02 Discrete-time signals in the time domain

02.01 Time domain representation

In digital signal processing, signals are sequences of numbers (called samples) function of an independent variable (called time), which is an integer in the interval $[-\infty, +\infty]$.

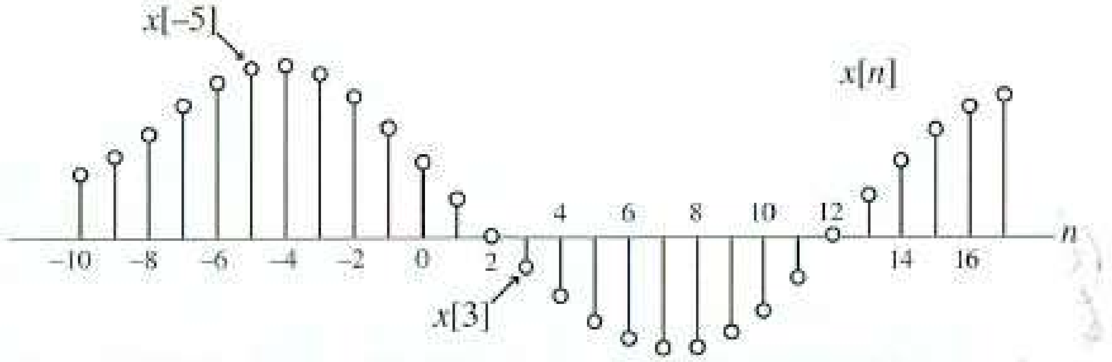


Figure 2.1: Graphical representation of a discrete-time sequence $\{x[n]\}$.

Figure 02.01: (From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

In the following, we will denote the generic sequence as $\{x(n)\}$, where $x(n)$ represents the sample of the sequence at time n [1]. Note that the sequence $x(n)$ is defined exclusively for integer values of n and *does not exist* for non-integer values of n.
We will represent or define a sequence through the use of

- a mathematical law:

$$\{x(n)\} = e^{|n|}$$

$$\{x(n)\} = \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \end{cases}$$

- a sequence of numbers between { }:

$$\{x(n)\} = \quad \{\ldots, 0.95, -0.2, \underset{\uparrow}{2.1}, 1.2, -3.2, \ldots\}$$

where the arrow denotes the element at $n = 0$, with elements to the left of the arrow corresponding to $n < 0$, and elements to the right corresponding to $n > 0$.

[1] Later, when there will be no ambiguity, we will directly represent our sequence as $x(n)$.

The sequence $\{x(n)\}$ is commonly generated by sampling a continuous-time signal $x_a(t)$ (an analog signal) at uniformly spaced intervals:

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT).$$

The interval time T that separates two consecutive samples is referred to as the *sampling period*. Its reciprocal is known as the *sampling frequency*:

$$F_T = \frac{1}{T}.$$

The sampling period is measured in seconds (s) and has the physical dimension of time. The sampling frequency is measured in cycles per second, denoted as Hertz (Hz).
In either scenario, whether the sequence $\{x(n)\}$ is derived from sampling a continuous-time signal or generated through alternative methods, $x(n)$ is referred to as the n-th sample of the sequence.
If the sequence takes only real values, it is referred to as a *real* sequence.
If the sequence assumes complex values, it is called a *complex* sequence.

Length of a discrete-time sequence
Discrete-time signals, i.e., sequences, possess either finite or infinite length.
A finite-length sequence is defined only within the interval

$$N_1 \leq n \leq N_2$$

where $-\infty < N_1 \leq N_2 < +\infty$, and the sequence has length (or duration):

$$N = N_2 - N_1 + 1.$$

A sequence of length N comprises only N samples. It can be transformed into an infinite-length sequence by assigning 0 values outside the $[N_1, N_2]$ interval. This operation is known as zero-padding.
There are three types of infinite-length sequences:

- *Causal* sequences, when $x(n) = 0\ \forall n < 0$. (The sequence has non-zero element only for $n \geq 0$).
- *Anti-causal* sequences, when $x(n) = 0\ \forall n > 0$.
- *Two-sided* sequences, with non-zero elements both for $n < 0$ and $n \geq 0$.

In the following, we will frequently examine finite-length causal sequences, which are defined solely in the interval $[0, N-1]$.

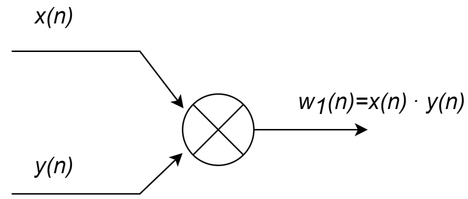
02.02 Operations on sequences

Given two sequences $x(n)$ and $y(n)$ we define the following operations:

- The *product* of two sequences:

$$w_1(n) = x(n) \cdot y(n),$$

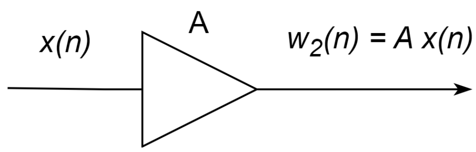
whose samples are given by the product of the corresponding samples of $x(n)$ and $y(n)$. This operation is also called *modulation*.



- The *scalar multiplication* of one sequence for a constant A:

$$w_2(n) = Ax(n)$$

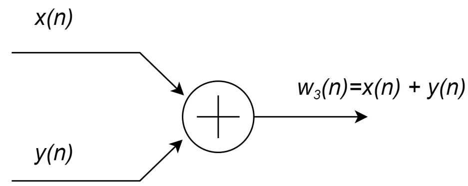
where each sample of $x(n)$ is multiplied for the scalar constant A.



- The *addition* of two sequences:

$$w_3(n) = x(n) + y(n),$$

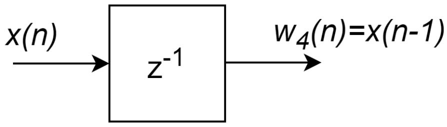
whose samples are given by the addition of the corresponding samples of $x(n)$ and $y(n)$.



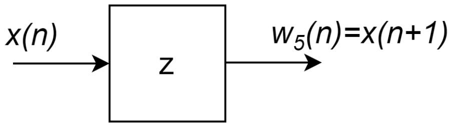
- The *time-shift* :

$$w_4(n) = x(n - N).$$

If $N > 0$, we say that the sequence has been delayed by N samples.
If $N < 0$, we say that the sequence has been time advanced of $|N|$ samples.
In systems processing real-time natural signals, it is possible to delay the signal, but time advancement is not feasible. Conversely, recorded signals can be time-shifted in both directions.
Unit delay:



Unit advance:



For examples of delayed signals, see Figure 02.02.

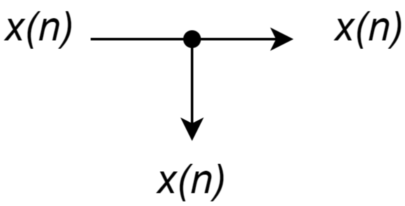
- The *time-reversal* or *folding* operation:

$$w_5(n) = x(-n)$$

which is the time-reversed version of the sequence.

For folding examples, see Figure 02.03.

In the block schemes we will study, there is another operation known as the *pick-off node*, which, when applied to a sequence, generates two other identical sequences.



Most digital signal processing systems are implemented using just these six operations on sequences

02.03 Classification of sequences

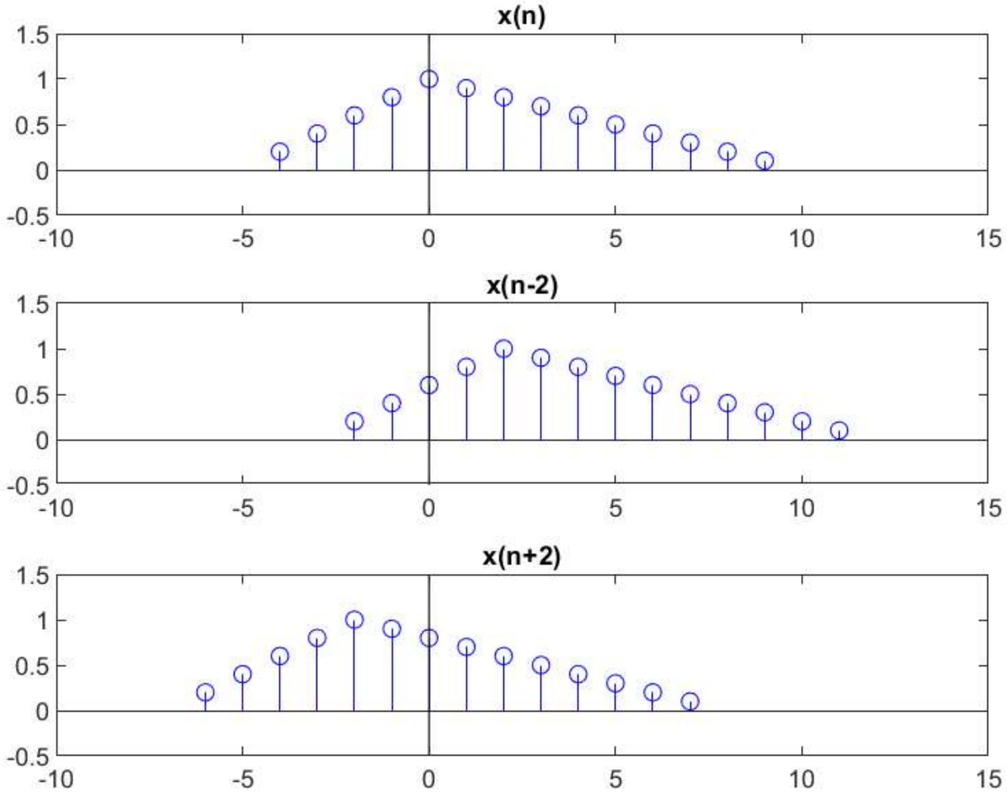
Sequences can be classified according to their symmetry properties:

- A real signal is called *symmetric* or *even* if:

$$x(n) = x(-n)$$

- A real signal is called *anti-symmetric* or *odd* if:

$$x(n) = -x(-n)$$



$$\{x(n)\} = \quad \{\dots, -3, -2, -1, \underset{\uparrow}{0}, 2, 4, \dots\}$$

$$\{x(n-1)\} = \quad \{\dots, -3, \underset{\uparrow}{-2}, -1, 0, 2, 4, \dots\}$$

$$\{x(n)\} = \quad \{\dots, -3, -2, -1, 0, 2, \underset{\uparrow}{4}, \dots\}$$

Figure 02.02: Delayed or time advanced sequences

- A real signal can be decomposed in the addition of an even and an odd signal:

$$x(n) = x_{\text{ev}}(n) + x_{\text{od}}(n)$$

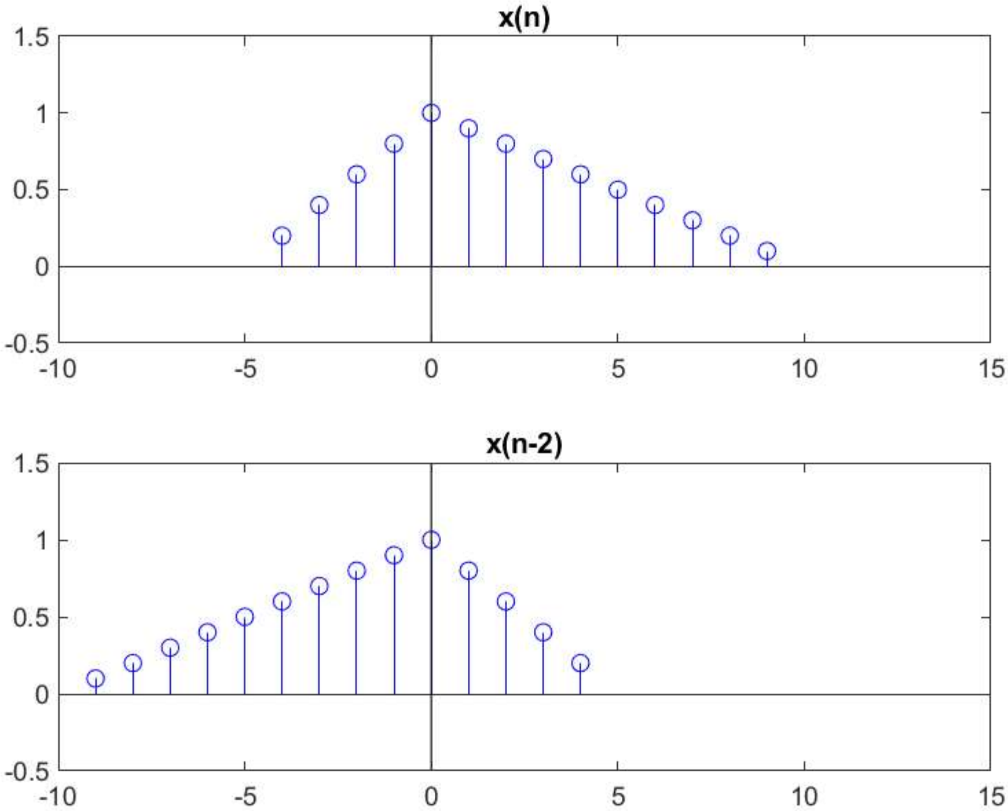
$$x_{\text{ev}}(n) = \frac{1}{2}\big[x(n) + x(-n)\big]$$

$$x_{\text{od}}(n) = \frac{1}{2}\big[x(n) - x(-n)\big]$$

- A complex signal is called *conjugate-symmetric* if

$$x(n) = x^*(-n),$$

which means that the real part of $x(n)$ is even and the imaginary part is odd.



$$\{x(n)\} = \{\ldots, -3, -2, -1, 0, 2, 4, \ldots\}$$
$$\uparrow$$

$$\{x(-n)\} = \{\ldots, 4, 2, 0, -1, -2, -3, \ldots\}$$
$$\uparrow$$

Figure 02.03: Time reversed sequences

- A complex signal is called *conjugate-antisymmetric* if

$$x(n) = -x^*(-n),$$

which means that the real part of $x(n)$ is odd and the imaginary part is even.

- A complex signal can be decomposed in the addition of a conjugate-symmetric and a conjugate-antisymmetric signal:

$$x(n) = x_{\mathrm{cs}}(n) + x_{\mathrm{ca}}(n)$$
$$x_{\mathrm{cs}}(n) = \frac{1}{2}\big[x(n) + x^*(-n)\big]$$
$$x_{\mathrm{ca}}(n) = \frac{1}{2}\big[x(n) - x^*(-n)\big]$$

Sequences can also be classified according to their periodicity or aperiodicity.

- A sequence such that $x_p(n) = x_p(n + kN)$ for all n, with $N \in \mathbb{N}$, $N > 0$, and $k \in \mathbb{Z}$, is called a *periodic sequence* with period N.
  The smallest $N > 0$ for which $x_p(n) = x_p(n + kN)$ is called *fundamental period* of the sequence.
  A sequence that is not periodic is called *aperiodic*.

Another classification comes from the energy and power content of the signals.

- The energy E_x of a signal $x(n)$ is:
  $$E_x = \sum_{n=-\infty}^{+\infty} \left|x(n)\right|^2.$$
  This definition applies both to real and to complex sequences.
  A finite length sequence has always finite energy. An infinite length sequence can have finite or infinite energy.
  For example, the sequence
  $$x_1(n) = \begin{cases} \dfrac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$
  has energy $E_x = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$. On the contrary, the sequence
  $$x_2(n) = \begin{cases} \dfrac{1}{\sqrt{n}} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$
  has energy $E_x = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right) = +\infty$.
- The *average power* of an aperiodic signal is:
  $$P_x = \lim_{K \to +\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} \left|x(n)\right|^2.$$
  The average power can be related to the energy by defining the energy in the interval $[-K, K]$:
  $$E_{x,K} = \sum_{n=-K}^{K} \left|x(n)\right|^2,$$
  $$P_x = \lim_{K \to +\infty} \frac{1}{2K+1} E_{x,K}.$$
  From this relation we see that a signal with fixed energy has zero average power.
  The average power of an infinite-length sequence can be finite or infinite.
  For example, the signal $x(n) = a$ for all n has average power $P_x = a^2$.
  The average power of a periodic signal $x_p(n)$ of period N is
  $$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left|x_p(n)\right|^2$$
  A signal with finite energy is called an *energy signal*.
  A signal with finite average power is called a *power signal*.

Other classifications that we will encounter in the upcoming lessons include the following:

- A sequence is called *bounded* if there exists a constant B_x such that

$$\left|x(n)\right| \leq B_x \qquad \forall n.$$

- A sequence is called *absolutely summable* if

$$\sum_{n=-\infty}^{+\infty} \left|x(n)\right| < +\infty.$$

- A sequence is called *square-summable* if

$$\sum_{n=-\infty}^{+\infty} \left|x(n)\right|^2 < +\infty.$$

  An example of a sequence that is square-summable but not absolutely summable is the *sinc* sequence:

$$x(n) = \begin{cases} \dfrac{\sin[\omega_c n]}{\pi n} & n \neq 0 \\ \dfrac{\omega_c}{\pi} & n = 0 \end{cases}$$

02.04 Basic sequences

- The *unit sample* sequence $\delta(n)$, also called *discrete-time impulse* or *unit impulse*, is defined by

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} .$$

  Thus,

$$\delta(n-k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} .$$

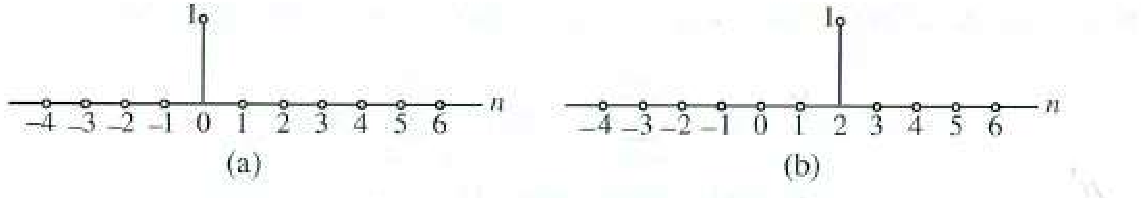


Figure 2.20: (a) The unit sample sequence $\{\delta[n]\}$ and (b) the shifted unit sample sequence $\{\delta[n-2]\}$.

(From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

Any sequence can be represented as the sum of infinite unit impulses, each shifted in time and appropriately weighted.
For example,

$$\begin{aligned}\{\ldots, 0.95, -0.2, 1.2, -3.2, 1.4\ldots\} &= \ldots 0.95\cdot\{\delta(n+2)\} - 0.2\cdot\{\delta(n+1)\} + 1.2\cdot\{\delta(n)\} \\ \uparrow \qquad\qquad & \quad -3.2\cdot\{\delta(n-1)\} + 1.4\{\delta(n-2)\} + \ldots\end{aligned}$$

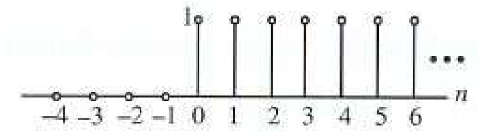
As a general rule, we have

$$\big\{a(n)\big\} = \sum_{m=-\infty}^{+\infty} a(m)\{\delta(n-m)\}$$

where $\delta(n-m)$ are the time-shifted unit impulses and $a(m)$ are the corresponding weights. In what follows we will neglect the curly brackets of the formula.

- The *unit step* sequence is defined by

$$\mu(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} .$$



(From S. K. Mitra, “Digital signal processing: a computer based approach”, McGraw Hill, 2011)

Note that:

$$\mu(n) = \sum_{m=0}^{+\infty} \delta(n-m)$$

$$\delta(n) = \mu(n) - \mu(n-1)$$

- The real *sinusoidal* sequence is defined by

$$x(n) = A\cos\big(\omega_0 n + \phi\big)$$

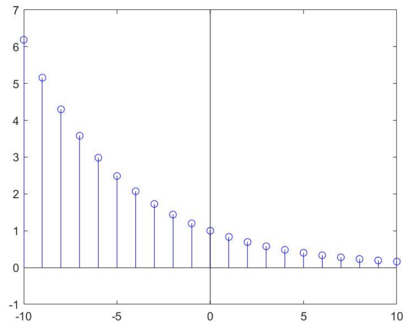
$$= A\cos\big(2\pi f_0 n + \phi\big)$$

$\omega_0 = 2\pi f_0$ is called *normalized angular frequency* or simply angular frequency.
f_0 is called *normalized frequency* or simply frequency.
ϕ is called *initial phase*.
A is the *amplitude* of the sinusoidal signal.

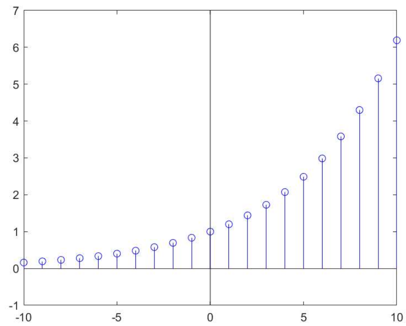
- The real *exponential* sequence is defined by

$$x(n) = Aa^n \qquad\qquad A, a \in \mathbb{R}$$

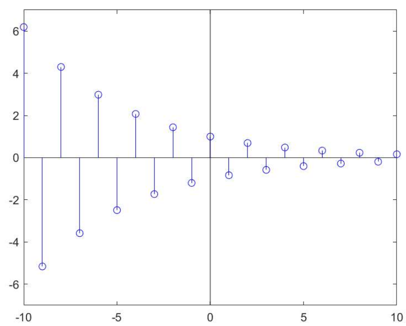
If $0 < a < 1$, it is an exponentially decreasing sequence.



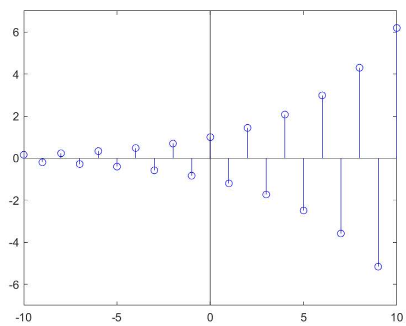
If $a > 1$, it is an exponentially increasing sequence.



If $-1 < a < 0$, it is an alternated exponentially decreasing sequence.



If $a < -1$, it is an alternated exponentially increasing sequence.



- The *complex exponential sequence* is defined by

$$x(n) = Aa^n \qquad a = r \cdot e^{j\omega_0} = e^{\sigma_0 + j\omega_0}$$

with $A, a \in \mathbb{C}$.
Since $A = \left|A\right| \cdot e^{j\phi}$, we can also write:

$$x(n) = \left|A\right| \cdot e^{\sigma_0 n} \cdot e^{j\left(\omega_0 n + \phi\right)}$$
$$= \left|A\right| e^{\sigma_0 n}\Big[\cos\left(\omega_0 n + \phi\right) + j\sin\left(\omega_0 n + \phi\right)\Big].$$

The real and imaginary parts of the complex exponential sequence are sinusoids with amplitude that increase or decrease exponentially.
A notable special case of the complex exponential sequence is the *generalized sinusoidal sequence* defined by

$$x(n) = e^{j\left(\omega_0 n + \phi\right)} = \cos\left(\omega_0 n + \phi\right) + j\sin\left(\omega_0 n + \phi\right).$$

- *Property:* A sinusoidal (or generalized sinusoidal) sequence is periodic if and only if the normalized frequency f_0 is a rational number, i.e., $f_0 \in \mathbb{Q}$.
  Proof: A sequence $x(n)$ is periodic if and only if $x(n) = x(n+N)$ for some $N > 0$ and for all n. Let us impose this equality. In our case:

$$A \cdot \cos\left[2\pi f_0 n + \phi\right] = A \cdot \cos\left[2\pi f_0 (n+N) + \phi\right]$$

Thus, the arguments can differ only by a multiple of 2π:

$$2\pi f_0 (n+N) + \phi = 2\pi f_0 n + \phi + 2\pi k$$

with $k \in \mathbb{Z}$. By simplifying the last identity we arrive to:

$$f_0 N = k \qquad \Longrightarrow \qquad f_0 = \frac{k}{N} \in \mathbb{Q}.$$

Q.E.D.

- *Property:* Two sinusoidal sequences with the same amplitude and phase, whose angular frequencies differ for a multiple of 2π, are equal.

  Proof: Let us consider

$$x_1(n) = A\cos\left(\omega_1 n + \phi\right)$$
$$x_2(n) = A\cos\left(\omega_2 n + \phi\right)$$

with $\omega_2 = \omega_1 + k \cdot 2\pi$ and $k \in \mathbb{Z}$. Thus

$$x_2(n) = A\cos\left(\omega_1 n + k2\pi n + \phi\right) =$$
$$= A\cos\left(\omega_1 n + \phi\right) = x_1(n)$$

Q.E.D.

- In the sinusoidal sequence

$$x(n) = A\cos\big(\omega_0 n + \phi\big),$$

the frequency of oscillations in $x(n)$ increases with ω_0 varying from 0 to π.
The frequency of oscillations in $x(n)$ is maximum for $\omega_0 = \pi$ (since $x(n) = \ldots, -A, +A, -A, +A, \ldots$).
The frequency of oscillations in $x(n)$ decreases with ω_0 varying from π to 2π.
Eventually, this behavior repeats (with period 2π) in the intervals $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, etc..

Let us consider the sinusoidal function

$$x_a(t) = A\cos(\Omega t + \phi) = A\cos(2\pi f t + \phi).$$

Let us sample it with a sampling frequency $F_s = \frac{1}{T}$:

$$x(n) = x_a(t)\Big|_{t=nT} = A\cos(2\pi f nT + \phi)$$
$$= A\cos(2\pi\frac{f}{F_s}n + \phi)$$

This is a sinusoidal sequence with normalized frequency $f_0 = \frac{f}{F_s}$. It explains the origin of the term 'normalized frequency': it is normalized with respect to the sampling frequency.
If $0 < 2\pi\frac{f}{F_s} < \pi$, i.e., $F_s > 2f$, the sinusoidal sequence follows the behavior of the sinusoidal function. In other words, as f increases, f_0 increases, and the frequency of oscillation increases.
On the contrary, if $2\pi\frac{f}{F_s} > \pi$, the sinusoidal sequence is unable to accurately follow the behavior of the sinusoidal function.
We will explore an important theorem, the sampling theorem, which states that a continuous-time signal can be faithfully reconstructed from his samples provided that the signal is sampled with a frequency at least twice its maximum frequency.

For more information study:

S. K. Mitra, "Digital Signal Processing: a computer based approach," 4th edition, McGraw-Hill, 2011
Chapter 2.1, pp. 41-45
Chapter 2.2, pp. 46-49
Chapter 2.3.3, pp. 58-62
Chapter 2.4, pp. 62-68