STATISTICAL METHODS WITH APPLICATION TO FINANCE

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Returns: Definitions and properties

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Financial markets and risk

The focus of the course is on common statistical techniques useful in **financial markets** data analysis. In many of the problems of interest in finance, the starting point is the analysis of price series.

A Stock market index shows how a specified portfolio of share prices changes over time, giving an indication of market trends (e.g. Dow Jones Industrial Average (DJIA), the NASDAQ, the Dax, the FTSE 100, Nikkei, etc.)

There are two main ingredients: *time* and *uncertainty*

- A stock market index shows how a specified portfolio of share prices changes over time
- Risk is related to the uncertainty in the return that we obtain from investing in stock shares (risky assets)

Financial markets and risk (cont)

Most financial studies involve returns, instead of prices, of assets (e.g. stocks, bonds, a portfolio of stocks and bonds) for two main reasons:

- return of an asset is a complete and scale-free summary of the investment opportunity
- return series have more attractive statistical properties

The return on an investment is simply its revenue expressed as a fraction of the initial investment. There are, however, several definitions of an asset return.

Financial markets and risk (cont)

We may model issues related to time and uncertainty within a mathematical framework, using tools from probability and statistics.

- Future returns cannot be known exactly and therefore are **random variables**
- Volatility, or the standard deviation of returns, is a common measure of risk. However, the use of volatility as a risk measure can lead to misleading conclusions!

  It can be appropriate as a risk measure only when financial returns are normally distributed, but this assumption is usually not supported by the data!

Time series

Much of the data from financial markets are time series, that is, sequences of data sampled over time

Most analysis of financial time series is done in the **time domain**, and we will restrict our attention to this.

Data visualization: a *time series plot* is the graph displaying time series observations in chronological order.

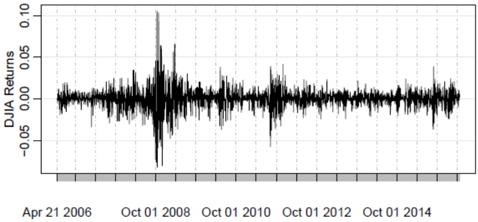

Figure 1: Daily returns of the Dow Jones Industrial Average (DJIA) from 2006 to 2016.

Time series (cont)

We concentrate on various types of statistical models widely used in econometrics, business forecasting, and many scientific applications.

The best-known examples of time series, which behave like *random walks*, are share prices on successive days. A model, which often gives a good approximation to such data, is

share price on day t = share price on day $(t-1)$ + random error

Simple linear models may be appropriate for many financial series, but the world is usually more complex!

Risk management

Econometrica, Vol. 50, No. 4 (July, 1982)

AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTICITY WITH ESTIMATES OF THE VARIANCE OF UNITED KINGDOM INFLATION[1]

BY ROBERT F. ENGLE

Traditional econometric models assume a constant one-period forecast variance. To generalize this implausible assumption, a new class of stochastic processes called autoregressive conditional heteroscedastic (ARCH) processes are introduced in this paper. These are mean zero, serially uncorrelated processes with nonconstant variances conditional on the past, but constant unconditional variances. For such processes, the recent past gives information about the one-period forecast variance.

A regression model is then introduced with disturbances following an ARCH process. Maximum likelihood estimators are described and a simple scoring iteration formulated. Ordinary least squares maintains its optimality properties in this set-up, but maximum likelihood is more efficient. The relative efficiency is calculated and can be infinite. To test whether the disturbances follow an ARCH process, the Lagrange multiplier procedure is employed. The test is based simply on the autocorrelation of the squared OLS residuals.

This model is used to estimate the means and variances of inflation in the U.K. The ARCH effect is found to be significant and the estimated variances increase substantially during the chaotic seventies.

1. INTRODUCTION

IF A RANDOM VARIABLE y_t is drawn from the conditional density function $f(y_t \mid y_{t-1})$, the forecast of today's value based upon the past information, under standard assumptions, is simply $E(y_t \mid y_{t-1})$, which depends upon the value of the conditioning variable y_{t-1}. The variance of this one-period forecast is given by $V(y_t \mid y_{t-1})$. Such an expression recognizes that the conditional forecast variance depends upon past information and may therefore be a random variable. For conventional econometric models, however, the conditional variance does not depend upon y_{t-1}. This paper will propose a class of models where the variance does depend upon the past and will argue for their usefulness in economics. Estimation methods, tests for the presence of such models, and an empirical example will be presented.

Consider initially the first-order autoregression

$$y_t = \gamma y_{t-1} + \epsilon_t$$

where ϵ is white noise with $V(\epsilon) = \sigma^2$. The conditional mean of y_t is γy_{t-1} while the unconditional mean is zero. Clearly, the vast improvement in forecasts due to time-series models stems from the use of the conditional mean. The conditional

[1] This paper was written while the author was visiting the London School of Economics. He benefited greatly from many stimulating conversations with David Hendry and helpful suggestions by Denis Sargan and Andrew Harvey. Special thanks are due Frank Srba who carried out the computations. Further insightful comments are due to Clive Granger, Tom Rothenberg, Edmond Malinvaud, Jean-Francois Richard, Wayne Fuller, and two anonymous referees. The research was supported by NSF SOC 78-09476 and The International Centre for Economics and Related Disciplines. All errors remain the author's responsibility.

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Nobel Prize winner Engle (1982) has determined a recent revolution in finance developing new models (ARCH Models) extended by Bollerslev (1986), that successfully capture changes in volatility

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Definition

Let P_t be the **price** of an asset at time index t. Let us consider a holding period from time $t-1$ to time t, and assume that the asset pays no dividends.

The *revenue* or profit during the holding period is

$$P_t - P_{t-1}.$$

Returns on an asset are changes in price expressed as a fraction of the initial price. In the following

- we will use the term "return" when the holding period may be arbitrary
- the term "rate of return" will refer to the case of annual returns
- we use "total return" when we want to emphasize a return including dividend income

Net returns

Definition (Net returns)

Assuming no dividends, the simple or **net return** over the holding period from time $t-1$ to time t is

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Because $P_t \geq 0$, then $R_t \geq -1$, so the worst possible return is -1, that is, a $100\,\%$ loss.
The revenue from holding an asset is then

revenue = initial investment $\times$ net return

Gross returns

The *gross return* is then

$$\frac{P_t}{P_{t-1}} = 1 + R_t \tag{1}$$

Thus, at the end of the holding period, we have

$$P_t = P_{t-1}(1 + R_t)$$

Example Assume $P_t = 2$ and $P_{t+1} = 2.1$. Then

$$1 + R_{t+1} = \frac{2.1}{2} = 1.05$$

and $R_{t+1} = 0.05$ meaning a net return of $5\,\%$.

Remark If W_0 is the initial wealth at time $t = 0$, then $W_0(1 + R_T)$ is the wealth at the end of the holding period $[0, T]$.

Log returns

An alternative return measure is the *continously compounded return*.

Definition (Log return)

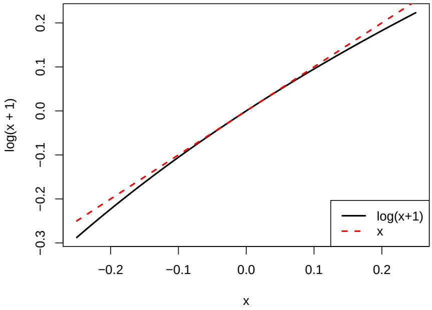
Log returns, also called *continously compounded returns*, are denoted by r_t and defined as

$$r_t = \log(1 + R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}$$

where $p_t = \log(P_t)$ is called the *log price* (we use $\log(x)$ to indicate the natural logarithm of x).

(Net) returns and log returns

- The difference between R_t and r_t is not large for daily returns: if x is small then $\log(1+x) \approx x$



(Net) returns and log returns

Example

- A $5\,\%$ return gives $\log(1+0.05)=0.0488$, that is a $4.88\,\%$ log return.
  A $-5\,\%$ return gives $\log(1-0.05)=-0.0513$, that is a $-5.13\,\%$.

- As the time between observations goes to zero, so does the difference between the two measures:

$$\lim_{\Delta t \to 0} r_t = R_t$$

  Returns are smaller in magnitude over shorter periods, hence returns and log returns can be similar for daily returns, less similar for yearly returns, and so on.

- The return and log return have the same sign:

$$R_t = (P_t/P_{t-1}) - 1 > 0 \;\rightarrow\; r_t = \log(1+R_t) > 0$$

Example: IBM returns

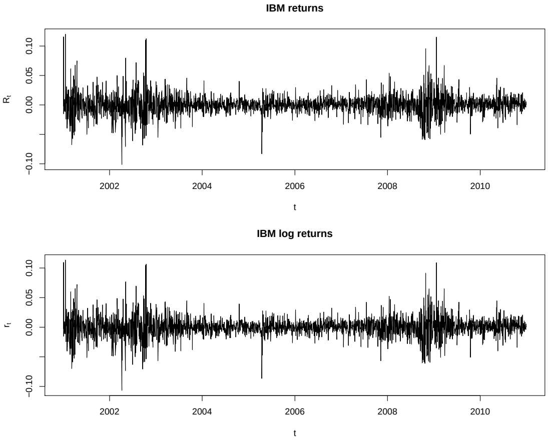


Figure 2: Time plots of daily simple and log returns of IBM stock from January 2, 2001 to December 31, 2010. There are 2515 observations. The correlation coefficient between the simple and log returns is 0.99.

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Multiperiod returns

We can use one-period returns (e.g. daily returns) to obtain monthly or annual returns.

Definition (Multiperiod gross return)

The *gross return over the most recent k periods* or k**-period return** is

$$1 + R_t(k) = \frac{P_t}{P_{t-k}} = \left(\frac{P_t}{P_{t-1}}\right)\left(\frac{P_{t-1}}{P_{t-2}}\right)\cdots\left(\frac{P_{t-k+1}}{P_{t-k}}\right)$$
$$= (1 + R_t)\cdots(1 + R_{t-k+1})$$

Thus, the k-period gross return is just the *product* of the k one-period gross returns involved.

Multiperiod returns: example

In this table, the prices at times $t-2$ to $t+1$ are given in the first row. The simple gross returns are in the second row, while gross returns over two or three periods are in the remaining rows.

Time	$t-2$	$t-1$	t	$t+1$
P	200	210	206	212
$1+R$		1.05	0.981	1.03
$1+R(2)$			???	1.01
$1+R(3)$				???

Multiperiod returns: example

$$1 + R_t(2) = \frac{P_t}{P_{t-2}} = \frac{206}{200} = 1.03$$
$$1 + R_{t+1}(3) = \frac{P_{t+1}}{P_{t-2}} = 1.05 \times 0.981 \times 1.03 = 1.06$$

Time	$t-2$	$t-1$	t	$t+1$
P	200	210	206	212
$1+R$		1.05	0.981	1.03
$1+R(2)$			1.03	1.01
$1+R(3)$				1.06

Remark Returns are *scale-free* but they are not unitless: their unit is time; they depend on the units of t (hour, day, month, etc.). In the example, if t is measured in years, then the net return at time $t-1$ is 5% per year.

Multiperiod returns

Example

Daily Closing Prices of Apple Stock from December 2 to 9, 2011

Date	12/02	12/05	12/06	12/07	12/08	12/09
Price($)	389.70	393.01	390.95	389.09	390.66	393.62

- The 1-day simple return of holding the stock from 12/08 to 12/09 is

$$R_t = (393.62/390.66) - 1 \approx 0.0076$$

- the weekly simple gross return of the stock is

$$1 + R_t(5) = 393.62/389.70 \approx 1.0101$$

so that the weekly net return is 1.01%.

Multiperiod log returns

Let us consider two one-period net returns of a stock share, R_t and R_{t-1}, over the two consecutive time periods. We multiply the corresponding gross returns to find the two-period gross return

$$1 + R_t(2) = (1 + R_t)(1 + R_{t-1})$$

The advantages of log returns become clear when considering multiperiod returns:

$$\log[(1 + R_t)(1 + R_{t-1})] = \log(1 + R_t) + \log(1 + R_{t-1}) = r_t + r_{t-1}$$

that is, log-returns are **additive**.

Multiperiod log returns

Definition (Multiperiod log return)

A **k-period log return** is given by the sum of the one-period log returns from time $t-k$ to time t):

$$\begin{aligned} r_t(k) &= \log\{1+R_t(k)\} \\ &= \log\{(1+R_t)\cdots(1+R_{t-k+1})\} \\ &= \log(1+R_t)+\cdots+\log(1+R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1} \end{aligned}$$

Log returns are in general more tractable, since it is much easier to derive the time series properties of sums than of products.

Example

Daily Closing Prices of Apple Stock from December 2 to 9, 2011

Date	12/02	12/05	12/06	12/07	12/08	12/09
Price($)	389.70	393.01	390.95	389.09	390.66	393.62

- The 1-day log return from December 8 to December 9 is

$$r_t = \log(393.62) - \log(390.66) \approx 0.75\%$$

- the weekly log return from 12/02 to 12/09 is

$$\begin{aligned} r_t(5) &= \log(P_{12/09}) - \log(P_{12/02}) \\ &= \log(393.62) - \log(389.70) = 0.01 \rightarrow 1\% \text{ weekly log return} \end{aligned}$$

or $r_t(5) = r_5 + r_4 + r_3 + r_2 + r_1$.

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Dividend payment

Many stocks pay dividends that must be accounted for when computing returns.

Adjustment for dividends

If a *dividend* (or interest) D_t is paid prior to time t, then the net return at time t becomes

$$(total\,return)\;R_t = \frac{P_t + D_t}{P_{t-1}} - 1 = \frac{D_t}{P_{t-1}} + \frac{P_t - P_{t-1}}{P_{t-1}}$$

The corresponding log return is

$$r_t = \log(P_t + D_t) - \log(P_{t-1})$$

Dividend payment

Multiple-period gross returns are products of one-period gross returns so that

$$
\begin{aligned}
1 + R_t(k) &= \left(\frac{P_t + D_t}{P_{t-1}}\right)\left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}}\right)\cdots\left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}}\right) \\
&= (1 + R_t)(1 + R_{t-1})\cdots(1 + R_{t-k+1})
\end{aligned}
$$

where, for any time s, $D_s = 0$ if there is no dividend between $s-1$ and s.

Similarly, a k-period log return is

$$
\begin{aligned}
r_t &= \log\{1 + R_t(k)\} = \log(1 + R_t) + \cdots + \log(1 + R_{t-k+1}) \\
&= \log\left(\frac{P_t + D_t}{P_{t-1}}\right) + \cdots + \log\left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}}\right)
\end{aligned}
$$

Dividend payment

Total return determines an investment's true growth over time.

Let's assume that all dividends are immediately reinvested and used to purchase additional shares of the same stock or security.
$\rightarrow$ we can compute the single-periods returns including dividends, and then compute the return over a longer horizon.

For example, if a stock pays dividends at the end of each quarter, with total returns R_{Q1}, R_{Q2}, R_{Q3}, R_{Q4}, then the **total annual return** is

$$R_{annual} = [(1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})(1 + R_{Q4})] - 1$$

Dividend payment: Example

Example

Stock prices and dividends for General Motors (GM) for 1999 and 2008.

date	price($)	dividend($)	date	price($)	dividend($)
31/12/98	71.56		31/12/07	24.89	
2/2/99	89.44	0.5	13/2/08	26.46	0.25
11/5/99	85.75	0.5	14/5/08	20.19	0.25
28/5/99	69.00	13.72	31/12/08	3.20	
10/8/99	60.81	0.5			
8/11/99	69.06	0.5			
31/12/99	72.69				

What were the annual returns on GM in this period?

Example / 2

We assume that the proceeds from the dividend payment were immediately reinvested in GM stock.

Step 1. One-period net returns must first be computed:

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1$$

For instance, the return over the period from 31/12/98 to 2/2/99 is

$$\frac{0.5 + 89.44}{71.56} - 1 = 0.2568 \text{ or } 25.68\%$$

and the return over the period from 8/11/99 to 31/12/99 is

$$\frac{72.69}{69.06} - 1 = 0.0526 \text{ or } 5.26\%$$

Example / 3

We obtain the following table

date	price	divid.	return	date	price	divid.	return
31/12/98	71.56			31/12/07	24.89		
2/2/99	89.44	0.5	0.2568	13/2/08	26.46	0.25	0.0731
11/5/99	85.75	0.5	-0.0357	14/5/08	20.19	0.25	-0.2275
28/5/99	69.00	13.72	-0.0353	31/12/08	3.20		-0.8415
10/8/99	60.81	0.5	-0.1114				
8/11/99	69.06	0.5	0.1439				
31/12/99	72.69		0.0526				

Step 2. Calculate the annual return as

$$R_{1999} = (1.2568)(0.9643)(0.9647)(0.8886)(1.1439)(1.0526) - 1 = 0.2509$$
$$R_{2008} = (1.0731)(0.7725)(0.1585) - 1 = -0.8686$$

1999: 25.09%; 2008: -86.86%.

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Portfolio returns

Consider returns on a set of assets.

A *portfolio* is simply a weighted average of the assets with weights that sum to one. The weights specify what fractions of the total investment are allocated to the assets.

For example, if a portfolio consists of

- 200 shares of Stock 1 selling at $88/share
- 150 shares of Stock 2 selling at $67/share

then the weights are

$$w_1 = \frac{(200)(88)}{(200)(88) + (150)(67)} = 0.637, \qquad w_2 = 1 - w_1$$

Portfolio returns / 2

Let p be a portfolio that places weight w_i on asset i, $i = 1, \dots N$.

- The (simple) net return of p at time t is the weighted sum of the returns of the individual assets

$$R_{t,p} = w_1 R_{t,1} + \cdots + w_N R_{t,N} = \sum_{i=1}^{N} w_i R_{t,i} \tag{2}$$

  where $R_{t,i}$ is the simple return of asset i.

- This does not work for the continuously compounded returns

$$r_{t,p} \neq \sum_{i=1}^{N} w_i r_{t,i} = \sum_{i=1}^{N} w_i \log(1 + R_{t,i})$$

  where $r_{t,i}$ is the log return of the portfolio at time t.

- If the $R_{t,i}$ are all small in magnitude, then we have

$$r_{t,p} \approx \sum_{i} w_i r_{t,i}$$

The effect of compounding

One euro invested for one year at a 5% rate of simple interest is worth $1 + 0.05 = €1.05$ at the end of one year.

If instead the 5% interest is compounded semi-annually, then the deposit after one year is

$$\left(1 + \frac{0.05}{2}\right)^2 = 1.050625,$$

given that the number of compounding periods is 2 and the stated interest rate is 5%. If the compounding is daily, then net value is

$$\left(1 + \frac{0.05}{365}\right)^{365} = 1.0512675.$$

after 1 year. If one compounded the interest every hour, then the worth after one year would be

$$\left(1 + \frac{0.05}{(24)(365)}\right)^{(24)(365)} = 1.0512709.$$

Continuous compounding

Discrete compounding means compounding at any fixed time interval (annual, semiannual, quarterly,ecc).

As m increases the net value approaches €1.051271, which is obtained by $\exp(0.05)$ and referred to as the *result of continuous compounding*:

$$\lim_{m\to\infty}\left(1+\frac{0.05}{m}\right)^m = e^{0.05} = 1.051271$$

If we invest C at a continuously compounded rate r for n years, the wealth after n years is

$$W = C\,e^{r\cdot n}$$

The return over the n years is $e^{r\cdot n}$ so that the log return is $\log(e^{r\cdot n}) = r\cdot n$.

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Average return

Example: Consider a holding period consisting of two consecutive years:
First year: return is $+10\%$;
Second year: return is -10%.

What was the "average" return?

The *arithmetic mean* is 0, however over the two years, the gross return was

$$1 + R(2) = (1 + 0.10)(1 - 0.10) = 0.99$$

i.e., we have lost money, since the net return was $R = -0.01$ that is -1%.

Average return / 2

Example (cont) The arithmetic average, $\bar{R}$, can only be used to approximate the annualized return.
However, to consider an *average over time*, we should deal with a sort of geometric average over k years. In the example, $k = 2$ and then

$$(1 + 0.10)(1 - 0.10) = (1 + \bar{R}_g)^2$$

$$\rightarrow \bar{R}_g = [(1 + 0.10)(1 - 0.10)]^{1/2} - 1 = -0.005013$$

thus the *annual growth rate* was $\bar{R}_g = -0.5013\%$ (average annual rate of return).

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Definition

If the asset was held for k years, then the **annualized (average) return** is defined as geometric mean of the k one-period simple gross returns involved:

$$\bar{R}_g(k) = [(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})]^{1/k} - 1$$

(Recall that the geometric mean of n numbers is the nth root of their product.)

Arithmetic vs geometric mean: example

Year End	S&P 500 Index	Dividends Paid *	S&P 500 Return (%)	Microsoft Return (%)
2001	1148.08			
2002	879.82	14.53	-22.1	-22.0
2003	1111.92	20.80	28.7	6.8
2004	1211.92	20.98	10.9	8.9
2005	1248.29	23.15	4.9	-0.9
2006	1418.30	27.16	15.8	15.8
2007	1468.36	27.86	5.5	20.8
2008	903.25	21.85	-37.0	-44.4
2009	1115.10	27.19	26.5	60.5
2010	1257.64	25.44	15.1	-6.5
2011	1257.60	26.59	2.1	-4.5

Table 1: Returns for the S&P 500 and Microsoft (2002-2011). Standard & Poor's 500 index represents a value-weighted portfolio of 500 of the largest U.S. stocks.

*Total dividends paid by the stocks in the portfolio, based on the n. of shares of each stock in the index, adjusted until the end of the year, assuming they were all reinvested.

Arithmetic vs geometric mean: example/ 2

Using the data in Table1, the arithmetic average return for the S&P 500 for the years 2002–2011 is

$$\bar{R}^{S\&P500} = \frac{1}{10}(-0.221 + 0.287 + \cdots + 0.021) = 0.0504 \to 5.04\%$$

while the annualized (average) return is

$$\bar{R}_g^{S\&P500} = [(1-0.221)(1+0.287)\cdots(1+0.021)]^{1/10} - 1 = 0.0293$$

That is, investing in the S&P 500 from 2002 to 2011 was equivalent to earning 2.93% per year over that time period.

Arithmetic vs geometric mean: example

Analogously, one can find that the sample mean for Microsoft returns is

$$\bar{R}^M = \frac{1}{10}(-0.220 + 0.068 + \cdots + 0.045) = 0.0435 \rightarrow 4.35\%$$

and the annualized (average) return is

$$\bar{R}_g^M = [(1 - 0.220)(1 + 0.068) \cdots (1 + 0.045)]^{1/10} - 1 = 0.0093$$

so that the geometric mean growth rate is 0.93%.

Note that in both cases the the annualized return is smaller than the arithmetic mean return $\bar{R}$.

To sum up

The average return is the simple mathematical average of a series of returns generated over a specified period of time.

The average return can help measure the past performance of a security or portfolio.

The average return is not the same as an annualized return, as it ignores compounding. The latter gives the fixed return on the asset or portfolio that would have been required to match the actual performance.

The geometric average is always lower than the average return.

The annualized return is sometimes approximated by the arithmetic average when returns are small.

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Main definitions

One-Period Simple Return is $R_t = P_t/P_{t-1} - 1$.

The relationships between simple return R_t and continuously compounded or **log returns** r_t are

$$r_t = \log(1 + R_t) \qquad R_t = e^{r_t} - 1$$

Temporal aggregation of the returns produces ***Multiperiod Returns***

$$1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) = P_t/P_{t-k}$$

$$r_t(k) = \log(1 + R_t(k)) = r_t + r_{t-1} + \cdots + r_{t-k+1}$$

Main definitions (cont)

If an asset pays dividends periodically

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1$$

$$r_t = \log\left(P_t + D_t\right) - \log\left(P_{t-1}\right)$$

If the continuously compounded interest rate is r per annum, X_0 is the initial capital, and t is the number of years, then the relationship between present and future values of an asset is

$$X_t = X_0 e^{rt}, \qquad X_0 = X_t e^{-rt}$$