03 Discrete-time signals in the frequency domain

We have observed that each sequence can be expressed in the time domain as the weighted sum of infinite impulse sequences shifted in time:

$$x(n) = \sum_{m=-\infty}^{+\infty} x(m)\delta(n-m).$$

In this chapter, we will explore an alternative representation of sequences through the weighted sum of infinite complex exponential sequences of the form $e^{-j\omega n}$, where ω represents the normalized angular frequency. This approach allows us to achieve a meaningful representation of sequences in the *frequency domain* and introduces the concept of signal *spectrum*.

We will initially explore continuous-time signals and introduce the Continuous-Time Fourier Transform (CTFT). Subsequently, we will transition to the discrete-time domain and introduce the Discrete-Time Fourier Transform (DTFT). Finally, we will examine the relationship between these transforms, particularly in the case of sampled signals.

Jean Baptiste Joseph Fourier, a French mathematician born in 1768, was the first to comprehend (in the early 1800s) that every periodic function can be represented as the sum of an infinite series of appropriately weighted sine and cosine functions at different frequencies.

03.01 The Fourier series

Let us assume that $x_p(t)$ is a complex function ($x_p(t) \in \mathbb{C}$), periodic with period T, continuous in t (with $t \in \mathbb{R}$). Then,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{j\frac{2\pi}{T}nt}$$

where

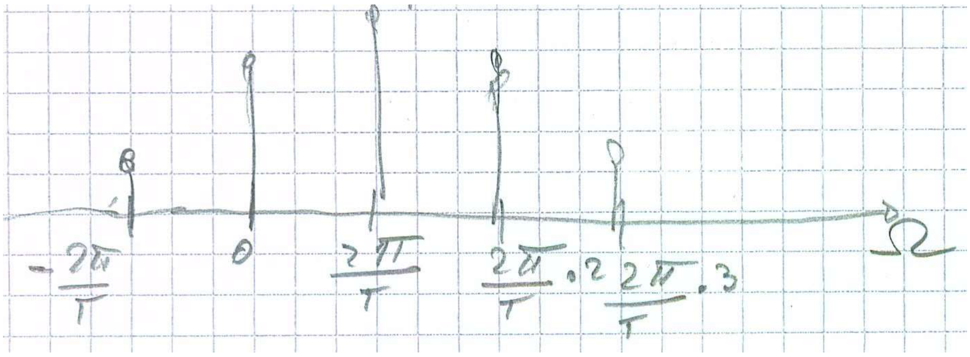
$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{-j\frac{2\pi}{T}nt}\, \mathrm{d}t$$

For the Euler's formula

$$e^{j\,\theta} = \cos\theta + j\,\sin\theta,$$

and $x_p(n)$ is the sum of infinite sine and cosine functions at different frequencies, with each sine/cosine function multiplied by an appropriate weight coefficient.

Note that we can represent the function $x_p(t)$ in the angular frequency domain $\frac{2\pi}{T}n = \Omega$ by associating to every discrete frequency $\frac{2\pi}{T}n$ the corresponding coefficient c_n:



The periodic signal is represented in the frequency domain by an infinite number of discrete "lines", corresponding to the coefficients of the Fourier series expansion of the signal. These lines are uniformly spaced and separated by $\frac{2\pi}{T}$.
Note that by increasing the period of the periodic function, the lines tend to become narrower (more frequent). Thus, we can understand that aperiodic functions (satisfying certain conditions we will see later) can also be expressed as the sum of sine and cosine functions, or, more precisely, as the sum of complex exponential functions $e^{j\Omega t}$ (whose frequency varies continuously) multiplied by a frequency-varying weight function.
Thus, we arrive at the Continuous-Time Fourier Transform (CTFT).

03.02 The Continuous-Time Fourier Transform

The frequency domain representation of a continuous-time signal $x_a(t)$ is given by the Continuous-Time Fourier Transform (CTFT), defined as[1]

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\Omega t} \mathrm{d}t$$

The CTFT is also referred to as the *Fourier spectrum*, or simply *spectrum*, of the continuous-time signal.
The continuous-time signal $x_a(t)$ can be reconstructed from its CTFT by means of the *inverse* continuous-time Fourier transform (ICTFT), defined as

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} \mathrm{d}\Omega.$$

Note the bijective mapping between the signal $x_a(t)$ and its transform:

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

Ω is a real variable representing the continuous-time angular frequency, measured in rad/s.
The inverse transform can be interpreted as the linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi} e^{j\Omega t} \mathrm{d}\Omega$.

[1] Why $X_a(j\Omega)$ and not $X_a(\Omega)$? It is simply a convention. Another transform, the Laplace transform $X_a(s)$, is defined on the entire complex plane, and when evaluated on the imaginary axis (i.e., for $s = j\Omega$), it yields the CTFT.

We can also express the transform in polar form:

$$X_a(j\Omega) = \big|X_a(j\Omega)\big| \cdot e^{j\theta_a(\Omega)},$$

where $\theta_a(\Omega) = \text{arg}\,\{X_a(j\Omega)\}$.
$\big|X_a(j\Omega)\big|$ is referred to as the *magnitude spectrum*, and $\theta_a(\Omega)$ is called the *phase spectrum*. Both $\big|X_a(j\Omega)\big|$ and $\theta_a(\Omega)$ are real functions of the angular frequency Ω.
Note that not all signals admit the CTFT. The integral $\int_{-\infty}^{+\infty} \cdot \mathrm{d}t$ may not converge. The CTFT exists if the continuous-time signal $x_a(t)$ satisfies the *Dirichlet conditions*:

1. The signal has a finite number of discontinuities and a finite number of maxima and minima in any finite interval.

2. The signal is absolutely integrable, i.e.,

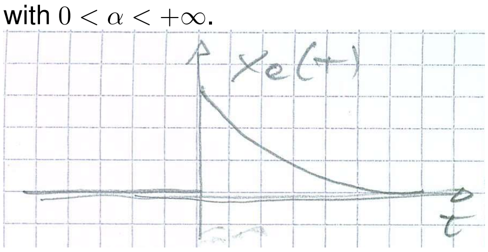
$$\int_{-\infty}^{+\infty} \big|x_a(t)\big| < +\infty$$

If these conditions are satisfied, $\int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t$ converges and $x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a(j\Omega)e^{j\Omega t}\mathrm{d}\Omega$ apart from the discontinuity points.

Example:

$$x_a(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases},$$

with $0 < \alpha < +\infty$.



This signal satisfies the Dirichlet conditions. Indeed, it has a unique discontinuity and

$$\int_{-\infty}^{+\infty} \big|x_a(t)\big| = \int_0^{+\infty} e^{-\alpha t}\mathrm{d}t = -\left.\frac{e^{-\alpha t}}{\alpha}\right|_0^{+\infty} = 0 - \left(-\frac{1}{\alpha}\right) = \frac{1}{\alpha}$$

$$\text{CTFT}[x_a(t)] = X_a(j\Omega) = \int_0^{+\infty} e^{-\alpha t}\,e^{-j\Omega t}\mathrm{d}t = \int_0^{+\infty} e^{-(\alpha+j\Omega)t}\mathrm{d}t$$
$$= -\left.\frac{1}{\alpha+j\Omega}e^{-(\alpha+j\Omega)t}\right|_0^{+\infty} = \frac{1}{\alpha+j\Omega} \quad \Big/ \cdot \frac{\alpha-j\Omega}{\alpha-j\Omega}$$
$$= \frac{\alpha-j\Omega}{\alpha^2+\Omega^2}$$

$$\big|X_a(j\Omega)\big| = \sqrt{\text{Re}\{\}^2 + \text{Im}\{\}^2} = \frac{1}{\sqrt{\alpha^2+\Omega^2}}$$
$$\theta_a(\Omega) = \text{arctan}\frac{\text{Im}\{\}}{\text{Re}\{\}} = -\text{arctan}\frac{\Omega}{\alpha}$$

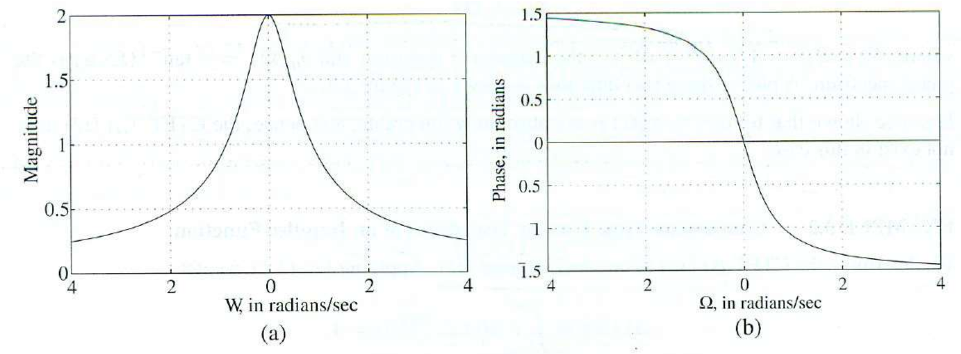


Figure 3.2: (a) Magnitude and (b) phase of $X_a(j\Omega) = 1/(0.5/\text{sec} + j\Omega)$.

(From S. K. Mitra, “Digital signal processing: a computer based approach”, McGraw Hill, 2011)

CTFT of a Dirac delta function

The *Dirac delta function* $\delta(t)$ is a function of the continuous-time variable t with notable properties, defined as:

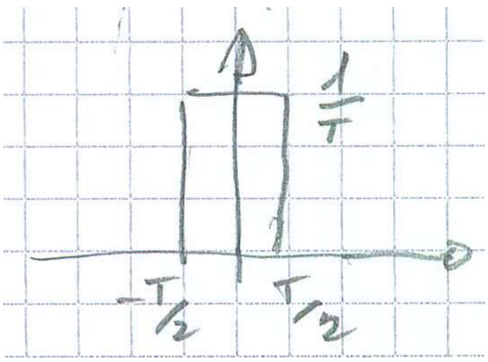
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ +\infty & t = 0 \end{cases}$$

with

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1.$$

It is the limit, as T approaches 0, of the rectangular pulse

$$\Delta_T(t) = \begin{cases} \dfrac{1}{T} & -\dfrac{T}{2} \leq t \leq +\dfrac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Properties of Dirac delta function:

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\mathrm{d}t = f(0)$$

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t-t_0)\mathrm{d}t = f(t_0)$$

$$\text{CTFT}\{\delta(t)\} = \Delta(j\Omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\Omega t}\mathrm{d}t = 1$$

$$\text{CTFT}\{\delta(t-t_0)\} = \int_{-\infty}^{+\infty} \delta(t-t_0)e^{-j\Omega t}\mathrm{d}t = e^{-j\Omega t_0}$$

Linearity of the CTFT

If $F_a(j\Omega)$ is the CTFT of $f_a(t)$ and $G_a(j\Omega)$ is the CTFT of $g_a(t)$, the CTFT of $x_a(t) = \alpha f_a(t) + \beta g_a(t)$, with α and β constants, is

$$X_a(j\Omega) = \alpha F_a(j\Omega) + \beta G_a(j\Omega).$$

Proof:

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\Omega t}\mathrm{d}t =$$

$$= \int_{-\infty}^{+\infty} [\alpha f_a(t) + \beta g_a(t)]\, e^{-j\Omega t}\mathrm{d}t =$$

$$= \alpha \int_{-\infty}^{+\infty} f_a(t)\, e^{-j\Omega t}\mathrm{d}t + \beta \int_{-\infty}^{+\infty} g_a(t)\, e^{-j\Omega t}\mathrm{d}t =$$

$$= \alpha F_a(j\Omega) + \beta G_a(j\Omega)$$

Q.E.D.

Time-shift property

If $G_a(j\Omega)$ is the CTFT of $g_a(t)$, the CTFT of $x_a(t) = g_a(t - t_0)$, with t_0 constant, is

$$G_a(j\Omega)e^{-j\Omega t_0}.$$

Proof:

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} g_a(t - t_0)\, e^{-j\Omega t}\mathrm{d}t =$$

Let us introduce the change of variables $t' = t - t_0$, i.e., $t = t' + t_0$:

$$= \int_{-\infty}^{+\infty} g_a(t')\, e^{-j\Omega(t'+t_0)}\mathrm{d}t' =$$

$$= \int_{-\infty}^{+\infty} g_a(t')\, e^{-j\Omega t'}\mathrm{d}t' e^{-j\Omega t_0} =$$

$$= G_a(j\Omega)e^{-j\Omega t_0}.$$

Q.E.D.

Symmetry property of the CTFT

The CTFT of a real signal $x_a(t) \in \mathbb{R}$ satisfies the property

$$X_a(-j\Omega) = X_a^*(j\Omega)$$

where x^* is the conjugate of x.

Proof:

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t$$

$$X_a^*(j\Omega) = \int_{-\infty}^{+\infty} x_a^*(t)e^{j\Omega t}\mathrm{d}t$$

$$= \int_{-\infty}^{+\infty} x_a(t)e^{-j(-\Omega)t}\mathrm{d}t$$
$$= X_a(-j\Omega).$$

Q.E.D.

Energy density spectrum

The total energy E_x of a continuous-time signal $x_a(t)$ is given by

$$E_x = \int_{-\infty}^{+\infty} \left|x_a(t)\right|^2 \mathrm{d}t.$$

An absolutely integrable signal, i.e., a signal for which $\int_{-\infty}^{+\infty} \left|x_a(t)\right| \mathrm{d}t < +\infty$, has finite energy, but there exist signals with finite energy which are not absolutely integrable. Moreover, there are signals with infinite energy (for example, periodic signals).

Consider a finite energy signal that admits CTFT. We can express the energy as a function of $X_a(j\Omega)$:

$$E_x = \int_{-\infty}^{+\infty} \left|x_a(t)\right|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} x_a(t)x_a^*(t)\mathrm{d}t =$$
$$\int_{-\infty}^{+\infty} x_a(t) \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a^*(j\Omega)e^{-j\Omega t}\mathrm{d}\Omega\right] \mathrm{d}t$$

where we have replaced $x_a^*(t)$ with the ICTFT. Let us exchange the two integrations:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a^*(j\Omega)\int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t\mathrm{d}\Omega$$

The inner integral is the CTFT of $x_a(t)$. Thus

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a^*(j\Omega)X_a(j\Omega)\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|X_a(j\Omega)\right|^2 \mathrm{d}\Omega$$

The integral from $-\infty$ to $+\infty$ of the squared magnitude spectrum equals the signal energy. Thus, the following theorem is proved.

Parseval's Theorem:

$$E_x = \int_{-\infty}^{+\infty} \left|x_a(t)\right|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|X_a(j\Omega)\right|^2 \mathrm{d}\Omega$$

$S_{xx}(\Omega) = \left|X_a(j\Omega)\right|^2$ is also called *Energy density spectrum* of the signal $x_a(t)$, and it provides the energy content of the signal at angular frequency Ω.

The energy of the signal $x_a(t)$ in a frequency range $\Omega_a \le \Omega \le \Omega_b$ can be computed by integrating $S_{xx}(\Omega)$ over the interval $[\Omega_a, \Omega_b]$:

$$E_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\mathrm{d}\Omega.$$

Band limited signals

A full-band continuous-time signal $x_a(t)$ has a spectrum occupying the entire frequency range. A signal is called *band-limited* if its spectrum occupies only a portion of the frequency range $-\infty < \Omega < +\infty$.

An *ideal band-limited signal* has spectrum that is zero outside a certain range $\Omega_a \leq |\Omega| \leq \Omega_b$, with $0 \leq \Omega_a, \Omega_b \leq +\infty$, i.e.,

$$X_a(j\Omega) = \begin{cases} 0 & 0 \leq |\Omega| < \Omega_a \\ 0 & \Omega_b < |\Omega| < +\infty \end{cases}$$

[Why do we consider $|\Omega|$ instead of Ω? Because if the signal $x_a(t)$ is real, $X_a(-j\Omega) = X_a^*(j\Omega)$, i.e., the spectrum has conjugate symmetry. If $X_a(j\Omega) \neq 0$ in a range $[\Omega_a, \Omega_b]$, then $X_a(j\Omega) \neq 0$ in $[-\Omega_b, -\Omega_a]$.]

It is not possible to generate an ideal band-limited signal, but in most applications, it suffices to ensure that the signal has sufficiently low energy outside the interval $[\Omega_a, \Omega_b]$ of interest.

A signal is called *low-pass* if its spectrum occupies the frequency range $0 \leq |\Omega| \leq \Omega_p$, where Ω_p is called the signal *bandwidth*.

A signal is called *high-pass* if its spectrum occupies the frequency range $\Omega_p \leq |\Omega| \leq +\infty$, and the signal bandwidth extends from Ω_p to ∞.

Eventually, a signal is called *passband* if its spectrum occupies the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < +\infty$, where $\Omega_H - \Omega_L$ is the signal bandwidth.

03.03 The Discrete-Time Fourier Transform

The frequency domain representation of a discrete-time signal is given by the *Discrete-Time Fourier Transform* (DTFT). This transform expresses a sequence as a weighted combination of complex exponential sequences of the form $e^{j\omega n}$, where ω is the *normalized angular frequency*, and $\omega \in \mathbb{R}$.

If it exists, the DTFT of a sequence is unique and the original sequence can be recovered from its transform with an inverse transform operation.

The DTFT $X(e^{j\omega})$ of a sequence $x(n)$ is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}.$$

Example:

Let us compute the DTFT of the unit impulse sequence $\delta(n)$,

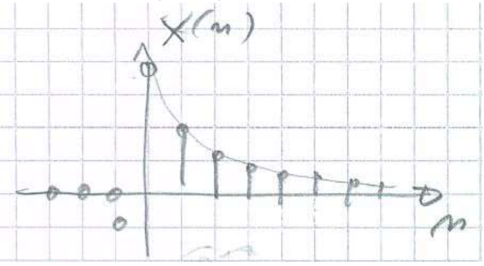
$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta(n)e^{-j\omega n} = 1 \cdot e^{-j\omega 0} = 1$$

Example:

Let us compute the DTFT of the causal exponential sequence $x(n) = \alpha^n \mu(n)$, where $|\alpha| < 1$ and $\mu(n)$

is the unit step sequence.



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \alpha^n \mu(n) e^{-j\omega n} =$$
$$= \sum_{n=0}^{+\infty} \alpha^n e^{-j\omega n} =$$
$$= \sum_{n=0}^{+\infty} \left(\alpha e^{-j\omega}\right)^n =$$
$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

since $|\alpha e^{j\omega}| = |\alpha| < 1$.

To derive this result, we have used the following identity:

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q} \qquad \text{if } |q| < 1.$$

We will encounter this exponential series frequently. The sum can be easily computed using the following steps:

$$S = \sum_{n=0}^{M-1} q^n = 1 + q + q^2 + \ldots + q^{M-1}$$
$$S \cdot q = q + q^2 + \ldots + q^M$$
$$S - S \cdot q = S(1-q) = 1 - q^M$$
$$S = \frac{1-q^M}{1-q}.$$

If $|q| < 1$ and $M \longrightarrow +\infty$, then $q^M \longrightarrow 0$ and $S \longrightarrow \dfrac{1}{1-q}$.

Q.E.D.

From the definition of the DTFT of a sequence $x(n)$, we can notice that it is a continuous function of the normalized angular frequency ω. Unlike the CTFT, the DTFT is a periodic function of ω with a period of 2π. Indeed,

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j(\omega+2\pi k)n} =$$
$$= \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n} \cdot e^{-j2\pi k n} =$$

$$= \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} = X(e^{j\omega}),$$

where we used the fact that $e^{-j2\pi kn} = 1$ for k and n integers.

Note that $\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ is the Fourier series expansion of the periodic function $X(e^{j\omega})$.

Thus, the coefficients of the Fourier series, $x(n)$, can be computed from $X(e^{j\omega})$ using the Fourier integral:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega,$$

which is the *Inverse Discrete-Time Fourier Transform* (IDTFT).

The IDTFT can be interpreted as the linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}\mathrm{d}\omega$ weighted by the complex function $X(e^{j\omega})$.

Let us verify that

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega.$$

By replacing the definition of $X(e^{j\omega})$ with

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} x(l)e^{-j\omega l}$$

we have

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{l=-\infty}^{+\infty} x(l)e^{-j\omega l}e^{j\omega n}\mathrm{d}\omega =$$

by interchanging $\int$ and $\sum$:

$$= \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} x(l)\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega$$

For computing this integral, we will consider two cases.

If $n \neq l$,

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega = \int_{-\pi}^{+\pi} \cos\left[\omega(n-l)\right]\mathrm{d}\omega + j\int_{-\pi}^{+\pi} \sin\left[\omega(n-l)\right]\mathrm{d}\omega =$$

$$= \left.\frac{\sin\left[\omega(n-l)\right]}{(n-l)}\right|_{-\pi}^{+\pi} =$$

$$= \frac{2\sin\left[\pi(n-l)\right]}{(n-l)} = 0.$$

Here, we have first exploited the fact that $\sin\left[\omega(n-l)\right]$ is an odd function and has $\int_{-\pi}^{+\pi}\mathrm{d}\omega = 0$. Finally, we have used the fact that $\sin\left[\pi(n-l)\right] = 0$.

If $n = l$,

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega = \int_{-\pi}^{+\pi} e^{j\omega 0}\mathrm{d}\omega = 2\pi.$$

Thus, in general, it is

$$\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega = 2\pi\delta(n-l)$$

where $\delta(n)$ is the unit impulse sequence.
Eventually, we have

$$\frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} x(l)\int_{-\pi}^{+\pi} e^{j\omega(n-l)}\mathrm{d}\omega = \sum_{l=-\infty}^{+\infty} x(l)\delta(n-l) = x(n)$$

Q.E.D.

In general, the DTFT of a sequence is a complex function of the real variable ω:

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$$

with $X_{\mathrm{re}}(e^{j\omega})$ and $X_{\mathrm{im}}(e^{j\omega}) \in \mathbb{R}$.

$$X_{\mathrm{re}}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{j\omega})\right],$$
$$X_{\mathrm{im}}(e^{j\omega}) = \frac{1}{2j}\left[X(e^{j\omega}) - X^*(e^{j\omega})\right].$$

The DTFT can be expressed in polar form as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where $\left|X(e^{j\omega})\right|$ is called *magnitude spectrum*, and $\theta(\omega)$ is called the *phase spectrum*:

$$\theta(\omega) = \text{arctan}\frac{X_{\mathrm{im}}(e^{j\omega})}{X_{\mathrm{re}}(e^{j\omega})}$$

Note that there is an indetermination of $2\pi k$, with $k \in \mathbb{Z}$, in the knowledge of $\theta(\omega)$. Indeed, $\theta(\omega)$ is an angle measured in radians.
Like $X(e^{j\omega})$, also $X_{\mathrm{re}}(e^{j\omega})$, $X_{\mathrm{im}}(e^{j\omega})$, $\left|X(e^{j\omega})\right|$, and $\theta(\omega)$ are periodic functions of ω with period 2π. Thus, it suffices to know these functions for $-\pi < \omega \leq \pi$ to know them everywhere.

Symmetry properties of the DTFT of a real sequence
If $x(n) \in \mathbb{R}$:

$$X(e^{-j\omega}) = X^*(e^{j\omega}).$$

In fact:

$$\begin{aligned} X^*(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x^*(n)e^{j\omega n} \\ &= \sum_{n=-\infty}^{+\infty} x(n)e^{-j(-\omega)n} \\ &= X(e^{-j\omega}). \end{aligned}$$

For this relation, if $x(n) \in \mathbb{R}$ we have that

$$X_{\mathrm{re}}(e^{-j\omega}) = X_{\mathrm{re}}(e^{j\omega}) \qquad \text{is an even function,}$$

$$X_{\mathrm{im}}(e^{-j\omega}) = -X_{\mathrm{im}}(e^{j\omega}) \qquad \text{is an odd function,}$$

$$\left|X(e^{-j\omega})\right| = \left|X(e^{j\omega})\right| \qquad \text{is an even function,}$$

$$\theta(-\omega) = -\theta(\omega) \qquad \text{is an odd function.}$$

If $x(n)$ is real and even ($x(n) = x(-n)$),

$$X(e^{j\omega}) = X_{\rm re}(e^{j\omega}) \qquad \text{is an real function.}$$

Indeed:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} x(n)e^{j\omega n} + x(0) + \sum_{n=1}^{+\infty} x(n)e^{j\omega n}$$

(Change the variable of the first sum as follows $n' = -n$)

$$= \sum_{n'=1}^{+\infty} x(-n')e^{-j\omega n'} + x(0) + \sum_{n=1}^{+\infty} x(n)e^{j\omega n}$$

$$= x(0) + \sum_{n=1}^{+\infty} x(n)\left[e^{-j\omega n} + e^{j\omega n}\right]$$

$$= x(0) + \sum_{n=1}^{+\infty} x(n) 2\cos(\omega n)$$

Q.E.D.

If $x(n)$ is real and odd ($x(-n) = -x(n)$),

$$X(e^{j\omega}) = jX_{\rm im}(e^{j\omega}) \qquad \text{is an imaginary function.}$$

Convergence conditions

The series that defines the Fourier transform could or could not converge. We say that the DTFT exists if its series converges according to some criteria.
Sufficient conditions for the existence of the DTFT of a sequence $x(n)$ are the following:

- $x(n)$ is absolutely summable, i.e.,

$$\sum_{-\infty}^{+\infty} |x(n)| < +\infty$$

  In this case, we talk about *uniform convergence*.

- $x(n)$ is a finite energy signal

$$\sum_{-\infty}^{+\infty} |x(n)|^2 < +\infty$$

  In this case, we talk about *mean square convergence*.

These two conditions are quite restrictive. Note that the first condition is stronger than the second. An absolutely summable signal is always a finite energy signal because:

$$\sum_{-\infty}^{+\infty} |x(n)|^2 \leq \left(\sum_{-\infty}^{+\infty} |x(n)|\right)^2.$$

Example

An absolutely summable signal is the following causal exponential sequence

$$x(n) = \alpha^n \mu(n)$$

for $|\alpha| < 1$.

Indeed,

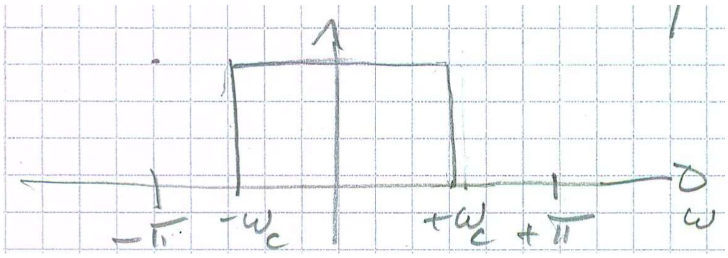
$$\sum_{n=-\infty}^{+\infty} |\alpha^n \mu(n)| = \sum_{n=0}^{+\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < +\infty.$$

We have already seen that its DTFT is $\frac{1}{1-\alpha e^{-j\omega}}$.

Example

A signal that is not absolutely summable (but has a DTFT) is the signal with the following ideal low-pass DTFT:

$$H_{\text{LP}}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



Let us compute the corresponding signal, which we will encounter frequently.

Let us compute the IDTFT:

$$h_{\text{LP}}(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H_{\text{LP}}(e^{j\omega}) e^{j\omega n} \mathrm{d}\omega,$$

for $-\infty < n < +\infty$ and $n \neq 0$:

$$\begin{aligned} h_{\text{LP}}(n) &= \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{j\omega n} \mathrm{d}\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right] \\ &= \frac{\sin(\omega_c n)}{\pi n}, \end{aligned}$$

for $n = 0$

$$h_{\text{LP}}(n) = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} \mathrm{d}\omega = \frac{\omega_c}{\pi}$$

Thus,

$$h_{\text{LP}}(n) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

Since

$$\frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n},$$

and since for the L'Hopital's rule $\lim_{x\to 0} \frac{\sin x}{x} = 1$, assuming by convention that $\frac{\sin(\omega_c 0)}{\omega_c 0} = 1$, we can write for all n

$$h_{\text{LP}}(n) = \frac{\sin(\omega_c n)}{\pi n}.$$

Note that $\sum_{-\infty}^{+\infty} |h_{\text{LP}}(n)| = +\infty$, i.e., the sequence is not absolutely summable, but it admits DTFT because it is a finite energy sequence.

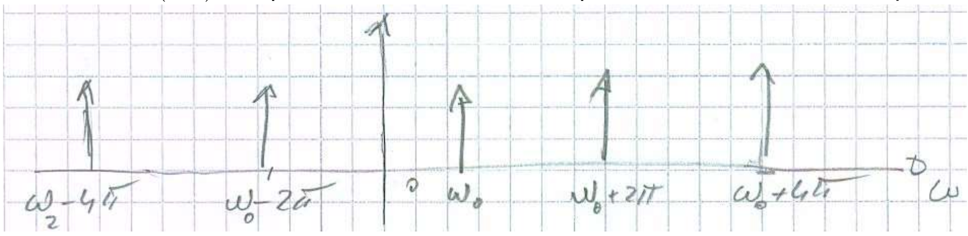
The DTFT can be defined also for a certain class of sequences that are not absolutely summable, nor have finite energy, e.g., the unit step sequence, the sinusoidal sequence, or the complex exponential sequence. In these cases the expression of the DTFT involves Dirac delta functions.

Example

The DTFT of the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$

The DTFT $X(e^{j\omega})$ is a periodic function in ω with period 2π and it is called a *periodic pulse train*.



Let us prove this relation by computing the IDTFT:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k) e^{j\omega n} \mathrm{d}\omega$$

If $-\pi < \omega_o \leq \pi$, the only non-null function in the interval $[-\pi, +\pi]$ is $\delta(\omega - \omega_0)$, thus

$$x(n) = \frac{1}{2\pi} 2\pi \int_{-\pi}^{+\pi} \delta(\omega - \omega_0) e^{j\omega n} \mathrm{d}\omega = e^{j\omega_0 n}$$

for the properties of $\delta(\omega)$.

Q.E.D

03.04 Properties of DTFT

Let us assume that $G(e^{j\omega})$ is the DTFT of a sequence $g(n)$ and that $H(e^{j\omega})$ is the DTFT of a sequence $h(n)$. The following properties hold.

Linearity property

If $x(n) = \alpha g(n) + \beta h(n)$, with α and β constants, the DTFT of $x(n)$ is $X(e^{j\omega}) = \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$.

$$\alpha g(n) + \beta h(n) \overset{DTFT}{\longleftrightarrow} \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} (\alpha g(n) + \beta h(n))\, e^{-j\omega n}$$
$$= \alpha \sum_{n=-\infty}^{+\infty} g(n) e^{-j\omega n} + \beta \sum_{n=-\infty}^{+\infty} h(n) e^{-j\omega n}$$
$$= \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

Q.E.D.

Time reversal

The DTFT of $g(-n)$ is $G(e^{-j\omega})$.

$$g(-n) \overset{DTFT}{\longleftrightarrow} G(e^{-j\omega})$$

Proof:

$$\sum_{n=-\infty}^{+\infty} g(-n) e^{-j\omega n} = \sum_{n'=-\infty}^{+\infty} g(n') e^{-j\omega(-n')} =$$

(where we have considered the following change of variables $n' = -n$)

$$= \sum_{n'=-\infty}^{+\infty} g(n') e^{-j(-\omega)n'}$$

Q.E.D.

Time-shifting

The DTFT of $g(n - n_0)$ is $e^{-j\omega n_0} G(e^{j\omega})$.

$$g(n - n_0) \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0} G(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{+\infty} g(n - n_0) e^{-j\omega n} = \sum_{n'=-\infty}^{+\infty} g(n') e^{-j\omega(n' + n_0)} =$$

(where we have considered the following change of variables $n' = n - n_0$)

$$= \sum_{n'=-\infty}^{+\infty} g(n') e^{-j\omega n'} e^{-j\omega n_0}$$

Q.E.D.

Frequency-shifting

The DTFT of $e^{j\omega_0 n} g(n)$ is $G(e^{j(\omega - \omega_0)})$.

$$e^{j\omega_0 n} g(n) \overset{DTFT}{\longleftrightarrow} G(e^{j(\omega - \omega_0)})$$

Proof:

$$\sum_{n=-\infty}^{+\infty} g(n)e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} g(n)e^{-j(\omega-\omega_0)n}$$

Q.E.D.

Frequency differentiation

The DTFT of $ng(n)$ is $j\frac{\mathrm{d}G(e^{j\omega})}{\mathrm{d}\omega}$.

$$ng(n) \overset{DTFT}{\longleftrightarrow} j\frac{\mathrm{d}G(e^{j\omega})}{\mathrm{d}\omega}$$

Proof: Omitted.

Modulation

The DTFT of $g(n) \cdot h(n)$ is $\frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\mathrm{d}\theta$.

$$g(n) \cdot h(n) \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\mathrm{d}\theta$$

Proof:

$$y(n) = g(n) \cdot h(n)$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} g(n)h(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta})e^{j\theta n}\mathrm{d}\theta \; h(n)e^{-j\omega n}$$

(where we have replaced $g(n)$ with its IDTFT. By exchanging $\sum$ and $\int$ ...)

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta}) \sum_{n=-\infty}^{+\infty} h(n)e^{-j\omega n}e^{j\theta n}\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta}) \sum_{n=-\infty}^{+\infty} h(n)e^{-j(\omega-\theta)n}\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\mathrm{d}\theta$$

Q.E.D.

The last integral is called *convolution integral*. We will introduce the concept of convolution in one of the following lessons.

Parseval theorem

$$\sum_{n=-\infty}^{+\infty} g(n)h^*(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})H^*(e^{j\omega})\mathrm{d}\omega$$

Proof: (Similar to the previous one)

$$\sum_{n=-\infty}^{+\infty} g(n)h^*(n) = \sum_{n=-\infty}^{+\infty} g(n)\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} H^*(e^{j\omega})e^{-j\omega n}\mathrm{d}\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{+\infty} g(n)e^{-j\omega n}\right)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})H^*(e^{j\omega})\mathrm{d}\omega$$

Q.E.D.

The linearity property, the time-shift property, the frequency shift property, the modulation property and the Parseval theorem find similar formulations also with the CTFT.

Examples using these theorems:

Let us compute the DTFT of the finite length exponential sequence:

$$y(n) = \begin{cases} \alpha^n & 0 \le n < M-1 \\ 0 & \text{otherwise} \end{cases}$$

with $|\alpha| < 1$.

$$y(n) = \alpha^n \mu(n) - \alpha^n \mu(n-M)$$

$$= \alpha^n \mu(n) - \alpha^M \alpha^{n-M} \mu(n-M)$$

We already know that:

$$\alpha^n \mu(n) \stackrel{DTFT}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

For the linearity and the time shift properties we have

$$Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} - \alpha^M \frac{e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1-\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

Let us compute the DTFT of the causal sequence $v(n)$ defined by the finite difference equation:

$$d_0 v(n) + d_1 v(n-1) = p_0 \delta(n) + p_1 \delta(n-1)$$

with $d_0 \neq 0$, and $d_1 \neq 0$.

Let us apply the DTFT to both terms of the finite difference equation. By exploiting the linearity and time-shit properties and remembering that DTFT$\{\delta(n)\} = 1$, we have

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

Thus,

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}.$$

Let us compute the DTFT of $y(n) = (-1)^n \alpha^n \mu(n)$ with $|\alpha| < 1$.

It is also $y(n) = e^{j\pi n}\left(\alpha^n \mu(n)\right) = e^{j\pi n} x(n)$, where we considered $x(n) = \alpha^n \mu(n)$.

We can apply the frequency-shifting property:

$$Y(e^{j\omega}) = X(e^{j(\omega-\pi)}) = \frac{1}{1-\alpha e^{-j(\omega-\pi)}} = \frac{1}{1+\alpha e^{-j\omega}}.$$

Energy density spectrum

The total energy of a sequence $g(n)$ is given by

$$E_g = \sum_{n=-\infty}^{+\infty} |g(n)|^2 .$$

By applying the Parseval theorem with $h(n) = g(n)$ we have

$$E_g = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left|G(e^{j\omega})\right|^2 \mathrm{d}\omega.$$

$S_{gg}(e^{j\omega}) = \left|G(e^{j\omega})\right|^2$ is called *energy density spectrum*, and it defines how the energy of the sequence is distributed over the frequency spectrum.

Example:

Let us compute the energy of the low-pass ideal signal

$$\sum_{n=-\infty}^{+\infty} |h_{\mathrm{LP}}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left|H_{\mathrm{LP}}(e^{j\omega})\right|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} \mathrm{d}\omega = \frac{\omega_c}{\pi} < +\infty$$

Band-limited signals

In general, the spectrum of a discrete-time signal is defined on the entire frequency range $-\pi < \omega \leq +\pi$.

A band-limited signal has a spectrum that is limited to part of this range.

An ideal *low-pass* signal has

$$X(e^{j\omega}) \begin{cases} \neq 0 & 0 \leq |\omega| \leq \omega_p \\ = 0 & \omega_p < |\omega| \leq \pi \end{cases}$$

ω_p is called signal *bandwidth*.

An ideal *high-pass* signal has

$$X(e^{j\omega}) \begin{cases} = 0 & 0 \leq |\omega| < \omega_p \\ \neq 0 & \omega_p \leq |\omega| \leq \pi \end{cases}$$

The signal bandwidth is given by $\pi - \omega_p$.

An ideal *passband* signal has

$$X(e^{j\omega}) \begin{cases} = 0 & 0 \leq |\omega| < \omega_a \\ \neq 0 & \omega_a \leq |\omega| \leq \omega_b \\ = 0 & \omega_b < |\omega| < \pi \end{cases}$$

The signal bandwidth is given by $\omega_b - \omega_a$.

03.05 The sampling theorem

Most of the signals we encounter in the real world are continuous-time signals (e.g. voice, music, images). Digital signal processing algorithms are often applied to process these continuous-time signals. To do that, signals are first sampled and coded with an A/D converter, then they are digitally processed, and finally, they are converted back to the analog form with a D/A converter.
We want to study the relations that link the continuous-time signals with the corresponding discrete-time signals.
Let us consider a continuous-time signal $g_a(t)$. We assume to uniformly sample it with sampling period T. We obtain the sequence:

$$g(n) = g_a(nT) \qquad -\infty < n < +\infty.$$

The sampling frequency is $F_T = \dfrac{1}{T}$. The continuous-time Fourier transform, CTFT, of $g_a(t)$ is

$$G_a(j\Omega) = \int_{-\infty}^{+\infty} g_a(t)e^{-j\Omega t}\mathrm{d}t,$$

and the DTFT of $g(n)$ is

$$G(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} g(n)e^{-j\omega n}.$$

We want to find the relation that exists between $G_a(j\Omega)$ and $G(e^{j\omega})$.
Mathematically, we can consider the sampling operations as the product of a signal $g_a(t)$ and a periodic pulse train $p(t)$, where

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT),$$

composed of an infinite sequence of Dirac impulses uniformly spaced in time and separated by the period T.
The multiplication between $g_a(t)$ and $p(t)$ results in a continuous-time function

$$g_p(t) = g_a(t) \cdot p(t) = \sum_{n=-\infty}^{+\infty} g_a(nT)\,\delta(t - nT),$$

which is also composed of infinite Dirac impulses, placed at time $t = nT$, and weighted by the sample value $g_a(nT)$.
We will provide two expressions for the CTFT of $g_p(t)$. The first one is derived directly from the last expression of $g_p(n)$ considering that the CTFT of $\{\delta(t - nT)\}$ is $e^{-j\Omega nT}$.
For the linearity property of the CTFT:

$$G_p(j\Omega) = \sum_{n=-\infty}^{+\infty} g_a(nT)e^{-j\Omega nT}.$$

By comparing this relation with the expression of $G(e^{j\omega})$, we see that

$$G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\frac{\omega}{T}}$$

$$G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}$$

The DTFT $G(e^{j\omega})$ coincides with the CTFT of $g_p(n)$, apart from a frequency axis normalization. This normalization maps the point at $\omega = 2\pi$ of $G(e^{j\omega})$ to the point at $\Omega_T = 2\pi F_T$ of $G_p(j\Omega)$.

In what follows, we find a second form for $G_p(j\Omega)$.

Assume you have two continuous time signals $a(t)$ and $b(t)$ with CTFT $A(j\Omega)$ and $B(j\Omega)$, respectively. The CTFT of the product $y(t) = a(t) \cdot b(t)$ is

$$Y(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(j\Psi) \cdot B\big(j(\Omega - \Psi)\big)\mathrm{d}\Psi$$

Proof:

$$Y(j\Omega) = \int_{-\infty}^{+\infty} a(t) \cdot b(t)e^{-j\Omega t}\mathrm{d}t =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(j\Psi)e^{j\Psi t}\mathrm{d}\Psi \cdot b(t)e^{-j\Omega t}\mathrm{d}t =$$

(obtained by replacing $a(t)$ with its ICTFT)

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(j\Psi)\int_{-\infty}^{+\infty} b(t)e^{-j(\Omega-\Psi)t}\mathrm{d}t\mathrm{d}\Psi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(j\Psi)B\big(j(\Omega - \Psi)\big)\mathrm{d}\Psi$$

Q.E.D.

We can use the property we have just proved to evaluate the CTFT of

$$g_p(t) = g_a(t) \cdot p(t).$$

We assume knowing the CTFT of $g_a(t)$. Let us compute the CTFT of the pulse train $p(t)$. In what follows, we prove that

$$P(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\Omega - k\frac{2\pi}{T})$$

i.e., the CTFT of $p(t)$ is also a uniformly spaced pulse train.

Proof:

Note that the pulse train $p(t)$ is a periodic signal with a period of T. Thus, we can expand $p(t)$ with the Fourier series:

$$p(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}$$

with

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)e^{-j\frac{2\pi}{T}kt}\mathrm{d}t =$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-j\frac{2\pi}{T}kt}\mathrm{d}t =$$

(because in the interval $[-\frac{T}{2}, +\frac{T}{2}]$ only one pulse, $\delta(t)$, is different from 0)

$$= \frac{1}{T} e^{-j\frac{2\pi}{T}k0} = \frac{1}{T}$$

Thus,

$$p(t) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt}$$

and for the linearity of the CTFT

$$P(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} \text{CTFT}\left\{e^{j\frac{2\pi}{T}kt}\right\}.$$

We can easily verify that

$$\text{CTFT}\left\{e^{j\frac{2\pi}{T}kt}\right\} = 2\pi\delta(\Omega - \frac{2\pi}{T}k)$$

Indeed, the IDFT of the right-hand side is

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\delta(\Omega - \frac{2\pi}{T}k) e^{j\Omega t} \mathrm{d}\Omega = e^{j\frac{2\pi}{T}kt}$$

(for the properties of the Dirac function).
Replacing CTFT $\left\{e^{j\frac{2\pi}{T}kt}\right\}$ in the expression of $P(j\Omega)$, we have

$$P(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\Omega - k\frac{2\pi}{T}).$$

We can now compute $G_p(j\Omega)$

$$G_p(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_a(j\Psi) \cdot P\big(j(\Omega - \Psi)\big) \mathrm{d}\Psi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_a(j\Psi) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\Omega - \Psi - k\frac{2\pi}{T}) \mathrm{d}\Psi =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_a(j\Psi) \cdot \delta(\Omega - \Psi - k\frac{2\pi}{T}) \mathrm{d}\Psi =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\big(j(\Omega - k\frac{2\pi}{T})\big) =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

with $\Omega_T = \frac{2\pi}{T}$.
Consequently,

$$G(e^{j\omega}) = G_p(j\Omega)|_{\Omega = \frac{\omega}{T}} =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\big[j\big(\frac{\omega}{T} + k\Omega_T\big)\big] =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\big[j\big(\frac{\omega + 2\pi k}{T}\big)\big].$$

The DTFT is given by the periodic repetition, with a period of 2π, of the continuous spectrum $G_a\left(j\frac{\omega}{T}\right)$. $G_a\left(j\frac{\omega}{T}\right)$ is identical to $G_a\left(j\Omega\right)$, but the frequency axis has been normalized such that $\omega = 2\pi$ corresponds to the angular sampling frequency $\Omega_T = \frac{2\pi}{T}$.
In order to avoid any overlap between the repeated spectra, it must be

$$G_a\left(j\frac{\omega}{T}\right) = 0 \quad \text{for} \quad |\omega| > \pi,$$

or, equivalently,

$$G_a\left(j\Omega\right) = 0 \quad \text{for} \quad |\Omega| > \frac{\pi}{T} = \frac{\Omega_T}{2}.$$

If this condition is satisfied, the discrete spectrum reproduces in the band $[-\pi, +\pi]$ the continuous spectrum.
If this condition is not satisfied, we have a distortion caused by the overlap of the tails of the spectrum. This distortion is called *aliasing*.
The frequency $\frac{1}{2T} = \frac{F_T}{2}$ is called *Nyquist frequency*.
Similarly, the angular frequency $\frac{\Omega_T}{2}$ is called *Nyquist angular frequency*.
Let us assume, for example, that our signal occupies the band $[-\Omega_m, +\Omega_m]$.
If $\Omega_T \geq 2\Omega_m$, the different repetitions of the spectrum do not overlap.
If $\Omega_T < 2\Omega_m$, the repetitions of the spectrum overlap, and we have an aliasing error.
See Figure 03.01.
If the continuous-time signal spectrum is preserved in the discrete domain (i.e., if we do not have aliasing), we can reconstruct the original signal from the samples $g(n) = g_a(nT)$.
For this purpose, let us build the signal:

$$g_p(t) = \sum_{n=-\infty}^{+\infty} g(n)\delta(t - nT) =$$

$$= \sum_{n=-\infty}^{+\infty} g_a(nT)\delta(t - nT)$$

We know its CTFT:

$$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} G_a\left[j(\Omega + k\Omega_T)\right].$$

This continuous spectrum is given by the periodic repetition of the spectrum $G_a(j\Omega)$ with period Ω_T.
If the signal has been sampled with a frequency $\Omega_T > 2\Omega_m$, where Ω_m is the maximum signal frequency of $g_a(t)$, we do not have aliasing, and the repetitions of $G_a(j\Omega)$ do not overlap. Thus, we can faithfully reconstruct the signal $g_a(t)$ by passing the signal $g_p(t)$ through a filter (i.e., a device, a circuit) that lets the spectrum in the band $[-\Omega_m, \Omega_m]$ pass without any alteration, while it stops all signal components outside that frequency range. We will see that such a filter is an ideal low-pass filter with bandwidth Ω_m.
We have just proved the theorem that is at the basis of all signal processing and modern telecommunications, the sampling theorem.

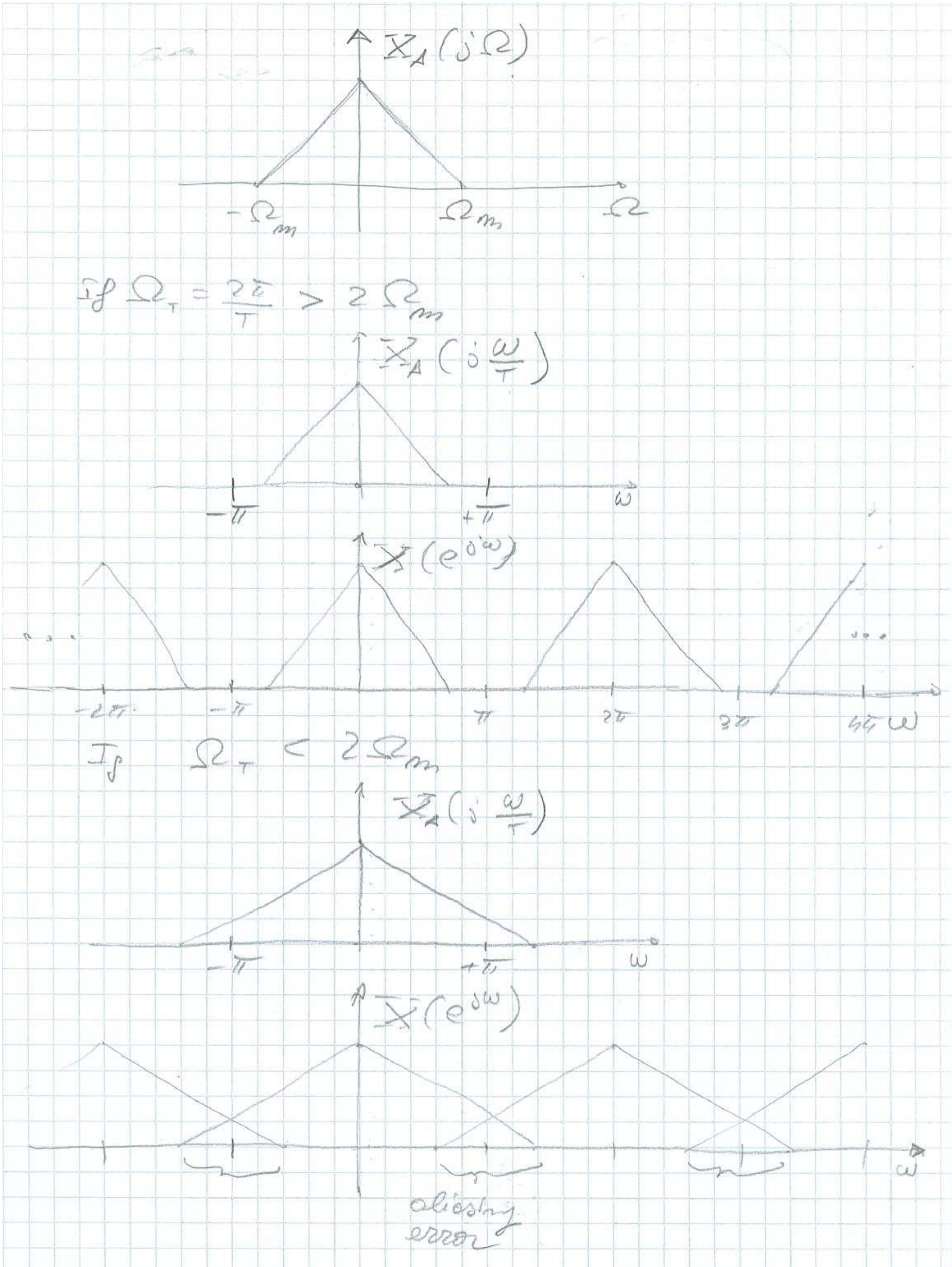


Figure 03.01: Illustration of sampling theorem.

The sampling theorem
(also called *Nyquist-Shannon theorem*)

Let $g_a(t)$ be a band-limited signal, with $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$. Then, $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, (i.e., can be faithfully reconstructed by its samples $g_a(nT)$,)

$-\infty < n < +\infty$, if the angular sampling frequency

$$\Omega_T \geq 2\Omega_m$$

with $\Omega_T = \dfrac{2\pi}{T}$.

In other words, if we want to be able to recover the band-limited signal $g_a(t)$ from its samples, we must sample the signal with a frequency at least twice the signal bandwidth.
The signal can be recovered by generating the signal

$$g_p(t) = \sum_{n=-\infty}^{+\infty} g(n)\delta(t - nT)$$

and by filtering this signal with an ideal low-pass filter.
In general, real-world signals are not band-limited. They occupy an infinite band, but most of their energy is concentrated at the low frequencies.
To avoid distortions due to aliasing, the signal is typically filtered with a low-pass filter before sampling. This filter cuts off high frequencies and generates a band-limited signal. Such a filter is called an *anti-aliasing* filter. After filtering, the signal can be safely sampled with a frequency greater than $2\Omega_m$ and processed as desired.

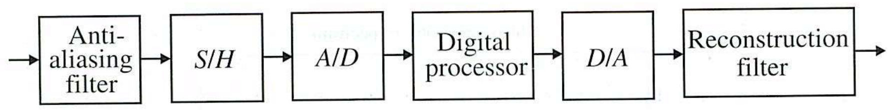


Figure 3.12: Block diagram representation of the discrete-time digital processing of a continuous-time signal.

(From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

Also the signal that comes out of the Digital to Analog (D/A) converter[2] is filtered with a low-pass filter, called the *reconstruction filter* or *anti-imaging filter*. In this way, all the frequencies (the images) outside the band of the original signal, $[-\Omega_m, +\Omega_m]$ are removed.
For example, to sample the voice in telephone applications, we exploit the fact that most of the voice energy falls between 300 and 3400 Hz. The voice is then sampled at 8kHz, which is greater than $2 \cdot 3.4$kHz.
In audio CDs, the musical signal has a bandwidth ranging from 0 to 20kHz and is sampled at 44.1kHz.
In DAT and DVDs, for the same musical signal, a larger bandwidth has been considered, and it has been sampled at 48kHz.

For more information study:

S. K. Mitra, "Digital Signal Processing: a computer based approach," 4th edition, McGraw-Hill, 2011
Chapter 3.1, pp. 89-93
Chapter 3.2-3.5, pp. 94-112
Chapter 3.8, pp. 115-124

[2] You can think of this signal as being $g_p(t)$, even though in reality, the Dirac pulses are replaced by rectangular pulses.