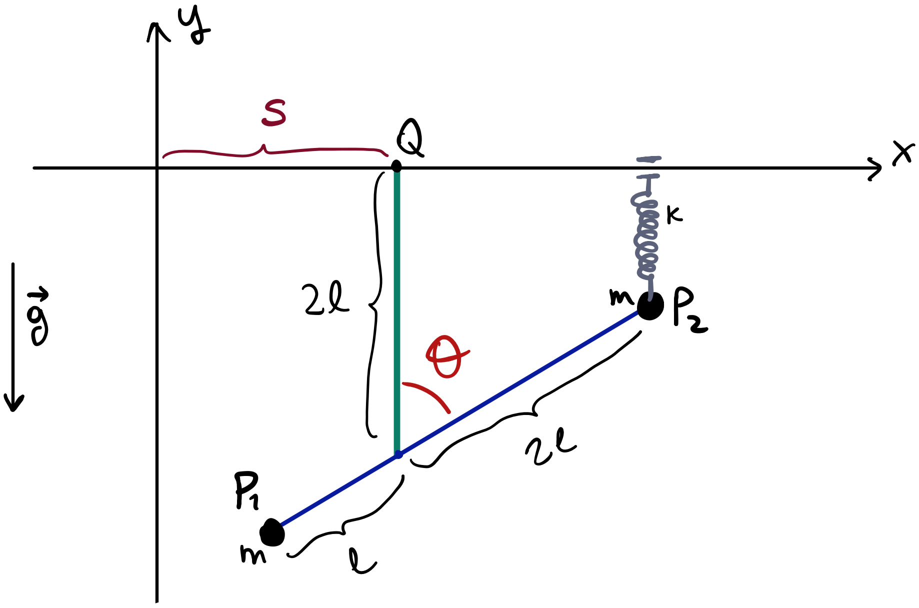
ES 2)



1,2) $x_1 = s - \ell \operatorname{sen}\theta \qquad \dot{x}_1 = \dot{s} - \ell\dot{\theta}\cos\theta$

$y_1 = -2\ell - \ell\cos\theta \qquad \dot{y}_1 = \ell\dot{\theta}\operatorname{sen}\theta$

$x_2 = s + 2\ell\operatorname{sen}\theta \qquad \dot{x}_2 = \dot{s} + 2\ell\dot{\theta}\cos\theta$

$y_2 = -2\ell + 2\ell\cos\theta \qquad \dot{y}_2 = -2\ell\dot{\theta}\operatorname{sen}\theta$

$$T = \frac{m}{2}\left(2\dot{s}^2 + 5\ell^2\dot{\theta}^2 + 2\ell\dot{s}\dot{\theta}\cos\theta\right) \rightarrow \mathcal{Q} = \begin{pmatrix} 2m & m\ell\cos\theta \\ m\ell\cos\theta & 5m\ell^2 \end{pmatrix}$$

$\operatorname{Tr}\mathcal{Q} > 0, \quad \det\mathcal{Q} = m^2\ell^2(10 - \cos\theta) > 0.$

$$V = mg(-4\ell + \ell\cos\theta) + \frac{K\,2\!\!/\,4\ell^2(1-\cos\theta)^2}{\cancel{2}}$$

$$= \text{cost.} + (mg\ell - 4K\ell^2)\cos\theta + 2K\ell^2\cos^2\theta$$

$$L = \frac{m}{2}\left(2\dot{s}^2 + 5\ell^2\dot{\theta}^2 + 2\ell\dot{s}\dot{\theta}\cos\theta\right) - (mg\ell - 4K\ell^2)\cos\theta - 2K\ell^2\cos^2\theta$$

3) $$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{d}{dt}\left(5m\ell^2\dot{\theta} + m\ell\dot{s}\cos\theta\right) = 5m\ell^2\ddot{\theta} + m\ell\ddot{s}\cos\theta - m\ell\dot{s}\dot{\theta}\operatorname{sen}\theta$$

$$\frac{\partial L}{\partial\theta} = -m\ell\dot{s}\dot{\theta}\operatorname{sen}\theta + (mg\ell - 4K\ell^2)\operatorname{sen}\theta + 4K\ell^2\cos\theta\operatorname{sen}\theta$$

$$5\ddot{\theta} + \frac{\ddot{s}}{\ell}\cos\theta = \left(\frac{g}{k} - 4\frac{k}{m}\right)\operatorname{sen}\theta + 4\frac{k}{m}\cos\theta\,\operatorname{sen}\theta$$

4) • Simm. di traslazione lungo l'asse x.
• Cost. del moto: componente lungo x della quantità di moto.

5) Coord. ciclica: s

$$L = \frac{m}{2}\left(2\dot{s}^2 + 5\ell^2\dot{\theta}^2 + 2\ell\dot{s}\dot{\theta}\cos\theta\right) - V(\theta)$$

$$\frac{\partial L}{\partial \dot{s}} = 2m\dot{s} + m\ell\dot{\theta}\cos\theta = p_s \to \dot{s} = \frac{p_s - m\ell\dot{\theta}\cos\theta}{2m}$$

$$L^{*} = L - p_s\dot{s}\Big|_{\dot{s}=\dots} =$$

$$= \frac{1}{2}\Big(2\dot{s}\big(-m\dot{s} + \underbrace{(2m\dot{s} + m\ell\dot{\theta}\cos\theta)}_{p_s}\big)\Big) - p_s\dot{s} + \frac{5}{2}m\ell^2\dot{\theta}^2 - V(\theta)$$

$$= -m\dot{s}^2 + \cancel{p_s\dot{s}} - \cancel{p_s\dot{s}} + \frac{5}{2}m\ell^2\dot{\theta}^2 - V(\theta)\Big|_{\dot{s}=\dots}$$

$$= \frac{5}{2}m\ell^2\dot{\theta}^2 - \frac{1}{4m}\left(p_s - m\ell\dot{\theta}\cos\theta\right)^2 - V(\theta)$$

Questo termine è cost. o proporzionale a $\dot{\theta}$

$$= \frac{1}{2}m\ell^2\dot{\theta}^2\left(5 - \frac{1}{2}\cos^2\theta\right) - V(\theta) + \text{derivata totale}$$

6) Config. equil. $\dot{\theta}=0$ e θ t.c. $V'(\theta)=0$

$$V(\theta) = (mgl - 4kl^2)\cos\theta + 2kl^2\cos^2\theta$$

$$V'(\theta) = -(mgl - 4kl^2)\,\mathrm{sen}\,\theta - 4kl^2\cos\theta\,\mathrm{sen}\,\theta$$

$$= -4kl^2\,\mathrm{sen}\,\theta\left(\cos\theta - \left(1 - \frac{mg}{4kl}\right)\right)$$

$\rightarrow$ $\theta_1 = 0$, $\theta_2 = \pi$

$\theta_{3,4} = \pm\theta^*$ con $\cos\theta^* = 1 - \dfrac{mg}{4kl}$

$\xi \equiv \dfrac{mg}{4kl}$

$\theta_{3,4}\ \exists$ se $-1 \leq 1 - \dfrac{mg}{4kl} \leq 1$ ok!

$\dfrac{mg}{4kl} \leq 2 \iff mg \leq 8kl$

$(\exists\ \xi \leq 2)$

$\cos\theta^* = 1-\xi$

$$V''(\theta) = (4kl^2 - mgl)\cos\theta + 4kl^2(1 - 2\cos^2\theta)$$
$$= 4kl^2\left[(1-\xi)\cos\theta + 1 - 2\cos^2\theta\right]$$

$$\frac{V''(\theta_1=0)}{4kl^2} = 1-\xi+1-2 = -\xi < 0 \qquad \text{INSTAB}$$

$$\frac{V''(\theta_2=\pi)}{4kl^2} = \xi - 1 + 1 - 2 = \xi - 2$$

$\xi > 2$ STAB.

$\xi < 2$ INSTAB

$$\frac{V''(\theta_{3,4}=\pm\theta^*)}{4kl^2} = (1-\xi)^2 + 1 - 2(1-\xi)^2 = 1-(1-\xi)^2 = 2\xi - \xi^2 = \xi(2-\xi)$$

STAB. quando esiste. $(\xi < 2)$

7) $K = \frac{mg}{9\ell} \Rightarrow \xi = \frac{mg}{4k\ell} = \frac{mg}{4\ell} \cdot \frac{9\ell}{mg} = \frac{9}{4} > 2$

STAB. è $\theta = \pi$

$T_{eff} = \frac{1}{4} m\ell^2 \dot{\theta}^2 (10 - \cos^2\theta) \leftarrow a_{eff} = \frac{m\ell^2}{2}(10 - \cos^2\theta)$

$\det(B - \omega^2 A) = 0 \rightsquigarrow \omega^2 = \frac{B}{A}$

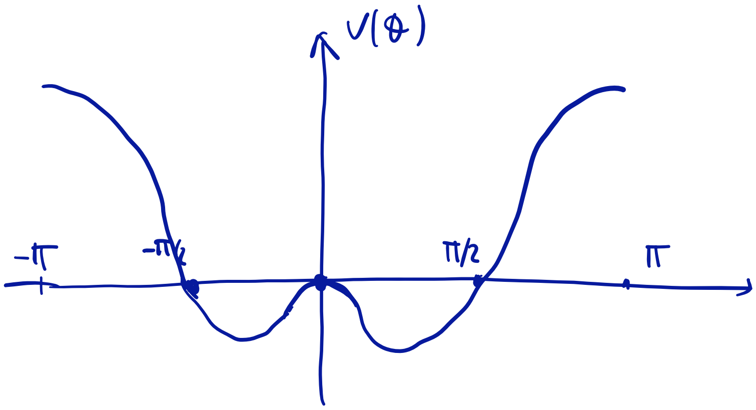
$B = V''_{eff}(\pi) = \left(\frac{9}{4} - 2\right) 4k\ell^2 = k\ell^2$

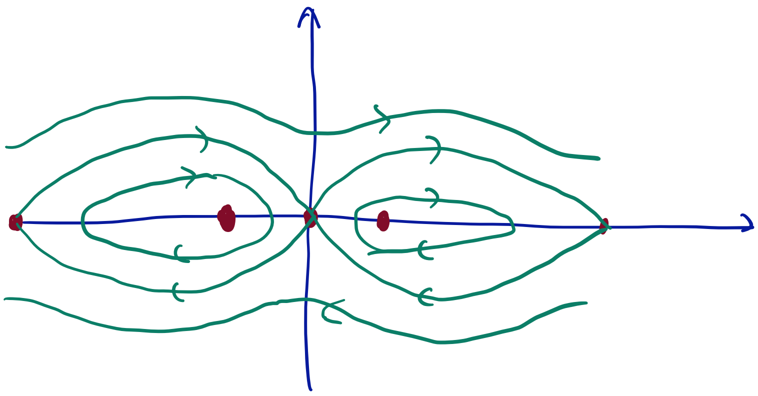
$A = a_{eff}(\pi) = \frac{9m\ell^2}{2}$

$\Rightarrow \omega^2 = \frac{2k}{9m} = \frac{2}{81} \frac{g}{\ell}$

8) $k = \frac{mg}{2l} \Rightarrow \xi = \frac{mg}{4kl} = \frac{mg}{4l} \cdot \frac{2l}{mg} = \frac{1}{2} < 2$

$$V(\theta) = (mgl - 4kl^2)\cos\theta + 2kl^2\cos^2\theta = mgl \cos\theta \, (\cos\theta - 1)$$





ES. 1

3) • Inv. per rotazioni attorno asse z:

$$\varphi_x(\alpha) = \cos\alpha\, x + \operatorname{sen}\alpha\, y$$
$$\varphi_y(\alpha) = -\operatorname{sen}\alpha\, x + \cos\alpha\, y$$
$$\varphi_z(\alpha) = z$$

$$\frac{\partial \bar{\varphi}}{\partial \alpha} = \begin{pmatrix} -\operatorname{sen}\alpha\, x + \cos\alpha\, y \\ -\cos\alpha\, x - \operatorname{sen}\alpha\, y \\ 0 \end{pmatrix}$$

$$P = \sum_h p_h \left.\frac{\partial \varphi_h}{\partial \alpha}\right|_{\alpha=0} =$$

$$= p_x y - p_y x \qquad p_x = m\dot{x} \quad p_y = m\dot{y}$$

$$= m(\dot{x}y - \dot{y}x)$$

• Invarianza per traslaz. lungo z.

$$\varphi_x(\alpha) = x$$
$$\varphi_y(\alpha) = y$$
$$\varphi_z(\alpha) = z + \alpha$$

$$\frac{\partial \bar{\varphi}}{\partial \alpha} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \sum_h p_h \frac{\partial \varphi_h(0)}{\partial \alpha} = p_z = m\dot{z}$$

5) Simm. per traslazioni

P_x genera trasl. lungo asse x: $x \mapsto x+\epsilon$, $y \mapsto y$, $z \mapsto z$

Infatti sotto P_x:

$$\delta x = \epsilon \{x, P_x\} = \epsilon$$

$$\delta y = \epsilon \{y, P_x\} = 0$$

$$\delta z = \epsilon \{z, P_x\} = 0$$

Analogam. per P_y e P_z.

6) Ci sono tre cost. del moto in involuzione:

H, M_z, $\vec{M}^2$.

ES. 3) $E_n = \dfrac{n^2\pi^2\hbar^2}{2m\ell^2}$

3) stato fondam. è autof. di E_n con autoval.

$$E_1 = \frac{\pi^2\hbar^2}{2m\ell^2}$$

$$\langle H \rangle_{st.f.} = (\psi_1, \underbrace{\hat{H}\psi_1}_{=E_1\psi_1}) = (\psi_1, E_1\psi_1) = E_1 \|\psi_1\|^2 = E_1$$

usiamo funz. normalizzate

4) Secondo livello energetico : $\psi_2 = \left(\frac{2}{\ell}\right)^{1/2} \operatorname{sen}\left(\frac{2\pi x}{\ell}\right)$

ψ_2 funz. pari nel dominio $x \in [-\frac{\ell}{2}, \frac{\ell}{2}]$

$\hat{P}\psi_2 = -i\hbar\,\psi_2'$ funz. dispari nel dominio

$$\langle P \rangle_{\psi_2} = \int_{-\ell/2}^{\ell/2} \underbrace{\psi_2^* \, \hat{P}\psi_2}_{\text{dispari}} \, dx = 0$$

pari dispari

5) $\text{Prob}\left(x \in [0, \frac{\ell}{2}]\right) = 1/2$

$$\int_0^{\ell/2} \underbrace{|\psi_n(x)|^2}_{\text{pari}} dx = \frac{1}{2} \underbrace{\int_{-\ell/2}^{\ell/2} |\psi_n(x)|^2 dx}_{= \|\psi_n\|^2} = \frac{1}{2}$$

(ψ_n ha parità definita)