Discrete-time signals in the time domain

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- In digital signal processing, signals are sequences of numbers (called samples) function of an independent variable (called time), which is an integer in the interval $[-\infty, +\infty]$.
- In the following, we will denote the generic sequence as $\{x(n)\}$, where $x(n)$ represents the sample of the sequence at time n. [Later, when there will be no ambiguity, we will directly represent our sequence as $x(n)$.]

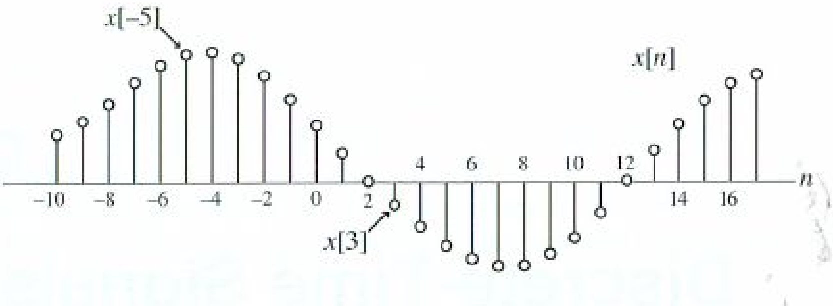


Figure 2.1: Graphical representation of a discrete-time sequence $\{x[n]\}$.

- We will represent or define a sequence through the use of
  - a mathematical law:

$$\{x(n)\} = e^{|n|}$$

$$\{x(n)\} = \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \end{cases}$$

  - a sequence of numbers between { }:

$$\{x(n)\} = \quad \{\ldots, 0.95, -0.2, \underset{\uparrow}{2.1}, 1.2, -3.2, \ldots\}$$

    where the arrow denotes the element at $n = 0$, with elements to the left of the arrow corresponding to $n < 0$, and elements to the right corresponding to $n > 0$.

- The sequence $\{x(n)\}$ is often generated by sampling a continuous-time signal $x_a(t)$ (an analog signal) at uniformly spaced intervals:

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT).$$

- The interval time T that separates two consecutive samples is referred to as the **sampling period**. Its reciprocal is known as the **sampling frequency** $F_T = \frac{1}{T}$.
- In either scenario, $x(n)$ is referred to as the n-**th sample** of the sequence.

Length of a discrete-time sequence

- Discrete-time signals, i.e., sequences, possess either **finite** or **infinite length**.
- A **finite-length sequence** is defined only within the interval

$$N_1 \leq n \leq N_2$$

where $-\infty < N_1 \leq N_2 < +\infty$, and the sequence has **length** (or duration):

$$N = N_2 - N_1 + 1.$$

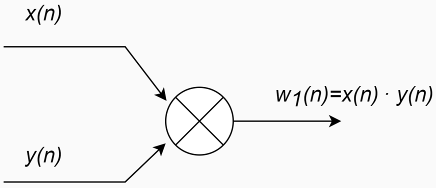
- A sequence of length N comprises only N samples. It can be transformed into an infinite-length sequence by assigning 0 values outside the $[N_1, N_2]$ interval. This operation is known as **zero-padding**.

- There are three types of infinite-length sequences:
  - **Causal** sequences, when $x(n) = 0 \ \forall n < 0$. (The sequence has non-zero element only for $n \geq 0$).
  - **Anti-causal** sequences, when $x(n) = 0 \ \forall n > 0$.
  - **Two-sided** sequences, with non-zero elements both for $n < 0$ and $n \geq 0$.
- In the following, we will frequently examine finite-length causal sequences, which are defined solely in the interval $[0, N-1]$.

- Given two sequences $x(n)$ and $y(n)$ we define the following operations:
- The **product** of two sequences:

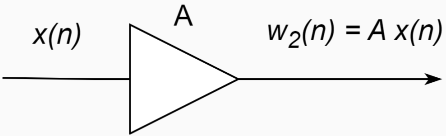
$$w_1(n) = x(n) \cdot y(n),$$

This operation is also called *modulation*.



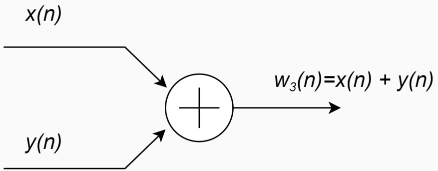
- The **scalar multiplication** of one sequence for a constant A:

$$w_2(n) = Ax(n)$$



- The **addition** of two sequences:

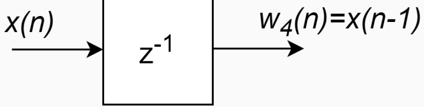
$$w_3(n) = x(n) + y(n),$$



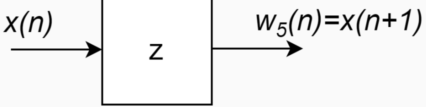
- The **time-shift** :

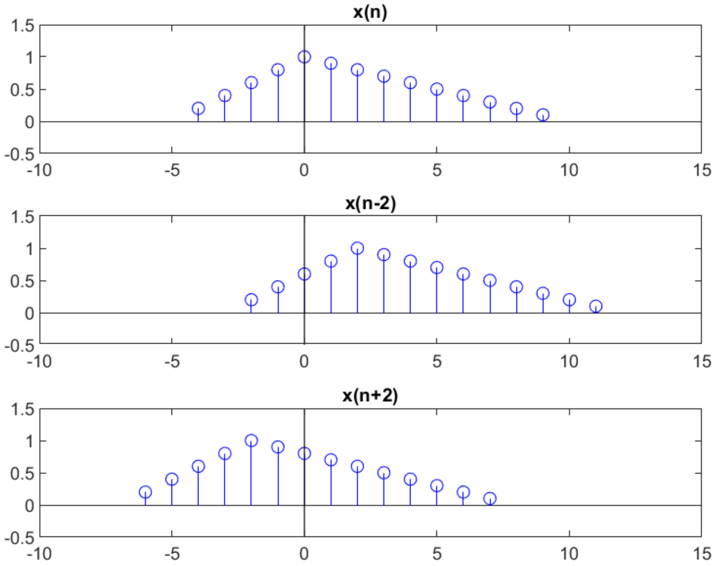
$$w_4(n) = x(n - N).$$

If $N > 0$, we say that the sequence has been delayed by N samples.
If $N < 0$, we say that the sequence has been time advanced of $|N|$ samples.
Unit delay:



Unit advance:




x(n)
1.5
1
0.5
0
-0.5
-10
-5
0
5
10
15
x(n-2)
x(n+2)

$$\{x(n)\} = \quad \{\ldots, -3, -2, -1, \underset{\uparrow}{0}, 2, 4, \ldots\}$$

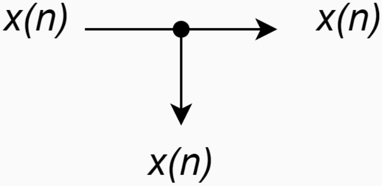
$$\{x(n-1)\} = \quad \{\ldots, -3, \underset{\uparrow}{-2}, -1, 0, 2, 4, \ldots\}$$

$$\{x(n)\} = \quad \{\ldots, -3, -2, -1, 0, 2, \underset{\uparrow}{4}, \ldots\}$$

- The **time-reversal** or **folding** operation:

$$w_5(n) = x(-n)$$

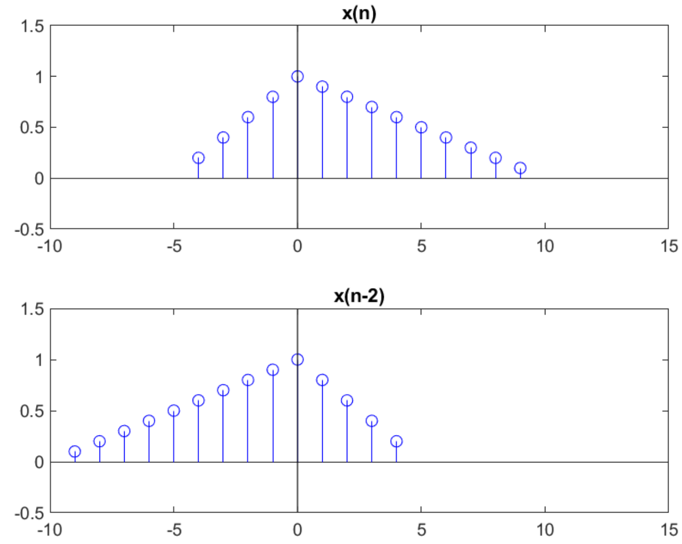
- The **pick-off node**,



- An example of time-reversal:

$$\{x(n)\} = \quad \{\ldots, -3, -2, -1, \underset{\uparrow}{0}, 2, 4, \ldots\}$$

$$\{x(-n)\} = \quad \{\ldots, 4, 2, \underset{\uparrow}{0}, -1, -2, -3, \ldots\}$$


x(n)
1.5
1
0.5
0
-0.5
-10
-5
0
5
10
15
x(n-2)
1.5
1
0.5
0
-0.5
-10
-5
0
5
10
15

- A real signal is called **symmetric** or **even** if:

$$x(n) = x(-n)$$

- A real signal is called **anti-symmetric** or **odd** if:

$$x(n) = -x(-n)$$

- A real signal can be decomposed in the addition of an even and an odd signal:

$$x(n) = x_{\mathrm{ev}}(n) + x_{\mathrm{od}}(n)$$

$$x_{\mathrm{ev}}(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

$$x_{\mathrm{od}}(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

- A complex signal is called **conjugate-symmetric** if

$$x(n) = x^*(-n),$$

  which means that the real part of $x(n)$ is even and the imaginary part is odd.

- A complex signal is called **conjugate-antisymmetric** if

$$x(n) = -x^*(-n),$$

  which means that the real part of $x(n)$ is odd and the imaginary part is even.

- A complex signal can be decomposed in the addition of a conjugate-symmetric and a conjugate-antisymmetric signal:

$$x(n) = x_{\mathrm{cs}}(n) + x_{\mathrm{ca}}(n)$$

$$x_{\mathrm{cs}}(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$$

$$x_{\mathrm{ca}}(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$$

- A sequence such that $x_p(n) = x_p(n + kN)$ for all n, with $N \in \mathbb{N}$, $N > 0$, and $k \in \mathbb{Z}$, is called a **periodic sequence** with period N.
- The smallest $N > 0$ for which $x_p(n) = x_p(n + kN)$ is called **fundamental period** of the sequence.
- A sequence that is not periodic is called **aperiodic**.

- The **energy** E_x of a signal $x(n)$ is:

$$E_x = \sum_{n=-\infty}^{+\infty} \left|x(n)\right|^2.$$

- A finite length sequence has always finite energy.
- An infinite-length sequence can have finite or infinite energy.
- For example, the sequence

$$x_1(n) = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

has energy $E_x = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$.

- The sequence

$$x_2(n) = \begin{cases} \frac{1}{\sqrt{n}} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

has energy $E_x = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right) = +\infty$.

- The **average power** of an aperiodic signal is:

$$P_x = \lim_{K \to +\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x(n)|^2.$$

- The average power can be related to the energy by defining the energy in the interval $[-K, K]$:

$$E_{x,K} = \sum_{n=-K}^{K} |x(n)|^2,$$

$$P_x = \lim_{K \to +\infty} \frac{E_{x,K}}{2K+1}.$$

  From this relation we see that a signal with fixed energy has zero average power.
- The average power of an infinite-length sequence can be finite or infinite.
- For example, the signal $x(n) = a$ for all n has average power $P_x = a^2$.
- The average power of a periodic signal $x_p(n)$ of period N is

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x_p(n)|^2$$

- A signal with finite energy is called an **energy signal**.
- A signal with finite average power is called a **power signal**.

- A sequence is called **bounded** if there exists a constant B_x such that

$$|x(n)| \leq B_x \qquad \forall n.$$

- A sequence is called **absolutely summable** if

$$\sum_{n=-\infty}^{+\infty} |x(n)| < +\infty.$$

- A sequence is called **square-summable** if

$$\sum_{n=-\infty}^{+\infty} |x(n)|^2 < +\infty.$$

An example of a sequence that is square-summable but not absolutely summable is the **sinc** sequence:

$$x(n) = \begin{cases} \dfrac{\sin[\omega_c n]}{\frac{\omega_c}{\pi}\pi n} & n \neq 0 \\ \dfrac{\omega_c}{\pi} & n = 0 \end{cases}$$

- The **unit sample** sequence $\delta(n)$, also called **discrete-time impulse** or **unit impulse**, is defined by

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} .$$

Thus,

$$\delta(n-k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} .$$

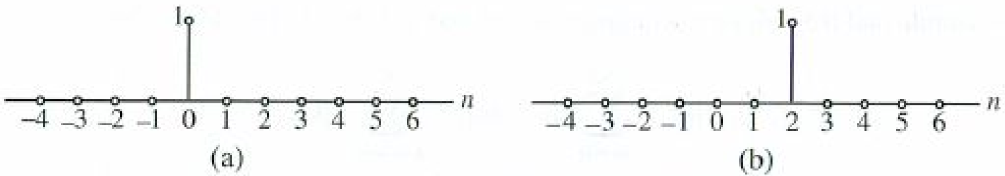


Figure 2.20: (a) The unit sample sequence $\{\delta[n]\}$ and (b) the shifted unit sample sequence $\{\delta[n-2]\}$.

- **Any sequence** can be represented as the **sum of infinite unit impulses**, each shifted in time and appropriately weighted.
- For example,

$$\{\dots, 0.95, -0.2, \underset{\uparrow}{1.2}, -3.2, 1.4 \dots\} \quad = \dots 0.95 \cdot \{\delta(n+2)\} - 0.2 \cdot \{\delta(n+1)\} + 1.2 \cdot \{\delta(n)\} \\ -3.2 \cdot \{\delta(n-1)\} + 1.4\{\delta(n-2)\} + \dots$$

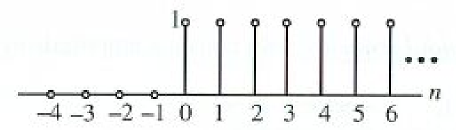
- As a general rule, we have

$$\{a(n)\} = \sum_{m=-\infty}^{+\infty} a(m)\{\delta(n-m)\}$$

where $\delta(n-m)$ are the time-shifted unit impulses and $a(m)$ are the corresponding weights.

- The **unit step** sequence is defined by

$$\mu(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} .$$



- Note that:

$$\mu(n) = \sum_{m=0}^{+\infty} \delta(n-m)$$

$$\delta(n) = \mu(n) - \mu(n-1)$$

- The real **sinusoidal** sequence is defined by

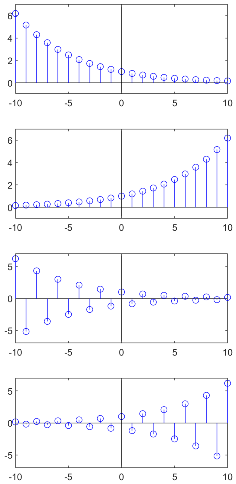
$$x(n) = A\cos\left(\omega_0 n + \phi\right)$$
$$= A\cos\left(2\pi f_0 n + \phi\right)$$

$\omega_0 = 2\pi f_0$ is called *normalized angular frequency* or simply angular frequency.
f_0 is called *normalized frequency* or simply frequency.
ϕ is called *initial phase*.
A is the *amplitude* of the sinusoidal signal.

- The real *exponential* sequence is defined by

$$x(n) = Aa^n \qquad\qquad A, a \in \mathbb{R}$$

- If $0 < a < 1$, it is an exponentially decreasing sequence.
- If $a > 1$, it is an exponentially increasing sequence.
- If $-1 < a < 0$, it is an alternated exponentially decreasing sequence.
- If $a < -1$, it is an alternated exponentially increasing sequence.



- The **complex exponential sequence** is defined by

$$x(n) = Aa^n \qquad\qquad a = r \cdot e^{j\omega_0} = e^{\sigma_0 + j\omega_0}$$

with $A, a \in \mathbb{C}$.

- Since $A = \left|A\right| \cdot e^{j\phi}$, we can also write:

$$x(n) = \left|A\right| \cdot e^{\sigma_0 n} \cdot e^{j\left(\omega_0 n + \phi\right)}$$

$$= \left|A\right| e^{\sigma_0 n} \left[\cos\left(\omega_0 n + \phi\right) + j \sin\left(\omega_0 n + \phi\right)\right].$$

- The real and imaginary parts of the complex exponential sequence are sinusoids with amplitude that increase or decrease exponentially.
  A notable special case of the complex exponential sequence is the **generalized sinusoidal sequence**

$$x(n) = e^{j\left(\omega_0 n + \phi\right)} = \cos\left(\omega_0 n + \phi\right) + j \sin\left(\omega_0 n + \phi\right).$$

- **Property:** A sinusoidal (or generalized sinusoidal) sequence is periodic if and only if the normalized frequency f_0 is a rational number, i.e., $f_0 \in \mathbb{Q}$.
- **Proof:** A sequence $x(n)$ is periodic if and only if $x(n) = x(n+N)$ for some $N > 0$ and for all n.
  Let us impose this equality. In our case:

$$A \cdot \cos\left[2\pi f_0 n + \phi\right] = A \cdot \cos\left[2\pi f_0 (n+N) + \phi\right]$$

  Thus, the arguments can differ only by a multiple of 2π:

$$2\pi f_0 (n+N) + \phi = 2\pi f_0 n + \phi + 2\pi k$$

  with $k \in \mathbb{Z}$. By simplifying the last identity we arrive to:

$$f_0 N = k \qquad \Longrightarrow \qquad f_0 = \frac{k}{N} \in \mathbb{Q}.$$

Q.E.D.

- **Property:** Two sinusoidal sequences with the same amplitude and phase, whose angular frequencies differ for a multiple of 2π, are equal.
- **Proof:** Let us consider

$$x_1(n) = A\cos(\omega_1 n + \phi)$$
$$x_2(n) = A\cos(\omega_2 n + \phi)$$

with $\omega_2 = \omega_1 + k \cdot 2\pi$ and $k \in \mathbb{Z}$. Thus

$$x_2(n) = A\cos(\omega_1 n + k2\pi n + \phi) =$$
$$= A\cos(\omega_1 n + \phi) = x_1(n)$$

Q.E.D.

- In the sinusoidal sequence

$$x(n) = A\cos(\omega_0 n + \phi),$$

the frequency of oscillations in $x(n)$ increases with ω_0 varying from 0 to π.
The frequency of oscillations in $x(n)$ is maximum for $\omega_0 = \pi$ (since $x(n) = \ldots, -A, +A, -A, +A, \ldots$).
The frequency of oscillations in $x(n)$ decreases with ω_0 varying from π to 2π.
Eventually, this behavior repeats (with period 2π) in the intervals $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, etc..

- Let us consider the sinusoidal function

$$x_a(t) = A\cos(\Omega t + \phi) = A\cos(2\pi f t + \phi).$$

Let us sample it with a sampling frequency $F_s = \dfrac{1}{T}$:

$$x(n) = x_a(t)\Big|_{t=nT} = A\cos(2\pi f n T + \phi)$$

$$= A\cos(2\pi \frac{f}{F_s} n + \phi)$$

This is a sinusoidal sequence with normalized frequency $f_0 = \dfrac{f}{F_s}$. It explains the origin of the term **'normalized frequency'**: it is normalized with respect to the sampling frequency.
If $0 < 2\pi \dfrac{f}{F_s} < \pi$, i.e., $F_s > 2f$, the sinusoidal sequence follows the behavior of the sinusoidal function: as f increases, f_0 increases, and the frequency of oscillation increases.
On the contrary, if $2\pi \dfrac{f}{F_s} > \pi$, the sinusoidal sequence is unable to accurately follow the behavior of the sinusoidal function.

Bibliography

- For more information study:

S. K. Mitra, "Digital Signal Processing: a computer based approach," 4th edition, McGraw-Hill, 2011
Chapter 2.1, pp. 41-45
Chapter 2.2, pp. 46-49
Chapter 2.3.3, pp. 58-62
Chapter 2.4, pp. 62-68

Unless otherwise specified, all images have either been originally produced or have been taken from S. K. Mitra, "Digital Signal Processing: a computer based approach."