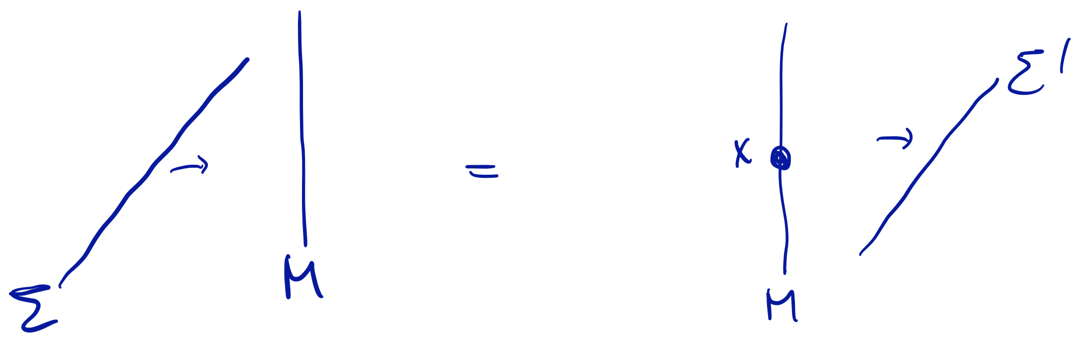
Action on extended operators

Consider a $\mathcal{O}(M_p)$ that is "irreducible" i.e. equivalently

- it cannot be expressed as a sum of two other p-operators;
- there are no topol. local op. that can be inserted at a pt. $x \in \Sigma_p$ except multiples of id.

Now, deforming $U_g(\Sigma_{d-p-1})$ across $\mathcal{O}(M_p)$ leaves behind a top. local op. $\mathcal{O}(x)$ at the inters. pt. $\Sigma \cap M$

$$U(\Sigma_{d-p-1})\,\mathcal{O}(M_p) = \mathcal{O}(x)\,\mathcal{O}(M_p)\,U(\Sigma'_{d-p-1})$$



Since $\mathcal{O}(x) = \phi(g)\,\mathbb{1}$, then

$$U_g(\Sigma_{d-p-1})\,\mathcal{O}(M_p) = \phi(g)\;\mathcal{O}(M_p)\,U_g(\Sigma'_{d-p-1})$$

Because of fusion rules

$$U_g(\Sigma)\,U_{g'}(\Sigma) = U_{gg'}(\Sigma)$$

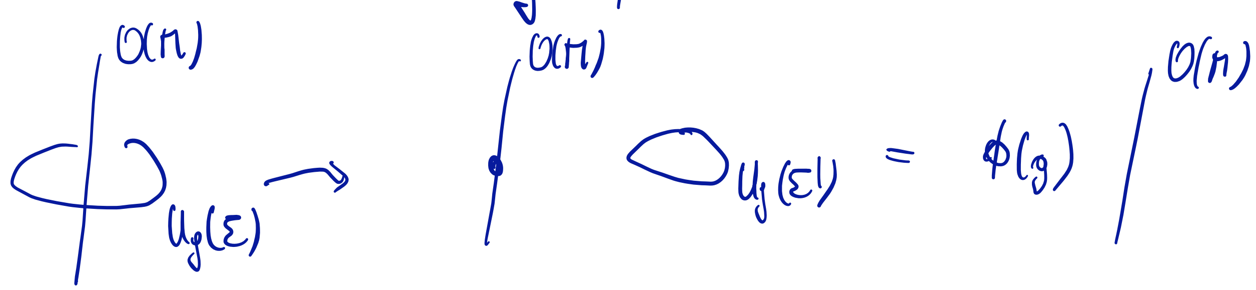
$$\longrightarrow \quad \phi(g)\,\phi(g') = \phi(gg')$$

I.e. ϕ furnishes a one-dim. rep. of the p-form symmetry group $G^{(p)}$.

$$\phi : G^{(p)} \longrightarrow \underset{U(1)}{\cancel{\mathbb{C}^*}} \qquad \text{“CHARACTER”}$$

if we require UNITARY REP'S

If we act via linking, the Σ' is contractible



$$\boxed{U_g(\Sigma)\, \mathcal{O}(M) = \phi(g)\, \mathcal{O}(M)} \; *$$

The characters of an abelian group form themselves an abelian group, with product being

$$\phi\phi'(g) = \phi(g)\,\phi'(g) \qquad \forall g \in G^{(p)}$$

$$\hat{G}^{(p)} = \{ \text{characters of } G^{(p)} \} \leftarrow \text{“PONTRYAGIN DUAL GROUP”}$$

* → The CHARGE carried by a (irreducible) p-dim. op. under $G^{(p)}$ is an element of $\hat{G}^{(p)}$.

(I.e. given $g \in G^{(p)}$, all possible phases $\phi(g)$ are given by $\hat{G}^{(p)}$.)

ES. $G^{(p)} = U(1) \longrightarrow \hat{G}^{(p)} = \mathbb{Z}$

$\quad\ \ \ni$

$g = e^{i\alpha} \qquad \alpha \in [0, 2\pi[$

Maxwell theory and 1-form sym.

$$\phi_n(g) = g^n = e^{in\alpha}$$

Dim. $\phi(e^{i\alpha}) = e^{i\theta(\alpha)}$ with $\theta(\alpha + 2\pi) = \theta(\alpha) + 2k\pi$

$b_1, b_2 \in \mathbb{Z}$

$$\phi\left((e^{i\alpha_1})^{b_1} (e^{i\alpha_2})^{b_2} \right) = \phi\left(e^{i(b_1\alpha_1 + b_2\alpha_2)} \right) =$$

$$= e^{i\theta(b_1\alpha_1 + b_2\alpha_2)}$$

$$= e^{ib_1\theta(\alpha_1)} e^{ib_2\theta(\alpha_2)}$$

$\Rightarrow \theta(b_1\alpha_1 + b_2\alpha_2) = b_1\theta(\alpha_1) + b_2\theta(\alpha_2) \mod 2\pi$

$\Rightarrow \theta(\alpha)$ is a linear function $\Rightarrow \theta(\alpha) = n\alpha \mod 2\pi$

$\rightarrow \phi_n(e^{i\alpha}) = e^{in\alpha}$

$\underline{n \in \mathbb{Z}}$ ← necessary for ϕ_n to be well defined. //

ES. $G^{(p)} = \mathbb{Z} \longrightarrow \hat{G}^{(p)} = U(1)$

$\ \ \ni$

$\ \ n$

$$\phi_\alpha(n) = e^{i\alpha n} \qquad \alpha \in [0, 2\pi[$$

Dim $\phi(n) = e^{i\theta(n)}$

$\phi(n_1 + n_2) = e^{i\theta(n_1 + n_2)}$

$\quad\ \ \|$

$e^{i\theta(n_1)} e^{i\theta(n_2)}$ $\Rightarrow \theta$ lin. $\Rightarrow \phi_\alpha(n) = e^{i\alpha n}$

$\alpha \in [0, 2\pi]$ for ϕ_α to be well defined

//

ES. $G^{(p)} = \mathbb{Z}_N \longrightarrow \hat{G}^{(p)} = \mathbb{Z}_N$

$\cup$

$g = e^{2\pi i \alpha / N}$ $\qquad \phi_\beta(g_\alpha) = e^{2\pi i \alpha \cdot \beta / N}$

$\alpha = 0, 1, \dots, N-1$ $\qquad \beta = 0, 1, \dots, N-1$

Dim. $\phi(e^{2\pi i \alpha / N}) = e^{i\theta(\alpha)}$

$\rightarrow \theta(\alpha)$ lin $\rightarrow$ $\phi_\beta(e^{2\pi i \alpha / N}) = e^{i \frac{2\pi i}{N} \beta \cdot \alpha}$

$\beta = 0, 1, \dots, N-1$ for ϕ_β to be well def.

//

ES. $G^{(p)}$ FINITE ABELIAN $\Rightarrow$ $\hat{G}^{(p)} = G^{(p)}$

These are products of $\mathbb{Z}_N$'s

One has : $\hat{\hat{G}}^{(p)} = G^{(p)}$

In fact $g \in G^{(p)}$ defines $h_g : \hat{G}^{(p)} \longrightarrow U(1)$

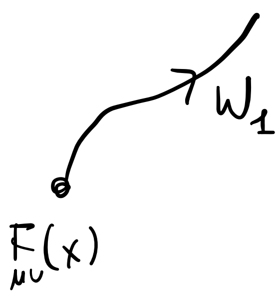
$\phi \mapsto \phi(g)$.

PURE SU(2) YM

- Wilson line operators parametrized by IRREP of SU(2)

$$W_j = \mathrm{Tr}_{R_j} P e^{i\int A} \qquad j \in \mathbb{Z}/2 \quad \text{is the "spin".}$$

We have charged operators on which these lines can end,



but this can happen only for integer spin!

$\Rightarrow$ The "unscreened" Wilson lines are

- the trivial one
- the one in fundam. rep.

$\Big\}$ they generate a $\mathbb{Z}_2$ group $= \hat{G}^{(1)}$

$\Rightarrow$ The one-form sym. of SU(2) is

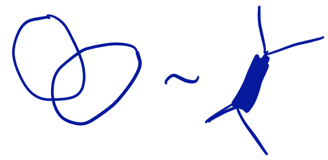
$$G_e^{(1)} = \hat{G}^{(1)} = \mathbb{Z}_2^{(1)} \quad \text{electric 1-form symmetry.}$$

- 't Hooft operators. For pure SU(2) there seem to be solitonic field configurations that have charges to screen all 't Hooft operators.

$$\Rightarrow \quad G_m^{(1)} = \{\mathbb{1}\}$$

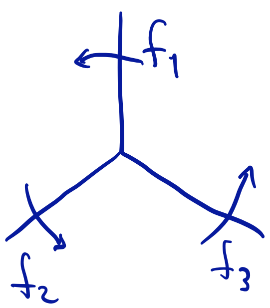
Aside: OBSTRUCTION CLASSES

Consider a principal bndl for $G = \tilde{G}/\Lambda$ $\quad \Lambda \subset Z(\tilde{G})$

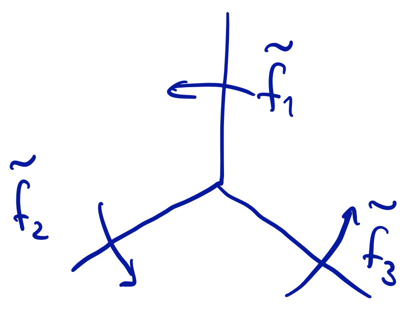
→ It can be described in terms of transition functions valued in G on codim 1 loci 

These codim 1 loci come together and form codim 2 junctions.

Consider the lift of these G-valued transition functions to $\tilde{G}$ valued transition functions. Before the lift, the product of transition function around a cod2 junction is 1



$$f_1 f_2 f_3 = 1 \in G$$



$$\tilde{f}_1 \tilde{f}_2 \tilde{f}_3 \in \Lambda \subset G$$

After the lift, the product will be the lift of 1, that is any element of Λ.

By Poincaré duality, this defines a Λ-valued 2-cochain w_2. The cohomology class of w_2 is knows as the obstruction class for lifting G-bndle to $\tilde{G}$-bndl.

It is indep. of the choice of lift.

For $G = SO(3)$ $\quad \tilde{G} = SU(2)$ $\quad \Lambda = \mathbb{Z}_2$

the class w_2 is known as the 2nd Stiefel-Whitney class.

PURE $SO(3)$ YM

Gauge group $G = SO(3) = SU(2)/\mathbb{Z}_2$ & no matter fields

- Wilson lines parametrized by irrep of $SO(3)$, that are fewer than in $SU(2)$: $j \in \mathbb{Z}$ (j is the spin of rep.)

Since all these WL can be screened $\Rightarrow G_e^{(1)} = \mathbb{1}$.

- 't Hoof lines : now not all TL can be screened, we are left with $\hat{G}_m^{(1)} = \mathbb{Z}_2 \Rightarrow G_m^{(1)} = \mathbb{Z}_2$.

The topological surface operator generating the magnetic 1-form symmetry can be expressed as

$$U(\Sigma_2) = e^{i\pi \int_{\Sigma_2} w_2}$$

Spontaneous breaking of high-form symmetries.

Let us consider gauge theories with Wilson Lines op.
These are charged (extended) op. under 1-form sym.
If their VEV is $\neq 0$ then the 1-form sym is SB.

$\langle W[C] \rangle$ typically depends on geometric properties of C:

$$\underset{(a)}{\langle W[C] \rangle \sim e^{-\text{Area}[C]}} \quad \text{or} \quad \underset{(b)}{\langle W[C] \rangle \sim e^{-\text{Perimeter}[C]}}$$

different phases

As we have seen, for large C

$$(a) \leftrightarrow \langle W[C] \rangle = 0$$

$$(b) \leftrightarrow \langle W[C] \rangle \neq 0$$

(this happens also for $V(r) \sim \frac{1}{r}$)

→ Interpret the problem of CONFINEMENT in terms of SPONTANEOUS SYMMETRY BREAKING of a 1-form sym.

What is the associated GOLDSTONE BOSON ?

For ordinary 0-form sym one starts from the WI

$$\partial_\mu \langle J^\mu(x)\, \phi(y) \rangle = -i\,\delta(x-y) \langle \delta\phi(y) \rangle$$

and do Fourier transform:

$$\int d^dx\, e^{ipx}\, \partial_\mu \langle J^\mu(x)\, \phi(y)\rangle = -i \int d^dx\, e^{ipx}\, \delta(x-y) \langle \delta\phi(y)\rangle$$

$$-i \int d^dx\, p_\mu\, e^{ipx} \langle J^\mu(x)\, \phi(y)\rangle = -i\, e^{ipy} \langle \delta\phi(y)\rangle$$

$$p_\mu \langle \tilde{J}^\mu(p)\, \phi(y)\rangle = e^{ipy} \langle \delta\phi(y)\rangle$$

F.T. in y

$$p_\mu \langle \tilde{J}^\mu(p)\, \tilde{\phi}(q)\rangle = \langle \delta\tilde{\phi}(p+q)\rangle$$

Set $q=-p$

Order param. for SSB.

$$p_\mu \langle \tilde{J}^\mu(p)\, \tilde{\phi}(-p)\rangle = \langle \delta\tilde{\phi}(0)\rangle \leftarrow \text{cost. in } p.$$

$\Rightarrow \quad \langle \tilde{J}^\mu(p)\, \tilde{\phi}(-p)\rangle$ MUST HAVE A POLE in $p=0$ if $\langle \delta\tilde{\phi}(0)\rangle \neq 0$ (i.e. in broken phase)

$$\Rightarrow \quad \langle \tilde{J}^\mu(p)\, \tilde{\phi}(-p)\rangle \sim \frac{p^\mu}{p^2} \leftarrow$$

This signals the presence of MASSLESS PHYSICA EXCITATIONS in the spectrum (see QTF II).

Let's repeat it for 1-form sym. We start from WI [Gomes]

$$\langle \partial_\mu J^{\mu\nu}(x)\, W[C]\rangle = -q_e \int_C dy^\nu\, \delta^{(d)}(x-y) \langle W[C]\rangle$$

$$(\delta W = -i q_e W)$$

Taking Fourier transform:

$$i p_\mu \langle \tilde{J}^{\mu\nu}(p)\, W[C] \rangle = q_e \underbrace{f^\nu(p;C)}_{= \int_C dy^\nu e^{ipy}} \langle W[C] \rangle$$

•) $f^\nu(0;C) \neq 0$

•) $p_\nu f^\nu(p;C) = \int_C dy^\nu p_\nu e^{ipy} =$

$= -i \int_C dy^\nu \partial_\nu e^{ipy} = 0$

C is closed

Take limit $p_\mu \to 0$:

$$\lim_{p \to 0} i p_\mu \langle \tilde{J}^{\mu\nu}(p)\, W[C] \rangle = q_e f^\nu(0;C) \langle W[C] \rangle$$

order param. for SSB

$\Rightarrow$ In broken phase $(\langle W[C] \rangle \neq 0)$ there is a pole

$$\langle \tilde{J}^{\mu\nu}(p)\, W[C] \rangle \sim \frac{p^\mu f^\nu(p;C) - p^\nu f^\mu(p;C)}{p^2}$$

$\Rightarrow$ There are MASSLESS EXCITATIONS and one can check that they have spin 1.

$\rightsquigarrow$ For continuous 1-form sym (like in Maxwell) with $\langle W[C] \rangle \sim e^{-\text{perim.}(C)} \neq 0$ (like in Maxwell) we expect spin 1 massless fields. This actually happens in Maxwell theory $\rightsquigarrow$ PHOTON is the GOLDSTONE BOSON of SSB of 1-form sym.

One can actually check that the photons ARE the Goldstone excitations.

- conserved current $J^{\mu\nu}$ creates Goldstone excitations from the vacuum in the broken phase

$$|\mathrm{Gold}\rangle \sim J^{\mu\nu}|0\rangle \qquad \text{(Like in QFT II.)}$$

- recall that $J^{\mu\nu} = F^{\mu\nu}$.
- using canonical quantiz. one can show that this actually creates one photon.