VECTORS, TENSORS AND INDEX NOTATION

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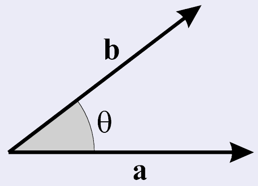
Review of vector algebra

The scalar product

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ (*dot product*) is usually written $\mathbf{a} \cdot \mathbf{b}$, and is defined as the scalar quantity:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$



Some properties of the scalar product

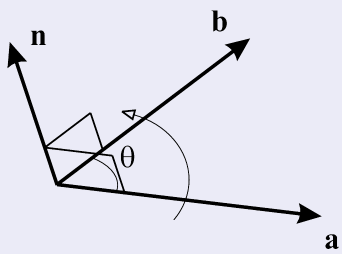
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2 \\ \mathbf{a} \cdot \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \cdot (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_x b_x + a_y b_y + a_z b_z \end{aligned}$$

Review of vector algebra - cont.

The vector product

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ (*cross product*) is usually written $\mathbf{a} \times \mathbf{b}$ (or $\mathbf{a} \wedge \mathbf{b}$), and is defined as the vector quantity

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\mathbf{n}$$



Some properties of the vector product

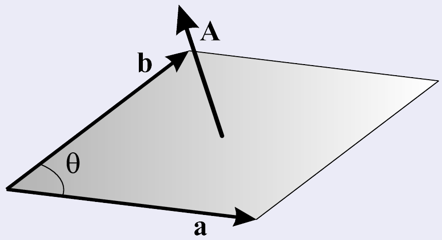
$$\begin{aligned} \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b} \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\ \mathbf{a} \times \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= (a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k} \end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Review of vector algebra - cont.

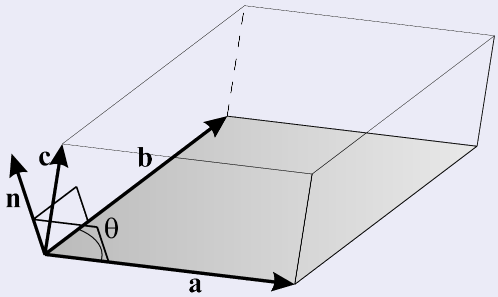
Vector area $\mathbf{A}$

$$\mathbf{A} = \mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\mathbf{n}$$



Volume of a parallelepiped

$$V = \mathbf{c}\cdot(\mathbf{a} \times \mathbf{b}) = \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = \mathbf{b}\cdot(\mathbf{c} \times \mathbf{a})$$



Scalar triple product

- We already saw that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

- A useful way of expressing the scalar triple product is in determinant form. Since

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

it follows that

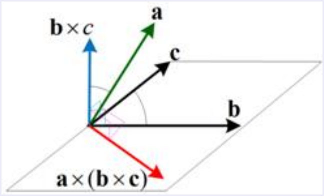
Scalar triple product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Vector triple product

Geometric interpretation

- From the properties of the cross-product, it follows that the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular both to $\mathbf{a}$ and to $(\mathbf{b} \times \mathbf{c})$.
- Since $(\mathbf{b} \times \mathbf{c})$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, it follows that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must lie in the plane containing both $\mathbf{b}$ and $\mathbf{c}$ (see figure).



Therefore we may write

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha \mathbf{b} + \beta \mathbf{c}$$

where α and β are yet to be determined. This can be done evaluating the vector-product on the left-hand side, and use the right-hand side to determine α and β.

Vector triple product - cont.

If $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$, and analogously for $\mathbf{b}$ and $\mathbf{c}$, then

$$\mathbf{b} \times \ \mathbf{c} = \mathbf{i} \left(b_y c_z - b_z c_y\right) - \mathbf{j} \left(b_x c_z - c_x b_z\right) + \mathbf{k} \left(b_x c_y - c_x b_y\right)$$

Using the *determinant* form of the vector product

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & c_x b_z - b_x c_z & b_x c_y - c_x b_y \end{vmatrix} \\
&= \mathbf{i} \left[a_y \left(b_x c_y - c_x b_y\right) - a_z \left(c_x b_z - b_x c_z\right)\right] \\
&\quad - \mathbf{j} \left[a_x \left(b_x c_y - c_x b_y\right) - a_z \left(b_y c_z - b_z c_y\right)\right] \\
&\quad + \mathbf{k} \left[a_x \left(c_x b_z - b_x c_z\right) - a_y \left(b_y c_z - b_z c_y\right)\right]
\end{aligned}$$

Regrouping according to the form $\alpha \mathbf{b} + \beta \mathbf{c}$

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = {} & \mathbf{i} b_x \left(a_y c_y + a_z c_z\right) + \mathbf{j} b_y \left(a_x c_x + a_z c_z\right) + \mathbf{k} b_z \left(a_x c_x + a_y c_y\right) \\
& - \mathbf{i} c_x \left(a_y b_y + a_z b_z\right) - \mathbf{j} c_y \left(a_x b_x + a_z b_z\right) - \mathbf{k} c_z \left(a_x b_x + a_y b_y\right)
\end{aligned}$$

Vector triple product - cont.

Adding and subtracting

$$\mathbf{i}\,(b_x a_x c_x) + \mathbf{j}\,(b_y a_y c_y) + \mathbf{k}\,(b_z a_z c_z)$$

to the previous expression finally gives

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \tag{1}$$

and therefore

$$\alpha = \mathbf{a} \cdot \mathbf{c}$$
$$\beta = \mathbf{a} \cdot \mathbf{b}$$

Gradient of a scalar

A vector operator, which is frequently used in fluid dynamics and heat transfer, is the *nabla*, (or *del*) operator, defined as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

When the *nabla* operator is applied on a scalar variable ϕ, it results in the *gradient* of ϕ, given by:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Therefore, the gradient of a scalar field is a vector field and it indicates that the value of ϕ changes in space in both magnitude and direction.

Gradient of a scalar - cont.

The projection of $\nabla\phi$ in the direction of the unit vector $\mathbf{e}_l$ is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}l} = \nabla\phi \cdot \mathbf{e}_l = |\nabla\phi| \cos\left(\nabla\phi, \mathbf{e}_l\right)$$

and is called *directional derivative* of ϕ along the direction of the unit vector $\mathbf{e}_l$.

- The maximum value of the directional derivative is $|\nabla\phi|$ and is achieved when $\cos(\nabla\phi, \mathbf{e}_l) = 1$, e.g. in the direction of $\nabla\phi$.
- Therefore, the gradient of a scalar field ϕ indicates the direction and magnitude of the largest variation of ϕ at every point in space.
- Finally, $\nabla\phi$ is *normal* to the ϕ = *const.* surface that passes through that point.

Operations on the ∇ operator

- The scalar product of the ∇ operator with a vector $\mathbf{u}$ of components u, v and w in the x, y and z direction, respectively, is the *Divergence* of the vector $\mathbf{u}$, which is a scalar quantity written as
$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$
- The divergence of the gradient of a scalar variable ϕ is denoted by the *Laplacian* operator and is a scalar given by
$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$
- The *Laplacian* of a vector follows from the above definition and is a vector given by
$$\nabla^2\mathbf{u} = \left(\nabla^2 u\right)\mathbf{i} + \left(\nabla^2 v\right)\mathbf{j} + \left(\nabla^2 w\right)\mathbf{k}$$

Additional vector operations

If ϕ is a scalar field, and $\mathbf{v}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are vector fields, then the following relations apply:

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{v}) &= 0 \\
\nabla \times (\nabla \phi) &= 0 \\
\nabla \cdot (\phi \mathbf{v}) &= \phi \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \phi \\
\nabla \times (\phi \mathbf{v}) &= \phi \nabla \times \mathbf{v} + \nabla \phi \times \mathbf{v} \\
\nabla (\mathbf{v}_1 \cdot \mathbf{v}_2) &= \mathbf{v}_1 \times (\nabla \times \mathbf{v}_2) + \mathbf{v}_2 \times (\nabla \times \mathbf{v}_1) + (\mathbf{v}_1 \cdot \nabla)\, \mathbf{v}_2 + (\mathbf{v}_2 \cdot \nabla)\, \mathbf{v}_1 \\
\nabla \cdot (\mathbf{v}_1 \times \mathbf{v}_2) &= \mathbf{v}_2 \cdot (\nabla \times \mathbf{v}_1) - \mathbf{v}_1 \cdot (\nabla \times \mathbf{v}_2) \\
\nabla \times (\mathbf{v}_1 \times \mathbf{v}_2) &= \mathbf{v}_1 (\nabla \cdot \mathbf{v}_2) - \mathbf{v}_2 (\nabla \cdot \mathbf{v}_1) + (\mathbf{v}_2 \cdot \nabla)\, \mathbf{v}_1 - (\mathbf{v}_1 \cdot \nabla)\, \mathbf{v}_2 \\
\nabla \times (\nabla \times \mathbf{v}) &= \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \\
(\nabla \times \mathbf{v}) \times \mathbf{v} &= \mathbf{v} \cdot (\nabla \mathbf{v}) - \nabla (\mathbf{v} \cdot \mathbf{v})
\end{aligned}
$$

Additional vector operations - *cont.*

Let's prove the second, e.g.

$$\nabla \times (\nabla \phi) = 0$$

It can be expressed as

$$
\begin{aligned}
\nabla \times (\nabla \phi) &= \nabla \times \left(\mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\
&= \mathbf{i} \left(\frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y} \right) - \mathbf{j} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial x} \right) + \mathbf{k} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} \right) \\
&= 0
\end{aligned}
$$

Operations on the ∇ operator - cont.

Another quantity of interest is the *curl* of a vector field $\mathbf{u}$, which is a vector obtained by the cross product of the ∇ operator and vector $\mathbf{u}$

$$
\begin{aligned}
\nabla \times \mathbf{u} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \times (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & z \end{vmatrix} \\
&= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k}
\end{aligned}
$$

The dyadic product

- A *dyad* is a tensor of order two and rank one, and is the result of the *dyadic product* (also called *outer product* or *tensor product*) of two vectors.
- If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then the dyadic product is indicated as $\mathbf{ab}$, $\mathbf{a} \otimes \mathbf{b}$ or $\mathbf{ab}^T$, where T means *transpose*.
- The dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a *Tensor* and is given, in matrix form, by

$$
\mathbf{ab} \equiv \mathbf{a} \otimes \mathbf{b} \equiv \mathbf{ab}^T = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} [b_1\ b_2\ b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}
$$

The vector product of a vector and a tensor

The *Vector Product* (or *Dot Product*) of a vector $\mathbf{a}$ and a tensor $\mathbf{T}$ is a vector and is given by

$$\mathbf{a}\cdot\mathbf{T} \equiv \left(\mathbf{a}^T\mathbf{T}\right)^T = \left([a_1\, a_2\, a_3]\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\right)^T$$

$$= [a_1T_{11} + a_2T_{21} + a_3T_{31} \quad a_1T_{12} + a_2T_{22} + a_3T_{32} \quad a_1T_{13} + a_2T_{23} + a_3T_{33}]^T$$

$$= \left\{ \begin{array}{c} a_1T_{11} + a_2T_{21} + a_3T_{31} \\ a_1T_{12} + a_2T_{22} + a_3T_{32} \\ a_1T_{13} + a_2T_{23} + a_3T_{33} \end{array} \right\}$$

A common mixed product

An important mixed product, which is frequently found in fluid mechanics, is the following

$$[(\mathbf{u}\cdot\nabla)\,\mathbf{u}] = (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{j} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{k}$$

$$= \left\{ \begin{array}{c} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{array} \right\}$$

A common mixed product - cont.

It is easy to see that the same result can be obtained considering the vector (dot) product of the vector $\mathbf{u}$ with the tensor obtained with the dyadic product of the divergence with the vector $\mathbf{u}$:

$$\mathbf{u}\cdot(\nabla\mathbf{u}) \equiv \left(\mathbf{u}^T(\nabla\mathbf{u})\right)^T = \left([u\,v\,w]\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}\right)^T$$

$$= \left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \quad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \quad u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]^T$$

$$= \begin{Bmatrix} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \end{Bmatrix}$$

Index notation: overview

- The notation adopted to describe vectors is perhaps the most widely accepted.
- There are a number of alternative notations, and one of the most useful is the *index notations*, also called *Cartesian notation*.
- The *index notation* is a powerful tool for manipulating multidimensional equations, and it enjoys a number of advantages in comparison to the traditional *vector notation*:
  - *Conciseness:* it allows standard results to be written in an immediately clear though compact form.
  - *Quick proof* of the results that we need.
- There are other advantages - and a couple of disadvantages - but in addition it is important to note that the *index approach* is the most *natural* notation when dealing with a generalization of the vector: the *tensor*.
- We will find convenient to relabel the coordinates (x, y, z) as (x_1, x_2, x_3) or simply as x_i, with the agreement that the index i can take any of the values $1, 2$ and 3.
- Using this index notation, a vector $\mathbf{a}$ may be denoted by a_i. Therefore the components of the vector a_i are (a_1, a_2, a_3).
- There are times when the more conventional vector notation is more useful. It is therefore important to be able to easily convert back and forth between the two.

Equality and summation convention

Equality

Two vectors a_i and b_i are equal if and only if

$$a_i = b_i \tag{2}$$

If the index is unspecified then the equation in which it occurs is assumed to be valid for each of the three possible values that the index can take, e.g. (2) is shorthand for

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3 \tag{3}$$

The summation convention

If an index is repeated in a term, then it is assumed that the term is summed as the repeated index takes the values 1, 2 and 3. Thus

$$a_i b_i \;\; \text{means} \;\; \sum_{i=1}^{3} a_i b_i \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{4}$$

Using this convention, if a_i and b_i are two vectors (representing $\mathbf{a}$ and $\mathbf{b}$), then the scalar product $\mathbf{a} \cdot \mathbf{b}$ is represented by $a_i b_i$.

Dummy index

Dummy index

A repeated index is called a *dummy index*, since the precise letter used is irrelevant: $a_i b_i$ has the same meaning as $a_j b_j$, $a_p b_p$, $a_s b_s$ etc., that is $a_1 b_1 + a_2 b_2 + a_3 b_3$.

A dummy index appears twice within an additive term of an expression. In the equation below, j and k are both dummy indices

$$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

The dummy index is *local* to an individual additive term. It may be renamed in one term - so long as the renaming doesn't conflict with other indices - and it does not need to be renamed in other terms (and, in fact, may not necessarily even be present in other terms).

The summation convention: examples

$$\lambda = a_i b_i \equiv \lambda = \sum_{i=1}^{3} a_i b_i \equiv \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$c_i = S_{ik} x_k \equiv c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \begin{cases} c_1 = S_{11} x_1 + S_{12} x_2 + S_{13} x_3 \\ c_2 = S_{21} x_1 + S_{22} x_2 + S_{23} x_3 \\ c_3 = S_{31} x_1 + S_{32} x_2 + S_{33} x_3 \end{cases}$$

$$\tau = S_{ij} S_{ij} \equiv \tau = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \equiv \tau = S_{11} S_{11} + S_{12} S_{12} + \ldots + S_{32} S_{32} + S_{33} S_{33}$$

$$C_{ij} = A_{ik} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \equiv [C] = [A]\,[B]$$

$$C_{ij} = A_{ki} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \equiv [C] = [A]^T\,[B]$$

Free index

Free index

An index that is not repeated (in a term) is called a *free index*.

In the equation below, i is a free index:

$$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

The same letter must be used for the free index in every additive term. The free index may be renamed if and only if it is renamed in every term.
Terms in an expression may have more than one free index so long as the indices are distinct.
For example the vector-notation expression

$$\mathbf{A} = \mathbf{B}^T$$

is written

$$A_{ij} = (B_{ij})^T = B_{ji}$$

in index notation.
This expression implies nine distinct equations, since i and j are both free indices.

Free index - cont.

The number of free indices in a term equals the rank of the term:

	Notation	Rank
scalar	a	0
vector	a_i	1
tensor	A_{ij}	2
tensor	A_{ijk}	3

Technically, a scalar is a tensor with rank 0, and a vector is a tensor of rank 1. Tensors may assume a rank of any integer greater than or equal to zero. You may only sum together terms with equal rank.
The first free index in a term corresponds to the row, and the second corresponds to the column. Thus, a vector (which has only one free index) is written as a column of three rows

$$\mathbf{a} = a_i = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\}$$

and a rank-2 tensor is written as

$$\mathbf{A} = A_{ij} = \left[\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right]$$

The Kronecker delta

- We define a two-indexed quantity δ_{ij} called the Kronecker delta by

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } \; i = j \\ 0 & \text{otherwise} \end{array} \right. \tag{5}$$

Thus $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$. Another name for this quantity is the *substitution operator*, since it can be used to substitute one index for another. This follows from the easiliy verified result

$$a_j = \delta_{ji} a_i \tag{6}$$

The right-hand side is $\delta_{j1}a_1 + \delta_{j2}a_2 + \delta_{j3}a_3$, so if $j = 1$ in (6)
$a_1 = \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = a_1$, with similar results for $j = 2$ and $j = 3$.

- Like in ordinary algebra, an equation will be formed from a collection of terms added together and equated to zero.
Each term will include one or more indexed objects multiplied together.

Rules

1. The same index (subscript) may not appear more than twice in a product of two (or more) vectors or tensors. Thus
$$A_{ik}x_k, \quad A_{ik}B_{kj}, \quad A_{ij}B_{ik}C_{nk}$$
are valid, but
$$A_{kk}x_k, \quad A_{ik}B_{kk}, \quad A_{ij}B_{ik}C_{ik}$$
are meaningless.
2. The number and type of free indices in any term of a given equation must be the same. Thus
$$x_i = u_i + c_i \equiv \mathbf{x} = \mathbf{u} + \mathbf{c}$$
$$a_i = A_{ki}B_{kj}x_j + C_{ik}u_k \equiv \mathbf{a} = \mathbf{A}^T\mathbf{B}\mathbf{x} + \mathbf{C}\mathbf{u}$$
are valid, but
$$x_i = A_{ij}$$
$$x_j = A_{ik}u_k$$
$$x_i = A_{ik}u_k + c_j$$
are meaningless.

Rules - cont.

3. Free and dummy indices may be changed without altering the meaning of an expression, provided that rules 1 and 2 are not violated. Thus
$$x_i = A_{ik}x_k \Leftrightarrow x_j = A_{jk}x_k \Leftrightarrow x_j = A_{ji}x_i$$

Examples

Example 1

If a_i and b_i correspond to vectors $\mathbf{a}$ and $\mathbf{b}$, the index form of the vector equation

$$\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = \mathbf{0}$$

is the following

$$a_i + a_j b_j b_i = 0$$

Example 2

Verify that $\delta_{ij}\delta_{ij} = 3$

$$\delta_{ij}\delta_{ij} = \sum_i \sum_j \delta_{ij}\delta_{ij}$$

$$= \delta_{11}\delta_{11} + \cancel{\delta_{12}\delta_{12}} + \cancel{\delta_{13}\delta_{13}} + \cancel{\delta_{21}\delta_{21}} + \delta_{22}\delta_{22} + \cancel{\delta_{32}\delta_{32}} + \cancel{\delta_{31}\delta_{31}} + \cancel{\delta_{32}\delta_{32}} + \delta_{33}\delta_{33}$$

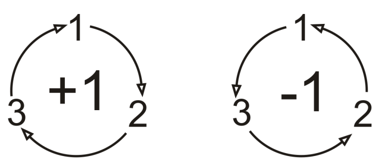
$$\delta_{ij}\delta_{ij} = 3$$

Permutation Symbol

The *Permutation Symbol* or *Levi-Civita symbol* [1], sometimes also called *Alternating Unit Tensor*, is defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is a non-cyclic permutation of } 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The alternating unit tensor is positive when the indices assume any clockwise cyclical progression, and it is negative when the indices assume any anticlockwise cyclic order, as shown in the figure:



Since each subscript can only take on values 1, 2 or 3, it follows that ϵ_{ijk} has 27 *components*.

[1] Tullio Levi-Civita, 29 March 1873 Padua - 29 December 1941 Rome, was an Italian mathematician, most famous for his work on absolute differential calculus (tensor calculus) and its applications to the theory of relativity.

Permutation Symbol - cont.

From the definition of ϵ_{ijk}, it follows that

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \tag{8}$$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1 \tag{9}$$

and all other $\epsilon_{ijk} = 0$, for example

$$\epsilon_{223} = \epsilon_{233} = \epsilon_{311} = 0 \tag{10}$$

These results imply the following identities

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \tag{11}$$

That is, a cyclic permutation of the indices on the permutation symbol does not alter its value. Also

$$\epsilon_{ijk} = -\epsilon_{kji}, \quad \epsilon_{ijk} = -\epsilon_{jik}, \quad \epsilon_{ijk} = -\epsilon_{ikj} \tag{12}$$

that is, interchange of any two indices on the permutation symbol introduces a minus sign. This corresponds to an anticyclic permutation of indices.

Permutation Symbol and vector product

With the introduction of the *permutation symbol* ϵ_{ijk} it can be shown that the *i*th component of the vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ (or a_i and b_i) is

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \tag{13}$$

The right-hand side of (13) involves two repeated indices and therefore a double summation.

1. Firstly summation over the index *j*:

$$\epsilon_{ijk} a_j b_k = \epsilon_{i1k} a_1 b_k + \epsilon_{i2k} a_2 b_k + \epsilon_{i3k} a_3 b_k$$

2. Summing over the index k (and using the result in (10) that ϵ_{ijk} is zero if it contains two identical indices),

$$\begin{aligned} \epsilon_{ijk} a_j b_k = & \cancel{\epsilon_{i11} a_1 b_1} + \epsilon_{i12} a_1 b_2 + \epsilon_{i13} a_1 b_3 \\ & + \epsilon_{i21} a_2 b_1 + \cancel{\epsilon_{i22} a_2 b_2} + \epsilon_{i23} a_2 b_3 \\ & + \epsilon_{i31} a_3 b_1 + \epsilon_{i32} a_3 b_2 + \cancel{\epsilon_{i33} a_3 b_3} \end{aligned}$$

Permutation Symbol and vector product - cont.

- Therefore if i takes the value 1:

$$\epsilon_{1jk} a_j b_k = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$$

- Similarly, if $i = 2$

$$\epsilon_{2jk} a_j b_k = \epsilon_{213} a_1 b_3 + \epsilon_{231} a_3 b_1 = a_3 b_1 - a_1 b_3$$

- and if $i = 3$

$$\epsilon_{3jk} a_j b_k = \epsilon_{312} a_1 b_2 + \epsilon_{321} a_2 b_1 = a_1 b_2 - a_2 b_1$$

But these three terms - just replace $1, 2, 3$ with x, y, z - are the components of $\mathbf{a} \times \mathbf{b}$, and thus (13) is verified.

The $\epsilon - \delta$ identity

There exists a basic identity (it is proved using the properties of determinants) connecting the ϵ symbol and the δ symbol. It is

$$\epsilon_{ijk} \epsilon_{ipr} = \delta_{jp} \delta_{kr} - \delta_{jr} \delta_{kp} \tag{14}$$

It has numerous applications when developing vector identities. For example, it is possible to re-prove, using the index approach, the identity (1), namely

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$

Now the ith component of the left-hand side is, using (13) twice

$$\epsilon_{ijk} a_j \left(\mathbf{b} \times \mathbf{c}\right)_k = \epsilon_{ijk} \epsilon_{klm} a_j b_l c_m$$

After a cyclic permutation on the first ϵ symbol on the right-hand side, it results

$$\begin{aligned} \epsilon_{kij} \epsilon_{klm} a_j b_l c_m &= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) a_j b_l c_m \\ &= a_j b_i c_j - a_j b_j c_i \\ &= (\mathbf{a} \cdot \mathbf{c})\, b_i - (\mathbf{a} \cdot \mathbf{b})\, c_i \\ &= \left\{(\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}\right\}_i \end{aligned}$$

which proves the result.

Total differential of a function and gradient

Consider the total differential of a function of three variables, $\phi(x_1, x_2, x_3)$. We have

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial \phi}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial \phi}{\partial x_3}\,\mathrm{d}x_3 \qquad (15)$$

In tensor notation, this is replaced by

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x_i \qquad (16)$$

Equation (16) can be thought of as the dot product of the *gradient* of ϕ, namely $\nabla\phi$, and the differential vector $\mathrm{d}\mathbf{x} = (\,\mathrm{d}x_1,\,\mathrm{d}x_2,\,\mathrm{d}x_3)$. Thus, the ith component of $\nabla\phi$, which we denote as $(\nabla\phi)_i$, is given by

Gradient

$$(\nabla\phi)_i = \frac{\partial \phi}{\partial x_i} = \phi_{,i} \qquad (17)$$

where a comma followed by an index is tensor notation for differentiation with respect to x_i.

Divergence and curl

The *divergence of a vector* $\mathbf{u}$ is given by

Divergence

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = u_{i,i} \qquad (18)$$

where again differentiation with respect to x_i is denoted by $,i$.
The *curl* of a vector $\mathbf{u}$ is

Curl

$$(\nabla \times \mathbf{u}) = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \epsilon_{ijk}u_{k,j} \qquad (19)$$