Ordinary (0-form) symmetries

Symmetry transf. in QFT

$$\langle U_g(\Sigma)\, \Phi^i(y)\rangle = R(g)^i{}_j \langle \Phi^j(y)\rangle$$

Since the sym. generators are CONSERVED / COMMUTE WITH HAMILTONIAN, $U_g(\Sigma)$ is "topological" (as we will see)

In Field Theory, if S is invariant under sym group G, then there exists a CONSERVED CURRENT $\partial_\mu j^\mu = 0$

j s.t. if we take local transf

$$S[\Phi^i + \underbrace{\epsilon(x) M^i{}_j}_{\text{generator}} \Phi^j] - S[\Phi^i] = -\int \epsilon(x)\, \partial_\mu j^\mu(x) \qquad (*)$$

there is a current associated with any gen.

In QFT $\rightsquigarrow$ WI

$$i\langle \partial_\mu j^\mu(x)\, \Phi^i(y)\rangle = \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y)\rangle \qquad (\circ)$$

Dim. $\langle \partial_\mu j^\mu(x)\, \Phi^i(y)\rangle = N\int \mathcal{D}\Phi\, \partial_\mu j^\mu(x)\, \Phi^i(y)\, e^{iS[\Phi]} =$

$$\overset{(*)}{=} -N\int \mathcal{D}\Phi\, \frac{\delta}{\delta\epsilon(x)} S[\Phi^k + \epsilon(x) M^k{}_j \Phi^j]\Big|_{\epsilon=0} \Phi^i(y)\, e^{iS[\Phi]} =$$

$$= -\frac{1}{i}\frac{\delta}{\delta\epsilon(x)}\left(N\int \mathcal{D}\Phi\, \Phi^i(y)\, e^{iS[\underbrace{\Phi^k + \epsilon M^k_j \Phi^j}_{=\Phi'^k \to\ \Phi^k = \Phi'^k - \epsilon M^k{}_j \Phi'^j}]}\right)\Big|_{\epsilon=0}$$

$$= i\,\frac{\delta}{\delta\epsilon(x)} N\int \mathcal{D}\Phi' \left(\Phi'^i(y) - \epsilon(y) M^i{}_j \Phi'^j(y)\right) e^{iS[\Phi']}\Big|_{\epsilon=0}$$

$$= -i\, \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y)\rangle \quad //$$

'tHooft ANOMALY:

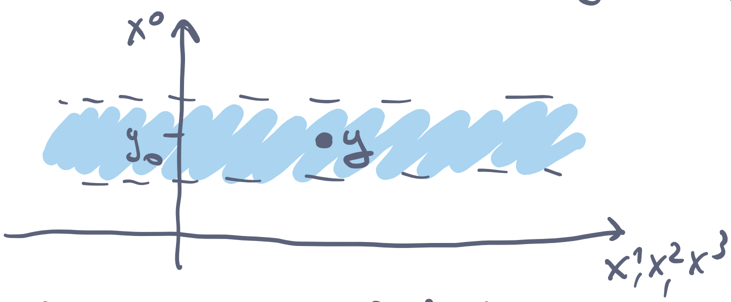
$$\partial_\mu^{x_1} \langle j_A^\mu(x_1)\, j^\alpha(x_2)\rangle = \frac{i}{\pi}\epsilon^{\alpha\beta}\partial_\beta \delta(x_1-x_2)$$

$\int dx_1 \downarrow = 0$

We can now integrate the WI (o) and obtain

$$i \langle [Q, \Phi^i(y)]_{\text{eq.time}} \rangle = M^i{}_j \langle \Phi^j(y) \rangle \qquad \text{(canonical quantization)}$$

Dim. Integrate $i \langle \partial_\mu j^\mu(x)\, \Phi^i(y) \rangle = \delta^4(x-y) M^i{}_j \langle \Phi^j(y) \rangle$ over the domain $\Omega_\varepsilon \equiv [y^0+\epsilon, y^0-\epsilon] \times \mathbb{R}^3$



$\int_{\mathbb{R}^3} \partial_i j^i = 0$

LHS: $$\int_{\Omega_\varepsilon} d^4x\, \partial_\mu j^\mu(x) = \int d^3x \left(j^0(y^0+\epsilon, \bar{x}) - j^0(y^0-\epsilon, \bar{x}) \right) =$$

$$= Q(y^0+\epsilon) - Q(y^0-\epsilon)$$

$$\langle (Q(y^0+\epsilon) - Q(y^0-\epsilon)) \Phi^i(y) \rangle = \langle 0 | T(Q(y^0+\epsilon) - Q(y^0-\epsilon)) \Phi^i(y) | 0 \rangle =$$

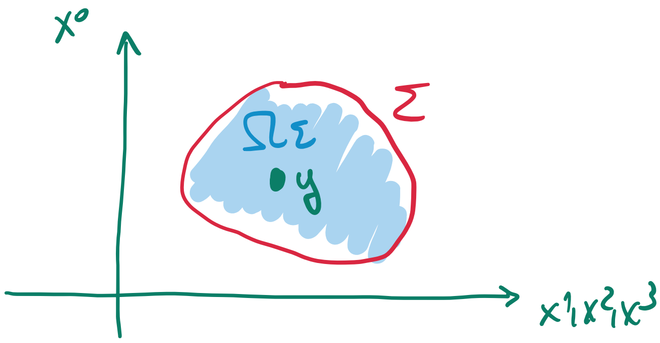
$$= \langle [\hat{Q}(y^0), \hat{\Phi}^i(y)] \rangle \qquad //$$

How does it work for extended objects?

?

Rewriting ordinary sym. transf.

$$i\langle Q(\Sigma)\,\Phi^i(y)\rangle = \mathrm{Link}(\Sigma, y)\, M^i{}_j \langle \Phi^j(y)\rangle$$



Charge Q on a time slice is generalized (Euclidean signature) to a charge $Q(\Sigma)$ on a 3d CLOSED subspace Σ

$$Q(\Sigma) \equiv \int_\Sigma *j$$

The commutation relations to LINK of Σ and y.
How do we derive this relation?

↓

Let's integrate w/ $(\circ)$ on Ω_Σ

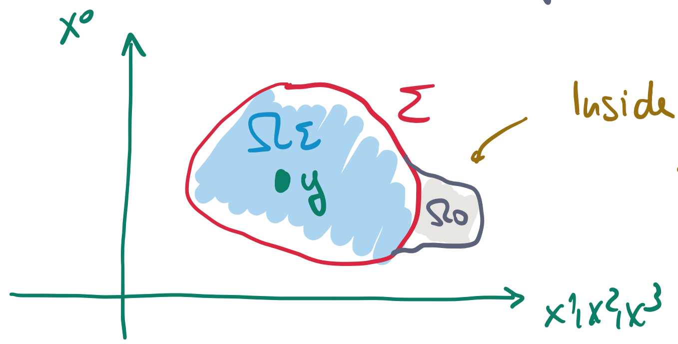
$$\text{LHS}: \int_{\Omega_\Sigma} \partial_\mu j^\mu d^4x = \int_{\Omega_\Sigma} d*j = \int_\Sigma *j = Q(\Sigma)$$

$$\hookrightarrow i\langle Q(\Sigma)\,\Phi^i(y)\rangle = \underbrace{\int_{\Omega_\Sigma} d^4x\, \delta^4(x-y)}_{\mathrm{Link}(\Sigma, y)}\, M^i{}_j \langle \Phi^j(y)\rangle$$

$\mathrm{Link}(\Sigma, y)$ ← TOPOLOGICAL INVARIANT

Also this is TOPOLOGICAL due to conserv. law:

under a contin. deform. $\Sigma \to \Sigma' = \Sigma + \partial\Omega_0 \quad y \in \Omega$



Inside Ω_0 there is NO INSERTION of local operators

$= 0$ in correlators $\longleftrightarrow \partial_\mu j^\mu = 0$

$$Q(\Sigma') = Q(\Sigma) + \int_{\partial\Omega_0} * j = Q(\Sigma) + \int_{\Omega_0} d * j = Q(\Sigma)$$

By exponentiating infinitesimal generators:

$$\langle U_g(\Sigma) \, \Phi^i(y) \rangle = R(g)^i{}_j \, \langle \Phi^j(y) \rangle \qquad (\text{if LINKED})$$

charged operator (0-dim $\rightsquigarrow$ 0-form sym.)

TOPOLOGICAL unitary operator depending on $g \in G$ & Σ

$$\left[\frac{d}{d\alpha} U_{e^{i\alpha}}(\Sigma) \Big|_{\alpha=0} = i \, Q(\Sigma) \right]$$

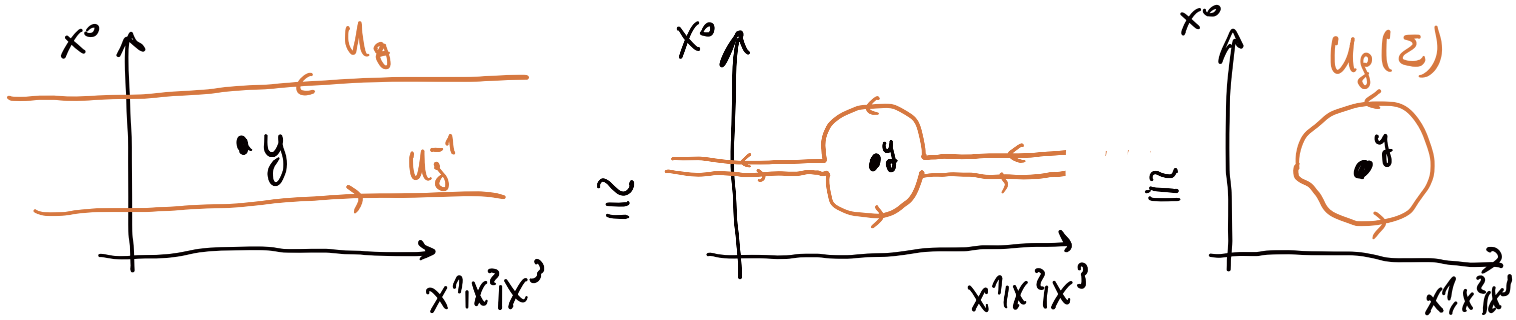
Discrete symmetries

- $g \in G$ discrete
- U_g : unitary operator commuting with Hamiltonian & momentum $\quad [U_g, P^\mu] = 0$
- $\langle U_g \, \Phi^i(y) \, U_g^{-1} \rangle = R(g)^i{}_j \, \langle \Phi^j(y) \rangle$

related to a TOPOL. OPERATOR $U_g(\Sigma)$ s.t.

$$\langle U_g(\Sigma) \, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \qquad (\text{if linked})$$

- $[U_g, P^\mu] = 0 \Rightarrow$ U_g can continuously move, i.e. is TOPOL.



- $U_g(\Sigma)\, U_{g'}(\Sigma) = U_{gg'}(\Sigma)$

Summary ORDINARY SYMMETRIES

$g \in G \longleftrightarrow$ TOPOL. OP. $U_g(\Sigma)$ $\qquad$ G cont/disc.

with

$$\langle U_g(\Sigma)\, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \quad \text{(if linked)} \qquad (*)$$

topological $\qquad$ not necessary topol.

Σ is $(d-1-\underline{\underline{0}})$-dim. $\qquad$ $\Phi^i(y)$ is $\underline{\underline{0}}$-dim $\quad \rightsquigarrow$ "0-form symmetry"

Reduced the problem of finding symmetries to the problem of finding topological operators.

$\downarrow$

This can be generalized to topological $(d-1-p)$-op.

i.e. p-form symmetries (we are now going to illustrate an example with $p=1$).

Observation: interpret $(*)$ as

$$\langle U_g(\Sigma)\, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle + 0 = R^i{}_j(g) \langle \Phi^j(y) \rangle + \langle U_g(\Sigma')\, \Phi^i(y) \rangle$$

with $\mathrm{Link}(\Sigma', y) = 0$

1-form symmetries in Maxwell theory

$$S[A] = -\frac{1}{2e^2}\int F\wedge *F = -\frac{1}{4e^2}\int d^4x\, F^{\mu\nu}F_{\mu\nu} \qquad (*)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\qquad$ A_μ COMPACT $U(1)$ gauge field

$\Rightarrow$ ELECTRIC CHARGES are QUANTIZED (seen through gauge transf. not-connected to identity)

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z}$$

proper normalization of $U(1)$ connection

$\rightarrow$ e is a meaningful coupling.

S-duality : Let's rewrite the action as

$$S[F,\tilde{A}] = \frac{1}{2e^2}\int F\wedge *F + \frac{1}{2\pi}\int F\wedge d\tilde{A}$$

now indep. variable

new 1-form

Lagrange multiplier

- e.o.m. of $\tilde{A}$: $dF = 0 \longrightarrow$ Bianchi id.
- e.o.m of F : $\frac{1}{e^2}*F = \frac{1}{2\pi}\tilde{F}$ $\qquad$ with $\tilde{F} = d\tilde{A}$

$\Rightarrow$ Integrating over $\tilde{A}$ $\rightarrow$ $S[A] = \frac{1}{2e^2}\int F\wedge *F$ with $F = dA$

Integrating over F $\rightarrow$ $S[\tilde{A}] = \frac{1}{2\tilde{e}^2}\int \tilde{F}\wedge *\tilde{F}$ with $\tilde{F} = d\tilde{A}$

and $\tilde{e}^2 = \frac{4\pi^2}{e^2}$

$\rightsquigarrow$ Example of DUALITY : the same theory has two equivalent presentations.

1-form symmetries

The e.o.m. of (★) are

$$\frac{1}{e^2}\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu (\star F)^{\mu\nu} = 0 \qquad \star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$d\star F = 0 \qquad\qquad dF = 0$$

F and $\star F$ are two-forms that are closed

$\Rightarrow$ they define TWO 1-form symmetries with currents $J_e = \frac{1}{e^2}F$ and $J_m = \frac{1}{2\pi}\star F$

- The corresponding CONSERVED CHARGES are

  - Electric flux

$$Q_e(\Sigma_2) = \frac{1}{e^2}\int_{\Sigma_2}\star F \sim \int_{\Sigma_2}\bar{E}\cdot d\bar{S} \leftrightarrow U(1)_e^{(1)}$$

  - Magnetic flux

$$Q_m(\Sigma_2) = \frac{1}{2\pi}\int_{\Sigma_2} F \sim \int_{\Sigma_2}\bar{B}\cdot d\bar{S} \longleftrightarrow U(1)_m^{(1)}$$

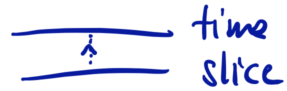
- Under S-duality

$$J_e \longleftrightarrow J_m$$

$$Q_e \longleftrightarrow Q_m$$

- Both $Q_e(S^2)$ & $Q_m(S^2)$ are TOPOLOGICAL under contin. deformations of Σ_2.

  $\Longrightarrow$ there should be corresponding symmetries (whose related conserved quantities are the topol. q's)



- Sym. operators

$$U_e(\alpha_e, \Sigma_2) = e^{i\alpha_e Q_e(\Sigma_2)}$$

$$U_m(\alpha_m, \Sigma_2) = e^{i\alpha_m Q_m(\Sigma_2)}$$

with $\alpha_e \sim \alpha_e + 2\pi$ and $\alpha_m \sim \alpha_m + 2\pi$.

Wilson Loop $\quad W(q_E, \gamma) = e^{iq_E \int_\gamma A}$

- Physically it is the worldline of a probe particle:

  Consider a particle with worldline $\mathcal{C}$ parametrized by x^0, s.t.

  $$\mathcal{C}: \quad x^i = y^i(x^0)$$

  Since $J^\mu = (\rho, \vec{j})$, then for one particle we have

  $$J^0(x^0, \bar{x}) = q_e \delta^{(3)}(\bar{x} - \bar{y}(x^0)) \qquad J^i(x^0, \bar{x}) = q_e \frac{dy^i}{dx^0} \delta^{(3)}(\bar{x} - \bar{y}(x^0))$$

  i.e.

  $$J^\mu = q_e \frac{dy^\mu}{dx^0} \delta^{(3)}(\bar{x} - \bar{y}(\tau))$$

  $$\Longrightarrow e^{i\int d^4x \, J^\mu A_\mu} = e^{i\int dx^0 \, q_e \frac{dy^\mu}{dx^0} \int d^3x \, A_\mu \delta^{(3)}(\bar{x}, \bar{y}(\tau))} =$$

  $$= e^{iq_e \int A_\mu(y) \dot{y}^\mu d\tau} = e^{iq_e \oint_{\mathcal{C}} A_\mu dx^\mu}$$

This means that

$$\langle W_{q_e}[C]\rangle = \int \mathcal{D}A \, e^{iq_e \oint_C A} e^{iS[A]} = \int \mathcal{D}A \, e^{iS[a] + i\int J^\mu A_\mu}$$

- gauge group $U(1) \ni e^{i\lambda}$ $\quad \lambda \sim \lambda + 2\pi$

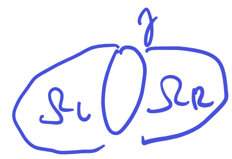
- $\lambda(x)$ can have <u>winding number on γ</u> : $\int_\gamma d\lambda = 2\pi w$ $\quad w \in \mathbb{Z}$

- Wilson loop gauge inv. $(A \to A + d\lambda)$

$$e^{iq_E \int_\gamma A} \stackrel{!}{=} e^{iq_E \int_\gamma A} e^{iq_E \int_\gamma d\lambda} \iff q_E 2\pi w \in 2\pi\mathbb{Z} \ \ \forall w \in \mathbb{Z}$$
$$\iff q_E \in \mathbb{Z}$$

i.e. Large gauge inv. of WL $\Rightarrow$ DIRAC quantization

On the other hand,



$$e^{iq_E \int_\gamma A} = e^{iq_E \int_{\Omega_L} F} = e^{iq_E \int_{\Omega_R} F}$$

$$\Rightarrow e^{iq_E \int_{S^2} F} = 1 \qquad S^2 = \Omega_L \cup \bar{\Omega}_R$$

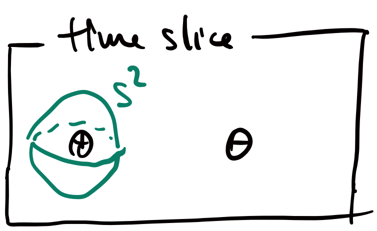
$$\Rightarrow \underbrace{\frac{1}{2\pi}\int_{S^2} F}_{q_m} \in \frac{\mathbb{Z}}{q_E} \quad \to \quad \text{Dirac quant.} \quad q_E q_m = 2\pi n \quad n \in \mathbb{Z}$$

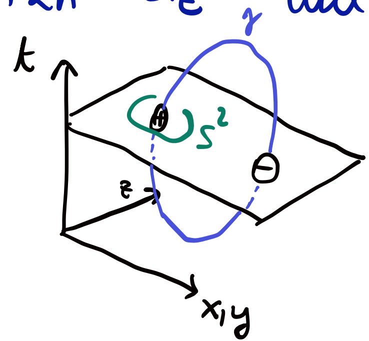
- Symmetry transformation

$$(\ast) \quad \langle \underbrace{U_{e^{i\alpha_E}}(S^2)}_{e^{i\alpha_E Q_E(S^2)}} \, e^{iq_E \int_\gamma A} \rangle = e^{i\alpha_E q_E \mathrm{Link}(S^2,\gamma)} \langle e^{iq_E \int_\gamma A} \rangle$$

$\mathrm{Link}(S^2,\gamma) \to$ Linking number between S^2 and γ

Sym. group is $U(1)$ $\quad \alpha_E + 2\pi \sim \alpha_E$, due to quant. of q_E.





Dim.
$$\langle U_{e^{i\alpha_E}}(S^2)\, e^{iq_E\int_\gamma A}\rangle = \int \mathcal{D}A\; e^{iS + i\alpha_E Q_E(S^2) + iq_E\int_\gamma A}$$

$$Q_E(S^2) = \frac{1}{e^2}\int_{S^2} *F = \frac{1}{e^2}\int_{B_3} d*F \equiv 2\int J_{B_3}\wedge d*F$$

$$S[\overbrace{A - \alpha_E J_{B_3}}^{\equiv A'}] = \int (dA - \alpha_E dJ_{B_3})\wedge *(dA - \alpha_E dJ_{B_3}) =$$

$$= S[A] + 2\alpha_E \int J_{B_3}\wedge d*F + \alpha_E^2 \int dJ_{B_3}\wedge * dJ_3$$

$= \int_{S^2} * dJ_3$ It can be regularized by a local counter-term

$$= S[A] + \alpha_E Q_E(S^2) + \dots$$

$$\Rightarrow \langle U_{e^{i\alpha_E}}(S^2)\, e^{iq_E\int_\gamma A}\rangle = \int \mathcal{D}A'\; e^{iS[A']}\, e^{iq_E\int_\gamma A'}\, e^{iq_E\alpha_E\int_\gamma J_{B_3}}$$

$$= e^{iq_E\alpha_3 \underbrace{\int_\gamma J_{B_3}}}\; \langle e^{iq_E\int_\gamma A}\rangle$$

Intersection number between γ and B_3, i.e. $\mathrm{Link}(S^2,\gamma)$

//

Summary:

- Sym op: $U_{e^{i\alpha_E}}(S^2) = e^{i\alpha_E Q_E(S^2)}$ 2d topol. op.
- Charged op: $e^{iq_E\int_\gamma A}$
- Sym. group: $e^{i\alpha_E} \in U(1)$

$\downarrow$

"ELECTRIC 1-form SYMMETRY"

't HOOFT LOOP $T(q_M, \gamma)$

- Probe magn. part. (monopole)
- Closed line $\Leftrightarrow$ gauge invariance of dual photon
- $q_M \in \mathbb{Z}$ (if $q_E = 1$)
- obtain same formal expression as before when we dualize electric $\longleftrightarrow$ magnetic.

$\downarrow$

"MAGNETIC 1-form SYMMETRY"

Generalisations

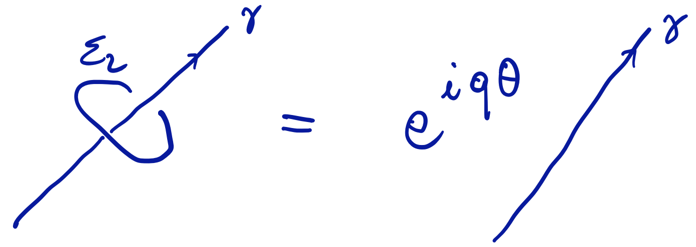
G p-form symmetry in d dim :

- Sym op. $U_g(\Sigma_{d-p-1})$
- Charged objects $W(q,\gamma_p)$
- Sym. transf. $\langle U_g(\Sigma_{d-p-1}) W(q,\gamma_p)\rangle = R(g)^q \langle W(q,\gamma_p)\rangle$ if linked

Take-home message : Existence of sym = Existence of TOPOLOGICAL OPERATORS

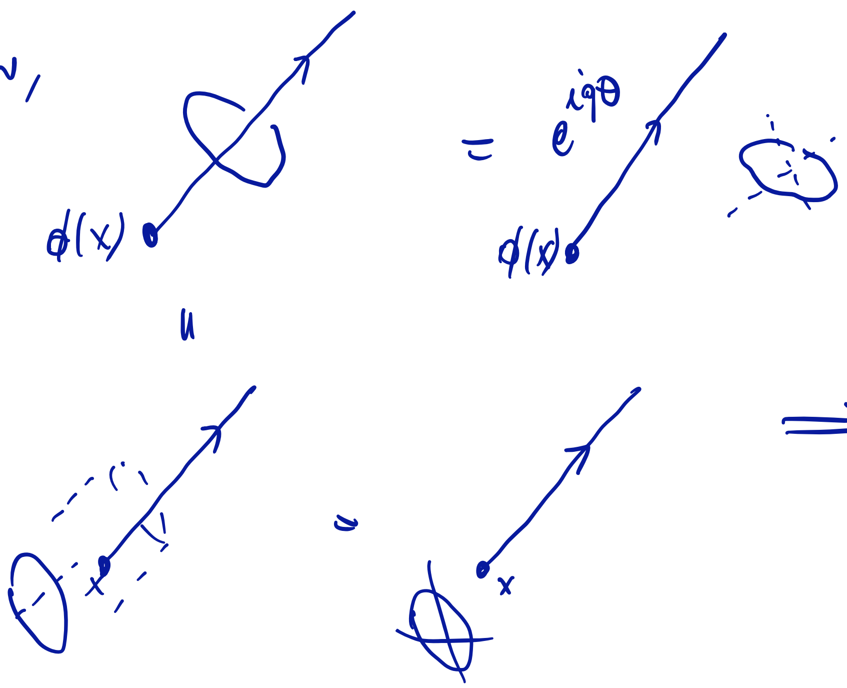
Adding matter

Let's remember how 1-form sym work:



If we now add charged fields $\phi(x)$ with charge q there will be gauge invariant lines that can end on the location x of the charged operator: in fact the gauge transf. on the extremum x of the WL is compensated by the gauge transf. of $\phi(x)$

Now,



$\Longrightarrow$ To be consistent:

$$e^{iq\theta} = 1$$

i.e. $\theta = \frac{2\pi k}{q}$

$$U(1)^{(1)} \to \mathbb{Z}_q^{(1)}$$

This is called "SCREENING" by lower dim. operators.
(Now line is uncharged.)

1-form symmetries in YM theory

$G = SU(N)$:

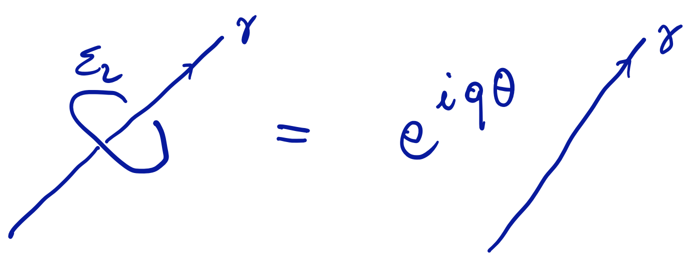
- Wilson lines $W(\gamma)$ can lie in all REPs of G
  (i.e. charges can be anywhere in weight lattice)

- There exist a SURFACE OPERATOR
  $U(\Sigma_2)$ that will be the generator of a
  $\mathbb{Z}_N$ one-form symmetry ; in fact

  $\langle U(\Sigma_2)\, W(\gamma) \dots \rangle$ and $\langle W(\gamma) \dots \rangle$ turn out to
  differ by a factor $e^{2\pi i/N \cdot \mathrm{Link}(\Sigma_2, \gamma)}$

  Equivalently $\mathbb{Z}_N$ 1-form sym shifts gauge field
  by a flat $\mathbb{Z}_N$ gauge connection

Why $\mathbb{Z}_N$? Let's remember how 1-form sym
work :



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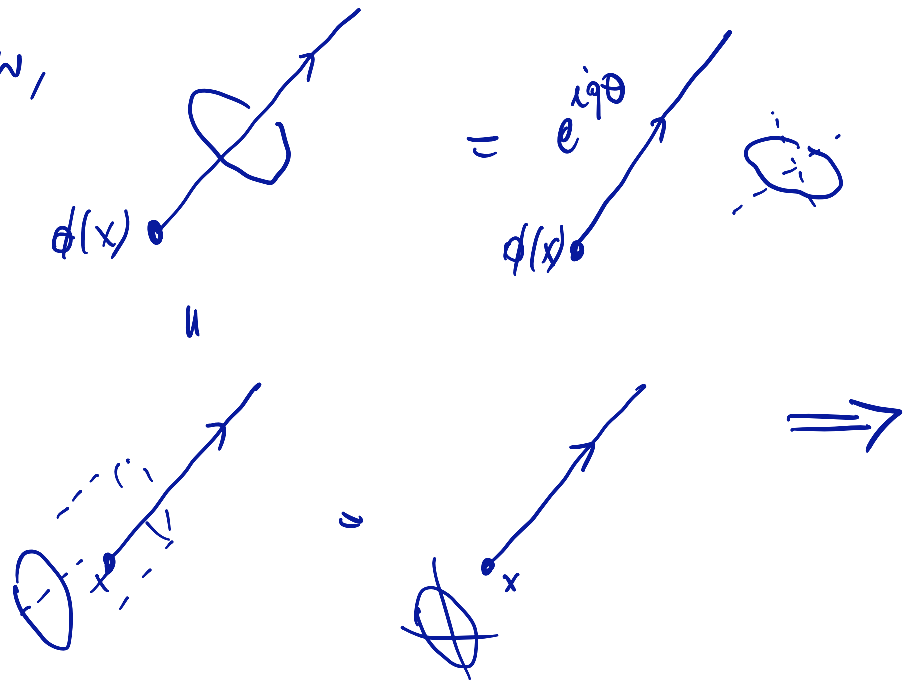

$\Rightarrow$ To be consistent:

$$e^{iq\theta} = 1$$

i.e. $\theta = \frac{2\pi k}{q}$

$$U(1)^{(1)} \to \mathbb{Z}_q^{(1)}$$

Wilson lines corresponding to probes with charges $\notin q\mathbb{Z}$ cannot end on $\phi(x)$ and have in fact a non trivial transformation under $\mathbb{Z}_q^{(1)}$

- In Maxwell theory there is no charged field, then WL for all probes have non-trivial $U(1)^{(1)}$ transformation.

- However, in YM there are ADJOINT FIELDS, i.e. the gluons gauge bosons.
  Probes in the adjoint rep produces WL that can end on the location of an adjoint field; then one can unlink the Σ_2 from the line and the corresponding WL must have zero charge

Only weights μ that are not in $\Lambda_{root}(G)$ give WL transforming non-trivially $\rightsquigarrow \mathbb{Z}_N^{(1)}$-sym.