Discrete-time signals in the frequency domain

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- We have observed that each sequence can be expressed in the **time domain** as the **weighted sum of infinite impulse sequences** shifted in time:

$$x(n) = \sum_{m=-\infty}^{+\infty} x(m)\delta(n-m).$$

- In this chapter, we will explore an **alternative representation** of sequences through the **weighted sum of infinite complex exponential sequences** of the form $e^{-j\omega n}$, where ω represents the normalized angular frequency.
- This approach allows us to achieve a meaningful representation of sequences in the **frequency domain** and introduces the concept of signal **spectrum**.

- Let us assume that $x_p(t)$ is a complex function ($x_p(t) \in \mathbb{C}$), periodic with period T, continuous in t (with $t \in \mathbb{R}$).
- Then,

$$\boxed{x_p(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{j\frac{2\pi}{T}nt}}$$

where

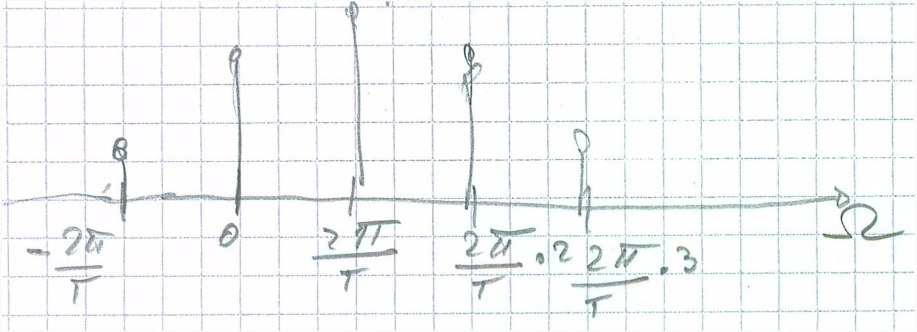
$$\boxed{c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{-j\frac{2\pi}{T}nt}\,\mathrm{d}t}$$

- For the Euler's formula

$$e^{j\,\theta} = \cos\theta + j\,\sin\theta,$$

and $x_p(n)$ is the sum of infinite sine and cosine functions at different frequencies, with each sine/cosine function multiplied by an appropriate weight coefficient.

- We can represent the function $x_p(t)$ in the angular frequency domain $\frac{2\pi}{T}n = \Omega$ by associating to every discrete frequency $\frac{2\pi}{T}n$ the corresponding coefficient c_n:



- The periodic signal is represented in the frequency domain by an infinite number of discrete "lines", corresponding to the coefficients of the Fourier series expansion of the signal.
- These lines are uniformly spaced and separated by $\frac{2\pi}{T}$.

- The frequency domain representation of a continuous-time signal $x_a(t)$ is given by the **Continuous-Time Fourier Transform (CTFT)**, defined as

$$\boxed{X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\Omega t} \mathrm{d}t}$$

- The CTFT is also referred to as the **Fourier spectrum**, or simply **spectrum**, of the continuous-time signal.
- The continuous-time signal $x_a(t)$ can be reconstructed from its CTFT by means of the **inverse continuous-time Fourier transform (ICTFT)**, defined as

$$\boxed{x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} \mathrm{d}\Omega.}$$

- Note the bijective mapping between the signal $x_a(t)$ and its transform:

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

- Ω is a real variable representing the continuous-time angular frequency, measured in rad/s.
- The inverse transform can be interpreted as the linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi} e^{j\Omega t} \mathrm{d}\Omega$.
- We can also express the transform **in polar form**:

$$X_a(j\Omega) = \left|X_a(j\Omega)\right| \cdot e^{j\theta_a(\Omega)},$$

where $\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$.

- $\left|X_a(j\Omega)\right|$ is referred to as the **magnitude spectrum**, and $\theta_a(\Omega)$ is called the **phase spectrum**.
- Both $\left|X_a(j\Omega)\right|$ and $\theta_a(\Omega)$ are real functions of the angular frequency Ω.

- Note that not all signals admit the CTFT. The integral $\int_{-\infty}^{+\infty} \cdot \mathrm{d}t$ may not converge.
- The CTFT exists if the continuous-time signal $x_a(t)$ satisfies the **Dirichlet conditions**:
  1. The signal has a finite number of discontinuities and a finite number of maxima and minima in any finite interval.
  2. The signal is absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} \left|x_a(t)\right| < +\infty$$

- If these conditions are satisfied, $\int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t$ converges and $x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a(j\Omega)e^{j\Omega t}\mathrm{d}\Omega$ apart from the discontinuity points.

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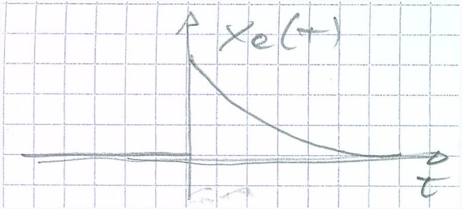
- $$x_a(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases},$$

  with $0 < \alpha < +\infty$.



- This signal satisfies the Dirichlet conditions: it has a unique discontinuity and

$$\int_{-\infty}^{+\infty} |x_a(t)| = \int_{0}^{+\infty} e^{-\alpha t} \mathrm{d}t = -\left.\frac{e^{-\alpha t}}{\alpha}\right|_0^{+\infty} = 0 - \left(-\frac{1}{\alpha}\right) = \frac{1}{\alpha}$$

$$\text{CTFT}[x_a(t)] = X_a(j\Omega) = \int_0^{+\infty} e^{-\alpha t}\, e^{-j\Omega t}\mathrm{d}t = \int_0^{+\infty} e^{-(\alpha+j\Omega)t}\mathrm{d}t = \frac{\alpha - j\Omega}{\alpha^2 + \Omega^2}$$

$$\left|X_a(j\Omega)\right| = \sqrt{\text{Re}\{\}^2 + \text{Im}\{\}^2} = \frac{1}{\sqrt{\alpha^2 + \Omega^2}}$$

$$\theta_a(\Omega) = \arctan\frac{\text{Im}\{\}}{\text{Re}\{\}} = -\arctan\frac{\Omega}{\alpha}$$

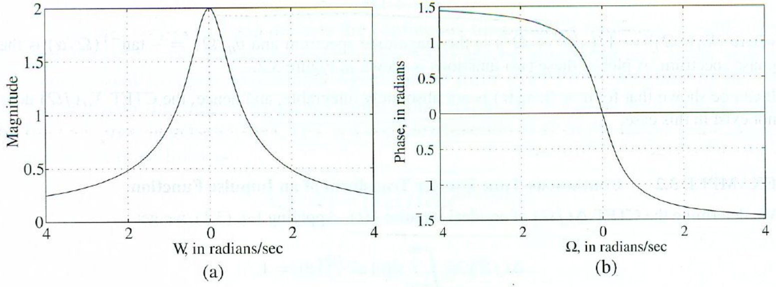


Figure 3.2: (a) Magnitude and (b) phase of $X_a(j\Omega) = 1/(0.5/\text{sec} + j\Omega)$.

- The **Dirac delta function** $\delta(t)$ is a function of the continuous-time variable t with notable properties.
- It is defined as:

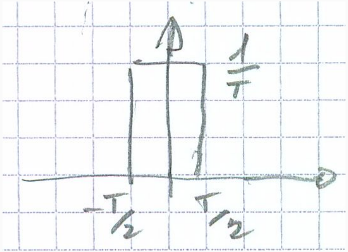
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ +\infty & t = 0 \end{cases}$$

with

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1.$$

- It is the limit, as T approaches 0, of the rectangular pulse

$$\Delta_T(t) = \begin{cases} \frac{1}{T} & -\frac{T}{2} \le t \le +\frac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$



$$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\mathrm{d}t = f(0)$$

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t-t_0)\mathrm{d}t = f(t_0)$$

$$\mathrm{CTFT}\{\delta(t)\} = \Delta(j\Omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\Omega t}\mathrm{d}t = 1$$

$$\mathrm{CTFT}\{\delta(t-t_0)\} = \int_{-\infty}^{+\infty} \delta(t-t_0)e^{-j\Omega t}\mathrm{d}t = e^{-j\Omega t_0}$$

- **Linearity of the CTFT:** If $F_a(j\Omega)$ is the CTFT of $f_a(t)$ and $G_a(j\Omega)$ is the CTFT of $g_a(t)$, the CTFT of $x_a(t) = \alpha f_a(t) + \beta g_a(t)$, with α and β constants, is

$$X_a(j\Omega) = \alpha F_a(j\Omega) + \beta G_a(j\Omega).$$

- **Time-shift property:** If $G_a(j\Omega)$ is the CTFT of $g_a(t)$, the CTFT of $x_a(t) = g_a(t - t_0)$, with t_0 constant, is

$$G_a(j\Omega)e^{-j\Omega t_0}.$$

- **Symmetry property of the CTFT:** The CTFT of a real signal $x_a(t) \in \mathbb{R}$ satisfies the property

$$X_a(-j\Omega) = X_a^*(j\Omega)$$

  where x^* is the conjugate of x.

- **Energy density spectrum:** The total energy E_x of a continuous-time signal $x_a(t)$ is given by

$$E_x = \int_{-\infty}^{+\infty} \left|x_a(t)\right|^2 \mathrm{d}t.$$

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- An **absolutely integrable signal**, i.e., a signal for which $\int_{-\infty}^{+\infty} \left|x_a(t)\right| \mathrm{d}t < +\infty$, has **finite energy**, but there exist signals with finite energy which are not absolutely integrable. Moreover, there are signals with infinite energy (for example, periodic signals).
- If $x_a(t)$ admits $X_a(j\Omega)$, then **Parseval's Theorem** holds:

$$\boxed{E_x = \int_{-\infty}^{+\infty} \left|x_a(t)\right|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|X_a(j\Omega)\right|^2 \mathrm{d}\Omega}$$

- $S_{xx}(\Omega) = \left|X_a(j\Omega)\right|^2$ is also called **Energy density spectrum** of the signal $x_a(t)$, and it provides the energy content of the signal at angular frequency Ω.
- The energy of the signal $x_a(t)$ in a frequency range $\Omega_a \leq \Omega \leq \Omega_b$ can be computed by integrating $S_{xx}(\Omega)$ over the interval $[\Omega_a, \Omega_b]$:

$$E_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) \mathrm{d}\Omega.$$

- A full-band continuous-time signal $x_a(t)$ has a spectrum occupying the entire frequency range.
- A signal is called **band-limited** if its spectrum occupies only a portion of the frequency range $-\infty < \Omega < +\infty$.
- An **ideal band-limited signal** has spectrum that is zero outside a certain range $\Omega_a \leq |\Omega| \leq \Omega_b$, with $0 \leq \Omega_a, \Omega_b \leq +\infty$, i.e.,

$$X_a(j\Omega) = \begin{cases} 0 & 0 \leq |\Omega| < \Omega_a \\ 0 & \Omega_b < |\Omega| < +\infty \end{cases}$$

- It is not possible to generate an ideal band-limited signal, but in most applications, it suffices to ensure that the signal has sufficiently low energy outside the interval $[\Omega_a, \Omega_b]$ of interest.
- A signal is called **low-pass** if its spectrum occupies the frequency range $0 \leq |\Omega| \leq \Omega_p$, where Ω_p is called the signal *bandwidth*.
- A signal is called **high-pass** if its spectrum occupies the frequency range $\Omega_p \leq |\Omega| \leq +\infty$, and the signal bandwidth extends from Ω_p to ∞.
- Eventually, a signal is called **passband** if its spectrum occupies the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < +\infty$, where $\Omega_H - \Omega_L$ is the signal bandwidth.

- The frequency domain representation of a discrete-time signal is given by the *Discrete-Time Fourier Transform* (DTFT).
- This transform expresses a sequence as a weighted combination of complex exponential sequences of the form $e^{j\omega n}$, where ω is the *normalized angular frequency*, and $\omega \in \mathbb{R}$.
- If it exists, the DTFT of a sequence is unique and the original sequence can be recovered from its transform with an inverse transform operation.
- The DTFT $X(e^{j\omega})$ of a sequence $x(n)$ is defined by:

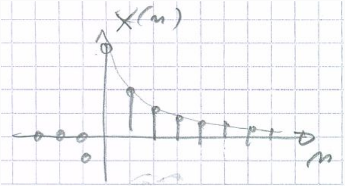
$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}.}$$

- Let us compute the DTFT of the unit impulse sequence $\delta(n)$,

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta(n) e^{-j\omega n} = 1 \cdot e^{-j\omega 0} = 1$$

- Let us compute the DTFT of the causal exponential sequence $x(n) = \alpha^n \mu(n)$, where $|\alpha| < 1$ and $\mu(n)$ is the unit step sequence.



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \alpha^n \mu(n) e^{-j\omega n} = \sum_{n=0}^{+\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{+\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1 - \alpha e^{-j\omega}}$$

since $|\alpha e^{j\omega}| = |\alpha| < 1$.

- To derive this result, we have used the following identity:

$$\boxed{\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q} \qquad \text{if } |q| < 1.}$$

- The DTFT is a **periodic function** of ω with a period of 2π:

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} \cdot e^{-j2\pi kn} = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} = X(e^{j\omega}),$$

where we used the fact that $e^{-j2\pi kn} = 1$ for k and n integers.

- Note that $\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ is the Fourier series expansion of the periodic function $X(e^{j\omega})$.
- The coefficients of the Fourier series, $x(n)$, can be computed from $X(e^{j\omega})$ using the Fourier integral:

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega,}$$

which is the **Inverse Discrete-Time Fourier Transform (IDTFT)**.

- The IDTFT can be interpreted as the linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}\mathrm{d}\omega$ weighted by the complex function $X(e^{j\omega})$.

- Let us verify that

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega.$$

- By replacing the definition of $X(e^{j\omega})$ with

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} x(l)e^{-j\omega l}$$

we have

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{l=-\infty}^{+\infty} x(l)e^{-j\omega l}e^{j\omega n}\mathrm{d}\omega =$$

and by interchanging $\int$ and $\sum$:

$$= \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} x(l)\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega$$

- For computing this integral, we will consider two cases.

- If $n \neq l$,

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)} \mathrm{d}\omega = \int_{-\pi}^{+\pi} \cos\left[\omega(n-l)\right] \mathrm{d}\omega + j \int_{-\pi}^{+\pi} \sin\left[\omega(n-l)\right] \mathrm{d}\omega = \left.\frac{\sin\left[\omega(n-l)\right]}{(n-l)}\right|_{-\pi}^{+\pi} = \frac{2\sin\left[\pi(n-l)\right]}{(n-l)} = 0.$$

- If $n = l$,

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)} \mathrm{d}\omega = \int_{-\pi}^{+\pi} e^{j\omega 0} \mathrm{d}\omega = 2\pi.$$

- Thus, in general, it is

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)} \mathrm{d}\omega = 2\pi\delta(n-l)$$

  where $\delta(n)$ is the unit impulse sequence.

- Eventually, we have

$$\frac{1}{2\pi} \sum_{l=-\infty}^{+\infty} x(l) \int_{-\pi}^{+\pi} e^{j\omega(n-l)} \mathrm{d}\omega = \sum_{l=-\infty}^{+\infty} x(l)\delta(n-l) = x(n)$$

Q.E.D.

- In general, the DTFT of a sequence is a complex function of the real variable ω:

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$$

with $X_{\mathrm{re}}(e^{j\omega})$ and $X_{\mathrm{im}}(e^{j\omega}) \in \mathbb{R}$.

$$X_{\mathrm{re}}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{j\omega})\right],$$
$$X_{\mathrm{im}}(e^{j\omega}) = \frac{1}{2j}\left[X(e^{j\omega}) - X^*(e^{j\omega})\right].$$

- The DTFT can be expressed in polar form as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where $\left|X(e^{j\omega})\right|$ is called **magnitude spectrum**, and $\theta(\omega)$ is called the **phase spectrum**:

$$\theta(\omega) = \arctan\frac{X_{\mathrm{im}}(e^{j\omega})}{X_{\mathrm{re}}(e^{j\omega})}$$

- Note that there is an indetermination of $2\pi k$, with $k \in \mathbb{Z}$, in the knowledge of $\theta(\omega)$.

- If $x(n) \in \mathbb{R}$:

$$X(e^{-j\omega}) = X^*(e^{j\omega}).$$

  In fact:

$$X^*(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x^*(n) e^{j\omega n} = \sum_{n=-\infty}^{+\infty} x(n) e^{-j(-\omega)n} = X(e^{-j\omega}).$$

- For this relation, if $x(n) \in \mathbb{R}$ we have that

$$X_{\mathrm{re}}(e^{-j\omega}) = X_{\mathrm{re}}(e^{j\omega}) \qquad \text{is an even function,}$$

$$X_{\mathrm{im}}(e^{-j\omega}) = -X_{\mathrm{im}}(e^{j\omega}) \qquad \text{is an odd function,}$$

$$\left|X(e^{-j\omega})\right| = \left|X(e^{j\omega})\right| \qquad \text{is an even function,}$$

$$\theta(-\omega) = -\theta(\omega) \qquad \text{is an odd function.}$$

- If $x(n)$ is real and even $(x(n) = x(-n))$,

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) \qquad \text{is an real function.}$$

- If $x(n)$ is real and odd $(x(-n) = -x(n))$,

$$X(e^{j\omega}) = jX_{\mathrm{im}}(e^{j\omega}) \qquad \text{is an imaginary function.}$$

- The series that defines the Fourier transform could or could not converge.
- We say that the DTFT exists if its series converges according to some criteria.
- Sufficient conditions for the existence of the DTFT of a sequence $x(n)$ are the following:
  - $x(n)$ is absolutely summable, i.e.,

    $$\sum_{-\infty}^{+\infty} |x(n)| < +\infty$$

    In this case, we talk about **uniform convergence**.
  - $x(n)$ is a finite energy signal

    $$\sum_{-\infty}^{+\infty} |x(n)|^2 < +\infty$$

    In this case, we talk about **mean square convergence**.
- These two conditions are quite restrictive. Note that the first condition is stronger than the second. An absolutely summable signal is always a finite energy signal because:

$$\sum_{-\infty}^{+\infty} |x(n)|^2 \leq \left(\sum_{-\infty}^{+\infty} |x(n)| \right)^2 .$$

- An absolutely summable signal is the following causal exponential sequence

$$x(n) = \alpha^n \mu(n)$$

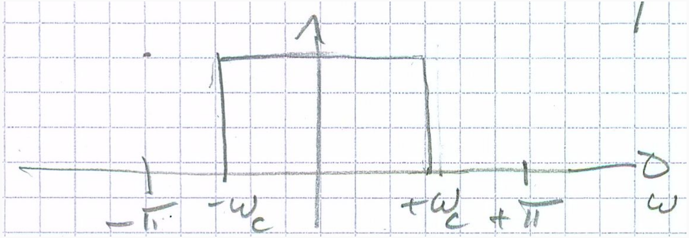
  for $|\alpha| < 1$.
- Indeed,

$$\sum_{n=-\infty}^{+\infty} |\alpha^n \mu(n)| = \sum_{n=0}^{+\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < +\infty.$$

  We have already seen that its DTFT is $\dfrac{1}{1-\alpha e^{-j\omega}}$.

- A signal that is not absolutely summable (but has a DTFT) is the signal with the following ideal low-pass DTFT:

$$H_{\mathrm{LP}}(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$



- Let us compute the corresponding signal, which we will encounter frequently.

- Let us compute the IDTFT:

$$h_{\mathrm{LP}}(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_{\mathrm{LP}}(e^{j\omega})e^{j\omega n}\mathrm{d}\omega,$$

- For $-\infty < n < +\infty$ and $n \neq 0$:

$$h_{\mathrm{LP}}(n) = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right] = \frac{\sin(\omega_c n)}{\pi n},$$

- For $n = 0$

$$h_{\mathrm{LP}}(n) = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} \mathrm{d}\omega = \frac{\omega_c}{\pi}$$

- Thus,

$$h_{\mathrm{LP}}(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

- Since

$$\frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n},$$

and assuming by convention that $\dfrac{\sin(\omega_c 0)}{\omega_c 0} = 1$, we can write for all n

$$h_{\mathrm{LP}}(n) = \frac{\sin(\omega_c n)}{\pi n}.$$

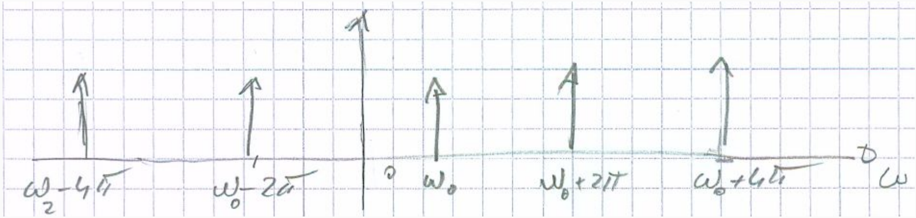
Further of DTFT convergence

- The DTFT can be defined also for a certain class of sequences that are not absolutely summable, nor have finite energy, e.g., the unit step sequence, the sinusoidal sequence, or the complex exponential sequence.
- In these cases the expression of the DTFT involves Dirac delta functions.

- The DTFT of the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)}$$

The DTFT $X(e^{j\omega})$ is a periodic function in ω with period 2π and it is called a **periodic pulse train**.



- Let us prove this relation by computing the IDTFT:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k) e^{j\omega n} \mathrm{d}\omega$$

- If $-\pi < \omega_o \leq \pi$, the only non-null function in the interval $[-\pi, +\pi]$ is $\delta(\omega - \omega_0)$, thus

$$x(n) = \frac{1}{2\pi} 2\pi \int_{-\pi}^{+\pi} \delta(\omega - \omega_0) e^{j\omega n} \mathrm{d}\omega = e^{j\omega_0 n}$$

- Let us assume that $G(e^{j\omega})$ is the DTFT of $g(n)$ and that $H(e^{j\omega})$ is the DTFT of $h(n)$.
- **Linearity property:** If $x(n) = \alpha g(n) + \beta h(n)$, with α and β constants, the DTFT of $x(n)$ is $X(e^{j\omega}) = \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$.
- **Time reversal:** The DTFT of $g(-n)$ is $G(e^{-j\omega})$.
- **Time-shifting:** The DTFT of $g(n - n_0)$ is $e^{-j\omega n_0} G(e^{j\omega})$.
- **Frequency-shifting:** The DTFT of $e^{j\omega_0 n} g(n)$ is $G(e^{j(\omega-\omega_0)})$.
- **Frequency differentiation:** The DTFT of $ng(n)$ is $j\dfrac{\mathrm{d}G(e^{j\omega})}{\mathrm{d}\omega}$.
- **Modulation:** The DTFT of $g(n) \cdot h(n)$ is $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{+\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\mathrm{d}\theta$.
- **Parseval theorem:**

$$\sum_{n=-\infty}^{+\infty} g(n)h^*(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})H^*(e^{j\omega})\mathrm{d}\omega$$

$$\alpha g(n) + \beta h(n) \overset{DTFT}{\longleftrightarrow} \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

$$g(-n) \overset{DTFT}{\longleftrightarrow} G(e^{-j\omega})$$

$$g(n - n_0) \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0} G(e^{j\omega})$$

$$e^{j\omega_0 n} g(n) \overset{DTFT}{\longleftrightarrow} G(e^{j(\omega-\omega_0)})$$

$$n g(n) \overset{DTFT}{\longleftrightarrow} j \frac{\mathrm{d}G(e^{j\omega})}{\mathrm{d}\omega}$$

$$g(n) \cdot h(n) \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{+\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) \mathrm{d}\theta$$

$$\sum_{n=-\infty}^{+\infty} g(n) h^*(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} G(e^{j\omega}) H^*(e^{j\omega}) \mathrm{d}\omega$$

Proof of Parseval theorem

$$\sum_{n=-\infty}^{+\infty} g(n)h^*(n) = \sum_{n=-\infty}^{+\infty} g(n)\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} H^*(e^{j\omega})e^{-j\omega n}\mathrm{d}\omega\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{+\infty} g(n)e^{-j\omega n}\right)\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})H^*(e^{j\omega})\mathrm{d}\omega$$

Q.E.D.

- Let us compute the DTFT of the finite length exponential sequence:

$$y(n) = \begin{cases} \alpha^n & 0 \leq n < M-1 \\ 0 & \text{otherwise} \end{cases}$$

  with $|\alpha| < 1$.

$$y(n) = \alpha^n \mu(n) - \alpha^n \mu(n-M) = \alpha^n \mu(n) - \alpha^M \alpha^{n-M} \mu(n-M)$$

- We already know that:

$$\alpha^n \mu(n) \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

- For the linearity and the time shift properties we have

$$Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} - \alpha^M \frac{e^{-j\omega M}}{1-\alpha e^{-j\omega}} = \frac{1-\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

- Let us compute the DTFT of the causal sequence $v(n)$ defined by the finite difference equation:

$$d_0 v(n) + d_1 v(n-1) = p_0 \delta(n) + p_1 \delta(n-1)$$

  with $d_0 \neq 0$, and $d_1 \neq 0$.
- Let us apply the DTFT to both terms of the finite difference equation.
- By exploiting the linearity and time-shit properties and remembering that $\text{DTFT}\{\delta(n)\} = 1$, we have

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

  Thus,

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}.$$

- Let us compute the DTFT of $y(n) = (-1)^n \alpha^n \mu(n)$ with $|\alpha| < 1$.
- It is also

$$y(n) = e^{j\pi n} \left(\alpha^n \mu(n)\right) = e^{j\pi n} x(n),$$

  where we considered $x(n) = \alpha^n \mu(n)$.
- We can apply the frequency-shifting property:

$$Y(e^{j\omega}) = X(e^{j(\omega-\pi)}) = \frac{1}{1-\alpha e^{-j(\omega-\pi)}} = \frac{1}{1+\alpha e^{-j\omega}}.$$

Energy density spectrum

- The **total energy** of a sequence $g(n)$ is given by

$$E_g = \sum_{n=-\infty}^{+\infty} |g(n)|^2 .$$

- By applying the Parseval theorem with $h(n) = g(n)$ we have

$$\boxed{E_g = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| G(e^{j\omega}) \right|^2 \mathrm{d}\omega}.$$

- $S_{gg}(e^{j\omega}) = \left| G(e^{j\omega}) \right|^2$ is called **energy density spectrum**, and it defines how the energy of the sequence is distributed over the frequency spectrum.

- Let us compute the energy of the low-pass ideal signal:

$$\sum_{n=-\infty}^{+\infty} |h_{\mathrm{LP}}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| H_{\mathrm{LP}}(e^{j\omega}) \right|^2 \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} \mathrm{d}\omega = \frac{\omega_c}{\pi} < +\infty$$

- In general, the spectrum of a discrete-time signal is defined on the entire frequency range $-\pi < \omega \leq +\pi$. A band-limited signal has a spectrum that is limited to part of this range.
- An ideal **low-pass** signal has

$$X(e^{j\omega}) \begin{cases} \neq 0 & 0 \leq |\omega| \leq \omega_p \\ = 0 & \omega_p < |\omega| \leq \pi \end{cases}$$

  ω_p is called signal *bandwidth*.
- An ideal **high-pass** signal has

$$X(e^{j\omega}) \begin{cases} = 0 & 0 \leq |\omega| < \omega_p \\ \neq 0 & \omega_p \leq |\omega| \leq \pi \end{cases}$$

  The signal bandwidth is given by $\pi - \omega_p$.
- An ideal **passband** signal has

$$X(e^{j\omega}) \begin{cases} = 0 & 0 \leq |\omega| < \omega_a \\ \neq 0 & \omega_a \leq |\omega| \leq \omega_b \\ = 0 & \omega_b < |\omega| < \pi \end{cases}$$

  The signal bandwidth is given by $\omega_b - \omega_a$.

- We want to study the relations that link the continuous-time signals with the corresponding discrete-time signals.
- Let us consider a continuous-time signal $g_a(t)$. We assume to uniformly sample it with sampling period T. We obtain the sequence:

$$\boxed{g(n) = g_a(nT)} \qquad -\infty < n < +\infty.$$

- The sampling frequency is $F_T = \frac{1}{T}$.
- The CTFT of $g_a(t)$ is

$$G_a(j\Omega) = \int_{-\infty}^{+\infty} g_a(t) e^{-j\Omega t} \mathrm{d}t,$$

and the DTFT of $g(n)$ is

$$G(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} g(n) e^{-j\omega n}.$$

- We want to find the relation that exists between $G_a(j\Omega)$ and $G(e^{j\omega})$.

- Mathematically, we can consider the sampling operations as the product of a signal $g_a(t)$ and a periodic pulse train $p(t)$, where

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT),$$

composed of an infinite sequence of Dirac pulses uniformly spaced in time and separated by the period T.

- The multiplication between $g_a(t)$ and $p(t)$ results in a continuous-time function

$$g_p(t) = g_a(t) \cdot p(t) = \sum_{n=-\infty}^{+\infty} g_a(nT)\,\delta(t - nT),$$

which is also composed of infinite Dirac impulses, placed at time $t = nT$, and weighted by the sample value $g_a(nT)$.

- We will provide two expressions for the CTFT of $g_p(t)$.

- The first one is derived directly from the last expression of $g_p(n)$ considering that the CTFT of $\{\delta(t-nT)\}$ is $e^{-j\Omega nT}$.
- For the linearity property of the CTFT:

$$G_p(j\Omega) = \sum_{n=-\infty}^{+\infty} g_a(nT)e^{-j\Omega nT}.$$

- By comparing this relation with the expression of $G(e^{j\omega})$, we see that

$$G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\frac{\omega}{T}}$$

$$G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}$$

- **The DTFT $G(e^{j\omega})$ coincides with the CTFT of $g_p(n)$, apart from a frequency axis normalization.**
- This normalization maps the point at $\omega = 2\pi$ of $G(e^{j\omega})$ to the point at $\Omega_T = 2\pi F_T$ of $G_p(j\Omega)$.

- In what follows, we find a second form for $G_p(j\Omega)$.
- Assume you have two continuous time signals $a(t)$ and $b(t)$ with CTFT $A(j\Omega)$ and $B(j\Omega)$, respectively. The CTFT of the product $y(t) = a(t) \cdot b(t)$ is

$$\boxed{Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(j\Psi) \cdot B\big(j(\Omega - \Psi)\big) \mathrm{d}\Psi}$$

- We can use the property we have just proved to evaluate the CTFT of

$$g_p(t) = g_a(t) \cdot p(t).$$

- We assume knowing the CTFT of $g_a(t)$. Let us compute the CTFT of the pulse train $p(t)$.
- In what follows, we prove that

$$\boxed{P(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\Omega - k\frac{2\pi}{T})}$$

i.e., the CTFT of $p(t)$ is also a uniformly spaced pulse train.

- Note that the pulse train $p(t)$ is a periodic signal with a period of T. Thus, we can expand $p(t)$ with the Fourier series:

$$p(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}$$

with

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)e^{-j\frac{2\pi}{T}kt}\mathrm{d}t = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-j\frac{2\pi}{T}kt}\mathrm{d}t = \frac{1}{T}e^{-j\frac{2\pi}{T}k0} = \frac{1}{T}$$

- Thus,

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt}$$

and for the linearity of the CTFT

$$P(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} \mathrm{CTFT}\left\{e^{j\frac{2\pi}{T}kt}\right\}.$$

- We can easily verify that

$$\text{CTFT}\left\{e^{j\frac{2\pi}{T}kt}\right\} = 2\pi\delta(\Omega - \frac{2\pi}{T}k)$$

- Indeed, the IDFT of the right-hand side is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\Omega - \frac{2\pi}{T}k)e^{j\Omega t}\mathrm{d}\Omega = e^{j\frac{2\pi}{T}kt}$$

- Replacing $\text{CTFT}\left\{e^{j\frac{2\pi}{T}kt}\right\}$ in the expression of $P(j\Omega)$, we have

$$P(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\Omega - k\frac{2\pi}{T}).$$

Q.E.D.

- We can now compute $G_p(j\Omega)$

$$G_p(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_a(j\Psi)\cdot P\big(j(\Omega-\Psi)\big)\mathrm{d}\Psi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_a(j\Psi)\cdot\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\Omega-\Psi-k\frac{2\pi}{T})\mathrm{d}\Psi =$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_a(j\Psi)\cdot\delta(\Omega-\Psi-k\frac{2\pi}{T})\mathrm{d}\Psi =$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} G_a\big(j(\Omega-k\frac{2\pi}{T})\big) =$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} G_a\big(j(\Omega-k\Omega_T)\big)$$

with $\Omega_T = \dfrac{2\pi}{T}$.

- Consequently,

$$G(e^{j\omega}) = G_p(j\Omega)|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} G_a[j(\frac{\omega}{T} + k\Omega_T)] = \frac{1}{T}\sum_{k=-\infty}^{+\infty} G_a[j(\frac{\omega + 2\pi k}{T})].$$

- **The DTFT is given by the periodic repetition, with a period of 2π, of the continuous spectrum $G_a\left(j\frac{\omega}{T}\right)$.**
- $G_a\left(j\frac{\omega}{T}\right)$ is identical to $G_a(j\Omega)$, but the frequency axis has been normalized such that $\omega = 2\pi$ corresponds to the angular sampling frequency $\Omega_T = \frac{2\pi}{T}$.
- In order to avoid any overlap between the repeated spectra, it must be

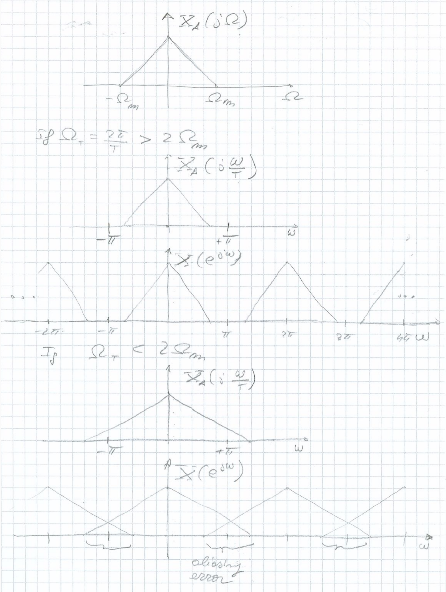
$$G_a\left(j\frac{\omega}{T}\right) = 0 \quad \text{for} \quad |\omega| > \pi,$$

or, equivalently,

$$G_a(j\Omega) = 0 \quad \text{for} \quad |\Omega| > \frac{\pi}{T} = \frac{\Omega_T}{2}.$$

- If this condition is satisfied, the discrete spectrum reproduces in the band $[-\pi, +\pi]$ the continuous spectrum.
- If this condition is not satisfied, we have a distortion caused by the overlap of the tails of the spectrum. This distortion is called **aliasing**.

- The frequency $\frac{1}{2T} = \frac{F_T}{2}$ is called **Nyquist frequency**.
- Similarly, the angular frequency $\frac{\Omega_T}{2}$ is called **Nyquist angular frequency**.
- Let us assume, for example, that our signal occupies the band $[-\Omega_m, +\Omega_m]$.
- If $\Omega_T \geq 2\Omega_m$, the different repetitions of the spectrum do not overlap.
- If $\Omega_T < 2\Omega_m$, the repetitions of the spectrum overlap, and we have an aliasing error.



- If the continuous-time signal spectrum is preserved in the discrete domain (i.e., if we do not have aliasing), we can reconstruct the original signal from the samples $g(n) = g_a(nT)$.
- For this purpose, let us build the signal:

$$g_p(t) = \sum_{n=-\infty}^{+\infty} g(n)\delta(t - nT) = \sum_{n=-\infty}^{+\infty} g_a(nT)\delta(t - nT)$$

- We know its CTFT:

$$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\left[j(\Omega + k\Omega_T)\right].$$

  This continuous spectrum is given by the periodic repetition of the spectrum $G_a(j\Omega)$ with period Ω_T.
- If the signal has been sampled with a frequency $\Omega_T > 2\Omega_m$ we do not have aliasing, and the repetitions of $G_a(j\Omega)$ do not overlap.
- Thus, we can faithfully reconstruct$g_a(t)$ by passing $g_p(t)$ through a filter that lets the spectrum in band $[-\Omega_m, \Omega_m]$ pass without any alteration, while stopping all signal components outside that frequency range.
- We will see that such a filter is an ideal low-pass filter with bandwidth Ω_m.
- **We have just proved** the theorem that is at the basis of all signal processing and modern telecommunications, **the sampling theorem**.

Let $g_a(t)$ be a band-limited signal, with $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$. Then, $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, (i.e., can be faithfully reconstructed by its samples $g_a(nT)$,) $-\infty < n < +\infty$, if the angular sampling frequency

$$\Omega_T \geq 2\Omega_m$$

with $\Omega_T = \dfrac{2\pi}{T}$.

- In other words, if we want to be able to recover the band-limited signal $g_a(t)$ from its samples, we must sample the signal with a frequency at least twice the signal bandwidth.
- The signal can be recovered by generating the signal

$$g_p(t) = \sum_{n=-\infty}^{+\infty} g(n)\delta(t - nT)$$

and by filtering this signal with an ideal low-pass filter.

- In general, real-world signals are not band-limited. They occupy an infinite band, but most of their energy is concentrated at the low frequencies.
- To avoid distortions due to aliasing, the signal is typically filtered with a low-pass filter before sampling.
- This filter cuts off high frequencies and generates a band-limited signal. Such a filter is called an **anti-aliasing** filter.
- After filtering, the signal can be safely sampled with a frequency greater than $2\Omega_m$ and processed as desired.

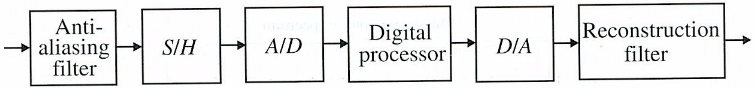


Figure 3.12: Block diagram representation of the discrete-time digital processing of a continuous-time signal.

- Also the signal that comes out of the Digital to Analog (D/A) converter is filtered with a low-pass filter, called the **reconstruction filter** or **anti-imaging filter**.
- In this way, all the frequencies (the images) outside the band of the original signal, $[-\Omega_m, +\Omega_m]$ are removed.

Bibliography

- For more information study:

S. K. Mitra, "Digital Signal Processing: a computer based approach," 4th edition, McGraw-Hill, 2011
Chapter 3.1, pp. 89-93
Chapter 3.2-3.5, pp. 94-112
Chapter 3.8, pp. 115-124

Unless otherwise specified, all images have either been originally produced or have been taken from S. K. Mitra, "Digital Signal Processing: a computer based approach."