \\ -x_2 + x_3 = -2 \end{cases}$$

Pongo $x_3 = t$; $x_2 = x_3 + 2 = t+2$

$x_1 = 3 - x_2 = 3 - t - 2 = 1 - t$

La generica soluzione è $s = \begin{pmatrix} 1-t \\ 2+t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$= \tilde{s} + s_0$

② $A \in M_{m,n}(K)$, $S \in GL(m, K)$, $T \in GL(n, K)$

$$L_A : K^n \to K^m, \quad L_A\left(\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right) := \underbrace{\underset{m \times n}{A} \cdot \underset{n \times 1}{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}}_{m \times 1} \in K^m$$

$\mathrm{rg}\, A = \mathrm{rg}\, L_A = \dim \mathrm{Im}\, L_A$

$\mathrm{rg}\,(S \cdot A \cdot T) = \mathrm{rg}\,(L_S \circ L_A \circ L_T) = \dim \mathrm{Im}\,(L_S \circ L_A \circ L_T)$

$L_S : K^m \to K^m$, $L_T : K^n \to K^n$

$$\begin{tikzcd} K^n \arrow[r, "L_T"'] & K^n \arrow[r, "L_A"'] & K^m \arrow[r, "L_S"'] & K^m \end{tikzcd}$$

$L_T : K^n \to K^n$; T invertibile $\Rightarrow L_T$ è isomorfismo

$L_A : K^n \to K^m$

$L_S : K^m \to K^m$; S invertibile $\Rightarrow L_S$ è isomorfismo

$\dim \mathrm{Im}\,(L_S \circ L_A \circ L_T) = \dim L_S(L_A(L_T(K^n)))$

L_T isomorfismo $\Rightarrow L_T(K^n) = K^n$

$= \dim L_S(L_A(K^n))$

L_S isomorfismo $\Rightarrow$ manda SSV in SSV della stessa dimensione

Infatti:

$L_S : \mathbb{K}^m \to \mathbb{K}^m$ isomorfismo;

$\mathrm{Im}\, L_A \subseteq \mathbb{K}^m$; $\dim(L_S(\mathrm{Im}\, L_A)) = ?$

Cons. $L_S|_{\mathrm{Im}\, L_A} : \mathrm{Im}\, L_A \longrightarrow \mathbb{K}^m$

In generale: se $f : V \longrightarrow V'$ AL

$W \subseteq V$ SSV

posso cons. $f|_W : W \longrightarrow V'$

$\forall w \in W$

$f|_W(w) = f(w)$

L_S isomorfismo $\Rightarrow \ker L_S = \{0_{\mathbb{K}^m}\}$

$\Rightarrow \ker(L_S|_{\mathrm{Im}\, L_A}) = \{v \in \mathrm{Im}\, L_A \mid L_S(v) = 0_{\mathbb{K}^m}\}$

$= \ker(L_S) \cap \mathrm{Im}\, L_A$

$= \{0_{\mathbb{K}^m}\} \cap \mathrm{Im}\, L_A = \{0_{\mathbb{K}^m}\}$

$\to L_S|_{\mathrm{Im}\, L_A}$ è iniettiva

$\Rightarrow$ Teo Dimensione $\underbrace{\dim \ker(L_S|_{\mathrm{Im}\, L_A})}_{=0} + \underbrace{\dim L_S|_{\mathrm{Im}\, L_A}(\mathrm{Im}\, L_A)}_{= \dim L_S(\mathrm{Im}\, L_A)} = \dim \mathrm{Im}\, L_A$

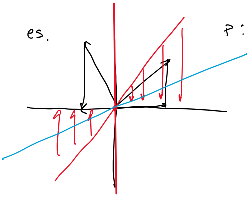
$= \dim L_A(\mathbb{K}^n) = \mathrm{rg}\, L_A = \mathrm{rg}\, A$

OSS: $f : V \longrightarrow V'$ AL

e supp. $\ker f \neq \{0_V\}$ ($\Rightarrow \dim V > \dim \mathrm{Im}\, f$)

$\dim \ker f \geq 1$

es.



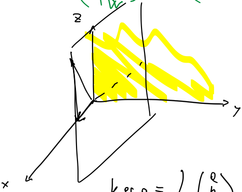
$p: \mathbb{R}^2 \to \mathbb{R}^2$

$p\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} a \\ 0 \end{pmatrix}$

$\ker p = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a = 0 \right\}$

$f|_W$ è iniettiva $\Longleftrightarrow W \cap \ker f = \{0_V\}$

$\ker(f|_W) = (\ker f) \cap W$



$p: \mathbb{R}^3 \to \mathbb{R}^3$

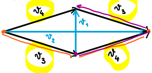
$p\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$

$\ker p = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a = 0 \right\}$

$= \left\{ \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \right\} = \left\{ b \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$= \text{Span}(e_2, e_3)$ è un piano vettoriale (piano yz)

③ ⓐ



$\|v_3\| = \|v_4\|$

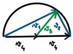
$v_1 = v_4 + (-v_3) = v_4 - v_3$

$v_2 = v_3 + v_4$

$\langle v_1, v_2 \rangle = \langle v_4 - v_3, v_3 + v_4 \rangle = -\langle v_3, v_3 \rangle + \langle v_3, v_4 \rangle - \underbrace{\langle v_4, v_3 \rangle}_{= \langle v_3, v_4 \rangle} + \langle v_4, v_4 \rangle$

$= -\|v_3\|^2 + \|v_4\|^2 = 0.$

ⓑ



$\|v_4\| = \|v_3\|$

$v_1 = v_4 + v_3$

$v_2 = -v_3 + v_4$

$\langle v_1, v_2 \rangle = \langle v_3 + v_4, -v_3 + v_4 \rangle$

$= -\langle v_3, v_3 \rangle - \langle v_4, v_3 \rangle + \langle v_3, v_4 \rangle + \langle v_4, v_4 \rangle$

$= -\|v_3\|^2 + \|v_4\|^2 = 0$

④ $f: V \to V, \quad g: V \to V$

$f \circ g = g \circ f; \quad g(\mathrm{Aut}_f(\lambda)) \subseteq \mathrm{Aut}_f(\lambda)$

$\mathrm{Aut}_f(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$

Sia $v \in \mathrm{Aut}_f(\lambda)$

devo dim $g(v) \in \mathrm{Aut}_f(\lambda)$

cioè $f(g(v)) = \lambda \cdot g(v)$

$= g(f(v)) \quad : f \circ g = g \circ f$

$$\boxed{\lambda g(v)} = g(\lambda v) = g(f(v)) = \boxed{f(g(v))}$$

$\Rightarrow g(v) \in \mathrm{Aut}_f(\lambda).$

5. Visto a lezione.

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} P_A(x) &= \det(A - x \cdot \mathbb{I}_3) \\ &= \det \begin{pmatrix} -x & 0 & 2 \\ 0 & 2-x & 0 \\ 2 & 0 & -x \end{pmatrix} \\ &= (-x) \cdot \big((2-x)(-x)\big) + 2 \cdot (0 - 2 \cdot (2-x)) \\ &= x^2(2-x) - 4(2-x) \\ &\overset{!}{=} (2-x)(x^2-4) = \underbrace{(2-x)}\,\underbrace{(x-2)}\,\underbrace{(x+2)} \end{aligned}$$

$\Rightarrow S_p(A) = \{2, -2\}$

$m_a(2) = 2, \quad m_a(-2) = 1$

Per Teo Spett. so che $m_g(2) = 2$

$m_g(-2) = 1$

Eq. per $\mathrm{Aut}(-2) = \ker(A - (-2)\mathbb{I}_3)$

$= \ker(A + 2\mathbb{I}_3)$

$= \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$

eq. cartesiane $\begin{cases} 2x_1 + 2x_3 = 0 \\ 4x_2 = 0 \end{cases}$ $\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases}$, $x_3 = t$, $x_1 = -t$

$\begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

base di $\mathrm{Aut}(-2)$ è $\left\{ \overset{=v_1}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \right\}$, $\|v_1\| = \sqrt{2}$

base ortonormale è $\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}$

$$\downarrow$$
$$\frac{v_1}{\|v_1\|}$$

$$\mathrm{Aut}(2) = \ker(A - 2I_3) = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

eq. cartesiane $-2x_1 + 2x_3 = 0$

ovvero $\boxed{-x_1 + x_3 = 0}$, $x_1 = x_3$

2 incognite libere $x_3 = t$, $x_2 = s$

$\Rightarrow x_1 = t$

$\rightarrow$ gen. solz. $\begin{pmatrix} t \\ s \\ t \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

base di $\mathrm{Aut}(2)$: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

oss. che $\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0$

$\Rightarrow$ sono ortogonali

basta normalizzare $\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{2}$

$\left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\| = 1$

$\Rightarrow$ base ortonormale di $\mathrm{Aut}(2)$ è $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow$ base ortonorm. di $\mathbb{R}^3$ formata da autovettori: faccio l'unione delle due basi

$$\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \mathcal{B}$$

In questa base $M_{\mathcal{B}}^{\mathcal{B}}(L_A) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$

$$D = \boxed{M_{\mathcal{B}}^{\mathcal{E}}(Id_{\mathbb{R}^3})} A \cdot \underline{M_{\mathcal{E}}^{\mathcal{B}}(Id_{\mathbb{R}^3})}$$

$$A = M_{\mathcal{E}}^{\mathcal{E}}(L_A) \qquad = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$M_{\mathcal{B}}^{\mathcal{E}}(Id_{\mathbb{R}^3}) = \left(\underline{M_{\mathcal{E}}^{\mathcal{B}}(Id_{\mathbb{R}^3})}\right)^{-1} = {}^t\!\left(M_{\mathcal{E}}^{\mathcal{B}}(Id_{\mathbb{R}^3})\right) = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

è una matrice ortogonale perché di passaggio da base ortonormale $\mathcal{B}$ a base ortonormale $\mathcal{E}$