07 Discrete-time LTI systems in the frequency domain

07.01 Ideal filters

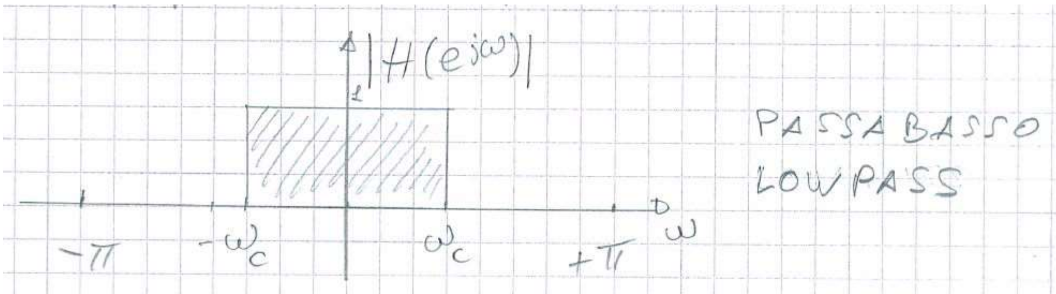
One of the most important applications of LTI systems is allowing the passage of certain frequency components of the signal without any distortion while simultaneously blocking all other frequency components. For this reason, LTI systems are also defined as *'filters.'* The two terms, 'LTI system' and 'filter,' can be considered synonymous. In fact, the term 'filter' is used not only for systems that are frequency-selective but also for all systems that realize an appropriate weighting (a 'spectral shaping') of the signal spectrum:

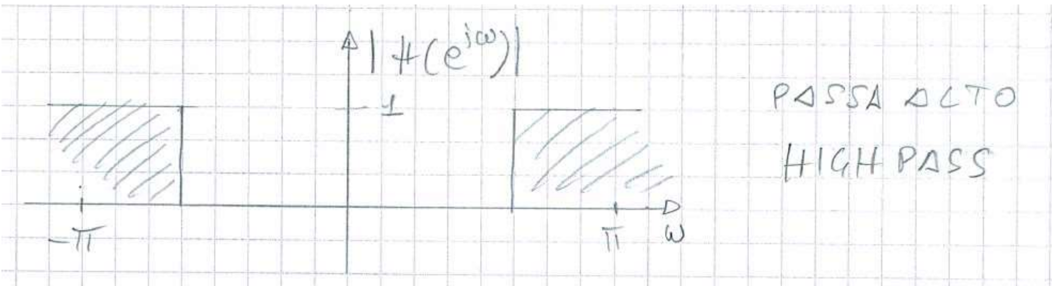
$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$$

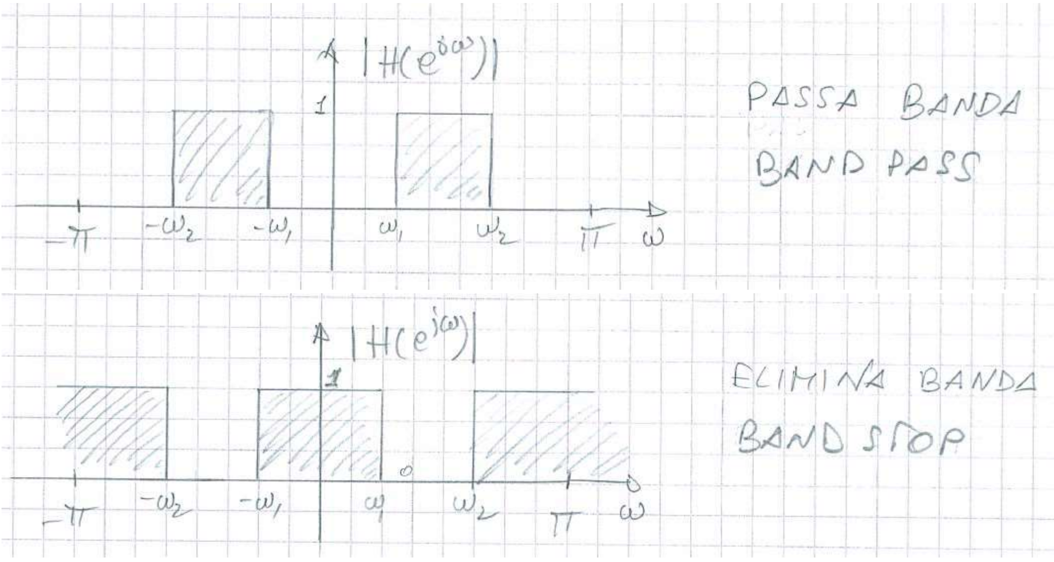
Filters can be classified according to their frequency domain characteristics as

- lowpass filters,
- highpass filters,
- bandpass filters,
- bandstop filters.

The ideal characteristics of the frequency responses of these filters are as follows:







An *ideal filter* allows certain frequency components to pass unaltered while completely eliminating all other frequency components. Therefore, an ideal filter exhibits a unit magnitude (or amplitude) response in the passband and a zero response in the stopband. Additionally, an ideal filter must feature linear phase in the passband.

Considering a signal $\{x(n)\}$ with a spectrum entirely within the band $\omega_1 \leq |\omega| \leq \omega_2$, let's filter it with a frequency response given by:

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_0} & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

In this case, the output signal has a spectrum given by:

$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega}).$$

This implies that:

$$y(n) = x(n - n_0).$$

The output signal coincides with the input signal except for a delay of n_0. Generally, a pure delay is tolerable and is not considered signal distortion. If the ideal filter has linear phase, the signal component in the passband is delayed without distortion. Thus, the ideal phase response is linear in the passband:

$$\theta(\omega) = -\omega n_0.$$

In practice, we are contented with imposing $-\frac{\mathrm{d}\theta}{\mathrm{d}\omega}$ to be constant in the passband, i.e., with imposing the group delay to be constant in the passband.

07.02 Phase delay and Group delay

Let's consider a LTI system with frequency response $H(e^{j\omega})$, and let $\theta(\omega)$ be its phase response,

$$H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\theta(\omega)}.$$

For simplicity, let's assume the system to be real, so that

$$|H(e^{j\omega})| = |H(e^{-j\omega})| \qquad \text{and} \qquad \theta(\omega) = -\theta(-\omega)$$

If we consider a sinusoidal sequence with normalized angular frequency ω_0 as system input:

$$x(n) = \cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2},$$

the output of the system is

$$y(n) = \frac{1}{2}|H(e^{j\omega_0})|e^{j\theta(\omega_0)}e^{j\omega_0 n} + \frac{1}{2}|H(e^{-j\omega_0})|e^{j\theta(-\omega_0)}e^{-j\omega_0 n} =$$

$$= \frac{1}{2}|H(e^{j\omega_0})| \left(e^{j(\omega_0 n+\theta(\omega_0))} + e^{-j(\omega_0 n+\theta(\omega_0))}\right) =$$

$$= |H(e^{j\omega_0})| \cos[\omega_0 n + \theta(\omega_0)] =$$

$$= |H(e^{j\omega_0})| \cos\left[\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right)\right] =$$

$$= |H(e^{j\omega_0})| \cos\left[\omega_0\left(n - t_p(\omega_0)\right)\right]$$

The system output is the same sinusoidal sequence delayed by

$$t_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}.$$

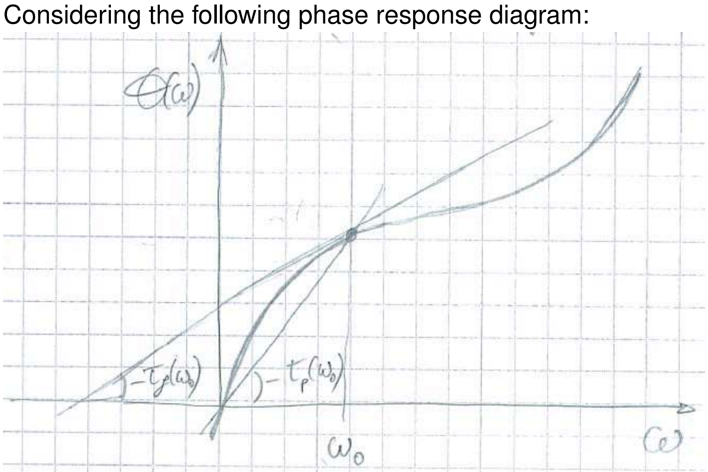
$t_p(\omega)$ is referred to as the *Phase Delay*, representing the delay of a sinusoidal component as it passes from the input to the output of the system.

However, when we consider a signal composed of multiple frequency components (several sinusoids), each component passing through the system experiences a different delay. In such cases, the delay introduced by the system on the signal is assessed using another parameter known as the *Group Delay*, defined as:

$$t_g(\omega) = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega}.$$

It's important to note that both $t_p(\omega)$ and $t_g(\omega)$ vary with frequency.

Considering the following phase response diagram:

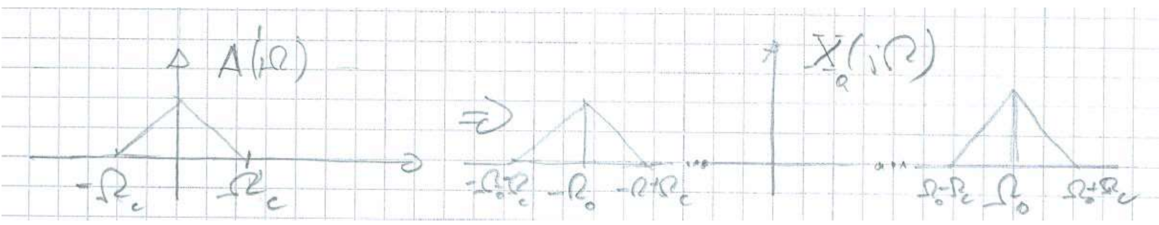


the group delay $t_g(\omega)$ corresponds to the opposite of the slope of $\theta(\omega)$ in ω_0. On the contrary, the phase delay $t_p(\omega)$ corresponds to the opposite of the slope of the line connecting the origin with the point $(\omega_0, \theta(\omega_0))$.

Why is the delay introduced by the system on the signal evaluated using the group delay $t_g(\omega)$?
This choice is particularly relevant in Amplitude Modulation (AM) systems, where the group delay represents the delay introduced on the modulated signal. In contrast, the phase delay corresponds to the delay of the modulating signal, i.e., the carrier.
Let's return to the continuous-time domain and consider a lowpass signal (for example, think about a musical signal), $a(t)$, with a passband $[-\Omega_c, \Omega_c]$. Now, let's modulate this signal with a carrier having an angular frequency $\Omega_0 \gg \Omega_c$. In other words, we multiply the signal by a sinusoidal signal with an angular frequency Ω_0:

$$x_a(t) = a(t) \cdot \cos(\Omega_0 t) = a(t) \cdot \left[e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right]/2$$

$$X_a(j\Omega) = \frac{1}{2}A\left(j(\Omega - \Omega_0)\right) + \frac{1}{2}A\left(j(\Omega + \Omega_0)\right)$$



The spectrum of the signal $a(t)$ is translated to $\pm\Omega_0$ and occupies the band[1]
$\pm\left[\Omega_0 - \Omega_c, \Omega_0 + \Omega_c\right]$.
Let us assume that the signal $x_a(t)$ passes through an LTI system with frequency response $H(j\Omega)$[2]. Since $\Omega_c \ll \Omega_0$, within the band $[\Omega_0 - \Omega_c, \Omega_0 + \Omega_c]$, we can assume the amplitude response of the LTI system to be constant (for simplicity, let's assume it equals 1). Additionally, we can approximate the phase response with a linear response:

$$\theta(\Omega) \simeq \theta(\Omega_0) + \left.\frac{\mathrm{d}\theta(\Omega)}{\mathrm{d}\Omega}\right|_{\Omega=\Omega_0}(\Omega - \Omega_0) =$$

$$= -t_p(\Omega_0) \cdot \Omega_0 - t_g(\Omega_0) \cdot (\Omega - \Omega_0)$$

For $\Omega > 0$, the output signal spectrum is:

$$Y_a(j\Omega) = \frac{1}{2}A\left(j(\Omega - \Omega_0)\right) e^{-jt_p(\Omega_0)\Omega_0} e^{-jt_g(\Omega_0)(\Omega - \Omega_0)}.$$

[1] With the AM modulation, the bandwidth of the signal is twice that of the "baseband" signal $a(t)$.

[2] Similar to discrete-time LTI systems, for continuous-time systems, the impulse response and the frequency response are sufficient to completely characterize the LTI system. The impulse response of a continuous-time system, denoted as $h(t)$, is the response to a Dirac pulse $\delta(t)$, and the system output is given by:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)\mathrm{d}\tau.$$

This expression is known as the convolution integral. The frequency response, $H(j\Omega)$, is the CTFT of the impulse response $h(t)$ and can be expressed as:

$$Y(j\Omega) = H(j\Omega)X(j\Omega)$$

where $X(j\Omega)$ and $Y(j\Omega)$ are the CTFTs of $x(t)$ and $y(t)$, respectively. The definitions of phase response, magnitude (or amplitude) response, phase delay, and group delay remain the same as in the discrete-time case.

For $\Omega < 0$, the output signal spectrum is the conjugate symmetric of the spectrum for $\Omega > 0$.
It is easy to verify that the system output is given by:

$$y_a(t) = a(t - t_g)\cos[\Omega_0(t - t_p)]$$

with $t_g = t_g(\Omega_0)$ and $t_p = t_p(\Omega_0)$. In fact, $a(t - t_g)$ has spectrum $A(j\Omega) \cdot e^{-jt_g\Omega}$. When $a(t - t_g)$ is multiplied by $\cos[\Omega_0(t - t_p)] = \dfrac{e^{j\Omega_0(t-t_p)} - e^{-j\Omega_0(t+t_p)}}{2}$, two components with conjugate symmetry are generated: one is centered at Ω_0, the other at $-\Omega_0$. Let's consider the component for $\Omega > 0$, originating from

$$\frac{1}{2}a(t - t_g)e^{j\Omega_0(t-t_p)} = \frac{1}{2}a(t - t_g)e^{-j\Omega_0 t_p}e^{j\Omega_0 t}$$

Due to the linearity and frequency shift properties of the Continuous-Time Fourier Transform (CTFT), the spectrum is:

$$\frac{1}{2}A\left(j(\Omega - \Omega_0)\right)e^{-j(\Omega-\Omega_0)t_g}e^{-j\Omega_0 t_p}$$

(because $e^{-j\Omega_0 t_p}$ is a constant, and the product $e^{j\Omega_0 t}K(t)$ has spectrum $K\left(j(\Omega - \Omega_0)\right)$). which is the expression of $Y_a(j\Omega)$ we has seen before.
Since $y_a(t) = a(t - t_g) \cdot \cos[\Omega_0(t - t_p)]$, we observe that the group delay (t_g) represents the delay of the baseband signal $a(t)$,while the phase delay (t_p) corresponds to the delay of the carrier $\cos[\Omega_0 t]$.

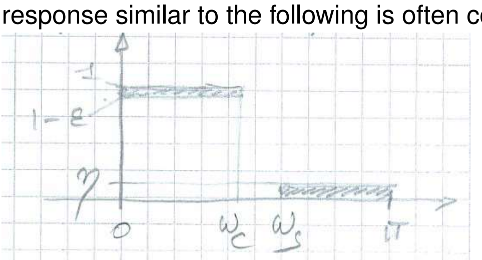
Let's redirect our focus to ideal filters. As discussed earlier, an ideal filter is characterized by a unit amplitude and linear phase in the passband, or, at the very least, it must exhibit a constant group delay in the passband to ensure that all signal components experience the same delay. Unfortunately, ideal filters are not realizable. To illustrate, consider the ideal lowpass filter with the frequency response:

$$H_{\mathrm{LP}}(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

This filter has an impulse response:

$$h_{\mathrm{LP}}(n) = \frac{\sin(\omega_c n)}{\pi n} \qquad\qquad -\infty < n < +\infty$$

It's essential to note that this filter is not causal and is, in fact, an unstable system because $h_{\mathrm{LP}}(n)$ is not absolutely summable.
To achieve stable and realizable filters, we relax the stringent conditions imposed by ideal filters. One key modification involves introducing a transition band between the passband and the stopband. This enables the magnitude response to gradually decay from its maximum value to zero. Additionally, the magnitude response is permitted to vary within specified bounds in both the passband and the stopband.
For instance, when designing a lowpass filter, a frequency mask that defines bounds for the frequency response similar to the following is often considered:

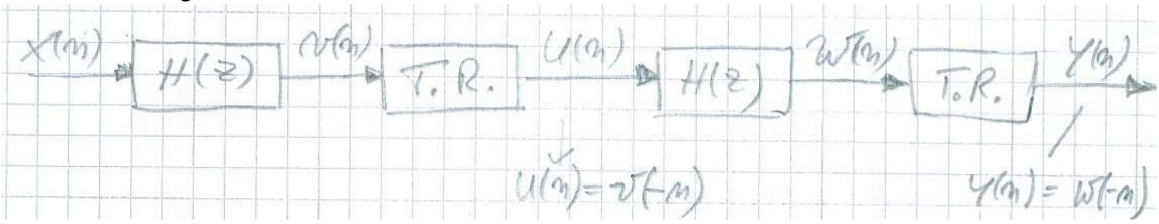


In practice, the following constraints are taken into account for $|H(e^{j\omega})|$:

$$\begin{cases} 1-\epsilon < |H(e^{j\omega})| < 1 & 0 \le \omega \le \omega_c \\ |H(e^{j\omega})| < \eta & \omega_s \le \omega \le \pi \end{cases}$$

07.03 Zero-phase filters

In many applications, it is crucial to design digital filters in a manner that introduces no phase distortion to the input signal components in the passband. One effective approach to avoiding phase distortions is the implementation of a *zero-phase filter*, characterized by a real positive frequency response. If we do not work in real-time and we process real sequences of finite duration, the zero-phase filtering can be easily implemented if we drop the hypothesis of system causality.
In this block diagram:



the input signal is processed with a filter $H(z)$ having real coefficients; the output of this filter is time-reversed and it is again filtered with the same filter $H(z)$, whose output is folded again. Let us prove that this is a zero-phase system.

$$V(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(e^{j\omega})$$

(because $v(n)$ is real)

$$W(e^{j\omega}) = H(e^{j\omega}) \cdot U(e^{j\omega})$$

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega}) \cdot U^*(e^{j\omega}) = H^*(e^{j\omega}) \cdot V(e^{j\omega}) =$$
$$= H^*(e^{j\omega}) \cdot H(e^{j\omega}) \cdot X(e^{j\omega}) = |H(e^{j\omega})|^2 \cdot X(e^{j\omega})$$

The system introduced above implements a filter with a frequency response $|H(e^{j\omega})|^2$, which is positive real, ensuring a zero-phase characteristic.
To design a zero-phase filter with a given magnitude response $A(e^{j\omega})$, one can design a filter with the magnitude response $\sqrt{A(e^{j\omega})}$, without imposing constraints on the phase. Subsequently, the technique described earlier can be applied. However, a notable drawback of this approach is that real-time signal processing becomes impossible. The entire sequence must be recorded before the technique can be applied.
For real-time processing systems, meeting our specifications while ensuring system causality often involves accepting a certain delay introduced by the filter and considering a linear phase response. We will explore that it is always possible to design linear phase FIR filters, whereas achieving linear phase IIR filters is impossible.

07.04 Linear phase FIR filters

In the upcoming discussion, we will demonstrate that a causal FIR filter with real coefficients, having a length of $N+1$, and a transfer function given by

$$H(z) = \sum_{n=0}^{N} h(n)z^{-n} = h(0) + h(1)z^{-1} + \ldots + h(n)z^{-N}$$

(note that now N is the exponent of the polynomial in z^{-1}), exhibits linear phase when the impulse response $h(n)$ is symmetric,

$$h(n) = h(N-n) \qquad \text{for } 0 \le n \le N,$$

or is antisymmetric,

$$h(n) = -h(N-n) \qquad \text{for } 0 \le n \le N.$$

Considering that the length can be either even or odd, we can categorize linear FIR filters with linear phase into four classes:

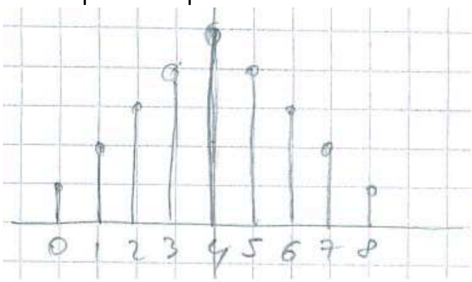
- TYPE 1: $h(n)$ is symmetric and has odd length,
- TYPE 2: $h(n)$ is symmetric and has even length,
- TYPE 3: $h(n)$ is antisymmetric and has odd length,
- TYPE 4: $h(n)$ is antisymmetric and has even length.

Let us analyze one by one these four classes.

TYPE 1: $h(n)$ is symmetric and has odd length. Thus, N is even.
Let us consider for example $N = 8$.

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \ldots + h(7)z^{-7} + h(8)z^{-8}$$

The impulse response could be the following,



Here, we have a symmetry axis for $\frac{N}{2} = 4$.
For the symmetry it is $h(0) = h(8)$, $h(1) = h(7)$, $h(2) = h(6)$, $h(3) = h(5)$, and

$$H(z) = h(0)(1 + z^{-8}) + h(1)(z^{-1} + z^{-7}) + h(2)(z^{-2} + z^{-6}) + h(3)(z^{-3} + z^{-5}) + h(4)z^{-4} \Big/ \cdot z^{4} \cdot z^{-4}$$

$$H(z) = z^{-4} \cdot \left[h(0)(z^4 + z^{-4}) + h(1)(z^3 + z^{-3}) + h(2)(z^2 + z^{-2}) + h(3)(z^1 + z^{-1}) + h(4)\right]$$

The frequency response is given by

$$H(e^{j\omega}) = e^{-j\omega 4} \cdot [2h(0)\cos(4\omega) + 2h(1)\cos(3\omega) + 2h(2)\cos(2\omega) + 2h(3)\cos(\omega) + h(4)]$$

where we have utilized the identity $e^{j\omega n} + e^{-j\omega n} = 2\cos(\omega n)$.

Note that the expression within the square brackets is real and can take both positive and negative values. Consequently, the phase of the frequency response is linear, given by

$$\theta(\omega) = -4\omega + \beta = -\frac{N}{2}\omega + \beta \qquad \text{with } \beta = 0 \text{ or } \pi$$

and the group delay is constant:

$$t_g = -\frac{\mathrm{d}\theta}{\mathrm{d}\omega} = 4 = \frac{N}{2}.$$

In general, for FIR filters of Type 1, the frequency response is given by

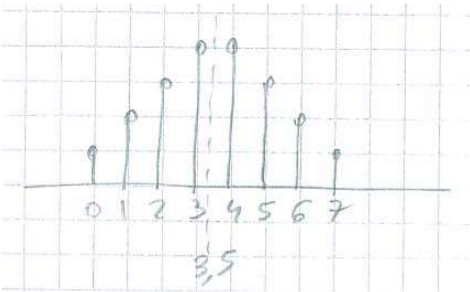
$$H(e^{j\omega}) = e^{-j4\omega} \cdot \overline{H}(\omega),$$

$$\overline{H}(\omega) = h(\frac{N}{2}) + 2\sum_{n=1}^{N/2} h(\frac{N}{2} - n)\cos(\omega n)$$

(which is the amplitude response apart from a ± 1 factor).

TYPE 2: $h(n)$ is symmetric and has even length. Thus, N is odd.

Let us consider the case where $N = 7$; here, we have a symmetry axis at $N/2 = 3.5$:



Let us proceed similarly to the previous case. For symmetry, we express $H(z)$ as:

$$H(z) = h(0)(1 + z^{-7}) + h(1)(z^{-1} + z^{-6}) + h(2)(z^{-2} + z^{-5}) + h(3)(z^{-3} + z^{-4}) \Big/_{\cdot z^{7/2} \cdot z^{-7/2}}$$

$$= z^{-7/2}\left[h(0)(z^{7/2} + z^{-7/2}) + h(1)(z^{5/2} + z^{-5/2}) + h(2)(z^{3/2} + z^{-3/2}) + h(3)(z^{1/2} + z^{-1/2})\right],$$

$$H(e^{j\omega}) = e^{-j\frac{7}{2}\omega}\left[2h(0)\cos(\frac{7}{2}\omega) + 2h(1)\cos(\frac{5}{2}\omega) + 2h(2)\cos(\frac{3}{2}\omega) + 2h(3)\cos(\frac{1}{2}\omega)\right].$$

Once again, the term within the square brackets is real, taking either positive or negative values. Consequently, the phase is given by

$$\theta(\omega) = -\frac{7}{2}\omega + \beta = -\frac{N}{2}\omega + \beta \qquad \text{with } \beta = 0 \text{ or } \pi$$

$$t_g = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega} = \frac{N}{2}$$

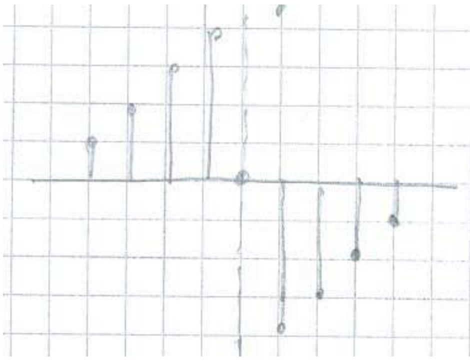
In general, it is

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}} \cdot \overline{H}(\omega)$$

$$\overline{H}(\omega) = 2 \sum_{n=1}^{(N+1)/2} h(\frac{N+1}{2} - n)\cos[\omega(n - \frac{1}{2})]$$

TYPE 3: $h(n)$ is antisymmetric and has odd length. Thus, N is even.

Here, we have an antisymmetry axis for $N/2$:



Given $h(N-n) = -h(n)$, for $n = \frac{N}{2}$, it follows that $h(\frac{N}{2}) = -h(\frac{N}{2}) = 0$.

Let's consider $N = 8$ and proceed similarly to the previous cases:

$$H(z) = h(0)(1 - z^{-8}) + h(1)(z^{-1} - z^{-7}) + h(2)(z^{-2} - z^{-6}) + h(3)(z^{-3} - z^{-5}) \Big/_{\cdot z^4 \cdot z^{-4}}$$

$$= z^{-4}\left[h(0)(z^4 - z^{-4}) + h(1)(z^3 - z^{-3}) + h(2)(z^2 - z^{-2}) + h(3)(z^1 - z^{-1})\right]$$

Since $e^{j\omega m} - e^{-j\omega m} = 2j\sin(\omega m) = 2e^{j\pi/2}\sin(\omega m)$,

$$H(z) = 2e^{-j4\omega}e^{j\pi/2}\left[h(0)\sin(4\omega) + h(1)\sin(3\omega) + h(2)\sin(2\omega) + h(4)\sin(\omega)\right]$$

$$\theta(\omega) = -4\omega + \frac{\pi}{2} + \beta = -\frac{N}{2}\omega + \frac{\pi}{2} + \beta \qquad \text{with } \beta = 0 \text{ or } \pi$$

$$t_g = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega} = \frac{N}{2}$$

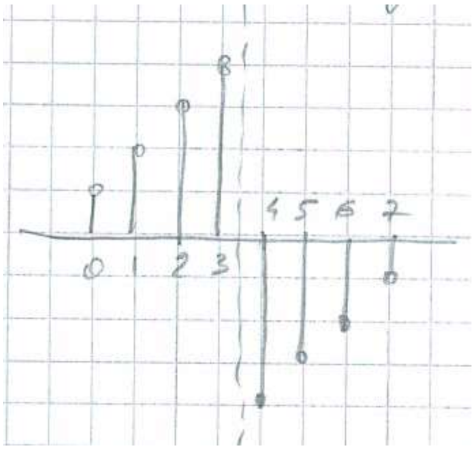
In general, it is

$$H(e^{j\omega}) = e^{-j\frac{N}{2}\omega}e^{j\pi/2} \cdot \overline{H}(\omega)$$

$$\overline{H}(\omega) = 2\sum_{n=1}^{N/2} h(\frac{N}{2} - n)\sin[\omega n]$$

TYPE 4: $h(n)$ is antisymmetric and has even length. Thus, N is odd.

We have an antisymmetry axis for $N/2$:



Let's proceed similarly to the previous cases, considering $N = 7$.

$$H(z) = h(0)(1 - z^{-7}) + h(1)(z^{-1} - z^{-6}) + h(2)(z^{-2} - z^{-5}) + h(3)(z^{-3} - z^{-4}) \Big/_{\cdot z^{7/2} \cdot z^{-7/2}}$$

$$= z^{-7/2}\left[h(0)(z^{7/2} - z^{-7/2}) + h(1)(z^{5/2} - z^{-5/2}) + h(2)(z^{3/2} - z^{-3/2}) + h(3)(z^{1/2} - z^{-1/2})\right]$$

$$H(e^{j\omega}) = e^{-j\frac{7}{2}\omega} e^{j\pi/2} 2\left[h(0)\sin(\frac{7}{2}\omega) + h(1)\sin(\frac{5}{2}\omega) + h(2)\sin(\frac{3}{2}\omega) + h(3)\sin(\frac{1}{2}\omega)\right]$$

Thus, the phase is

$$\theta(\omega) = -\frac{7}{2}\omega + \frac{\pi}{2} + \beta = -\frac{N}{2}\omega + \frac{\pi}{2} + \beta \qquad \text{with } \beta = 0 \text{ or } \pi$$

$$t_g = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega} = \frac{7}{2} = \frac{N}{2}$$

In general, it is

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}} \cdot e^{j\pi/2} \cdot \overline{H}(\omega)$$

$$\overline{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h(\frac{N+1}{2} - n)\sin[\omega(n - \frac{1}{2})]$$

Note that $\overline{H}(\omega)$ can take on negative values for certain ω. It represents the amplitude response, with the inclusion of a multiplicative term ± 1. Negative values of $\overline{H}(\omega)$ are commonly observed especially in the stopband.

Zeros' position in linear FIR filters

For the symmetric filters:

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n} = \sum_{n=0}^{N} h(N-n) z^{-n}$$

By introducing the variable change $m = N - n$ in the second equality, we get:

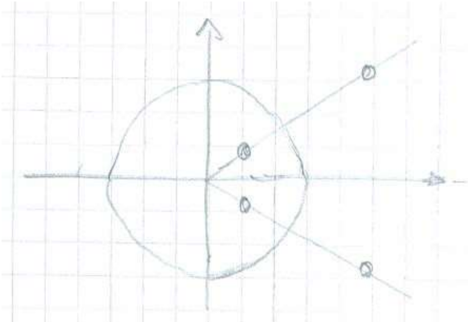
$$H(z) = \sum_{m=0}^{N} h(m) z^{-N+m} = z^{-N} H(z^{-1})$$

Similarly, for the antisymmetric filters, we have

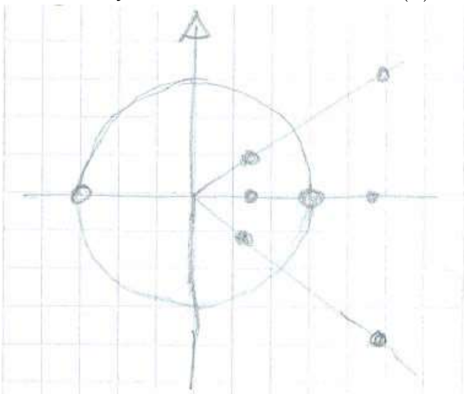
$$H(z) = -z^{-N} H(z^{-1}).$$

A polynomial with constant coefficients that satisfies the condition $H(z) = z^{-N} H(z^{-1})$ is referred to as a *mirror image* polynomial. Conversely, a polynomial with constant coefficients satisfying the condition $H(z) = -z^{-N} H(z^{-1})$ is termed an *antimirror image* polynomial.

For these two properties, if $z = \xi_0$ is a zero of $H(z)$ ($H(\xi_0) = 0$) then $z = \xi_0^{-1}$ is also a zero of $H(z)$. In other words, symmetric and antisymmetric FIR filters exhibit zeros with reciprocal symmetry, known as *mirror image symmetry* with respect to the unit circle. Additionally, if the filter has real coefficients, the zeros also possess conjugate symmetry. If a filter has a pair of conjugate symmetric zeros in $z = re^{\pm\theta}$, then, due to the *mirror image symmetry* property of the zeros, it must also have a pair of zeros a $z = r^{-1}e^{\pm\theta}$:



For a zero on the unit circle $z = e^{j\theta}$, its reciprocal coincides with the conjugate. Consequently, the filter can have pairs of zeros on the unit circle in $e^{\pm j\theta}$. For every real zero $z = r$, the filter must have also the reciprocal zero in $z = r^{-1}$. Additionally, zeros in $z = \pm 1$ are reciprocal of themselves and may appear individually in the set of zeros of $H(z)$.



Note that an FIR filter of Type 2 ($h(n)$ symmetric and of even length, N odd) must have at least a zero at -1. This is evident from the condition:

$$H(z) = z^{-N}H(z^{-1})$$

$$H(-1) = (-1)^{-N}H(-1) = -H(-1) = 0$$

Similarly, in Type 3 and 4 FIR filters, there must be at least one zero at $z = +1$ due to the antisymmetry condition:

$$H(z) = -z^{-N}H(z^{-1})$$

$$H(1) = -1^{N}H(1) = -H(1) = 0$$

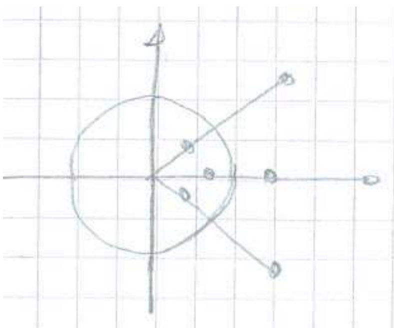
Furthermore, Type 3 FIR filters, which have an odd length and N even, must also have at least one zero at -1.

$$H(-1) = -(-1)^{N}H(-1) = -H(-1) = 0$$

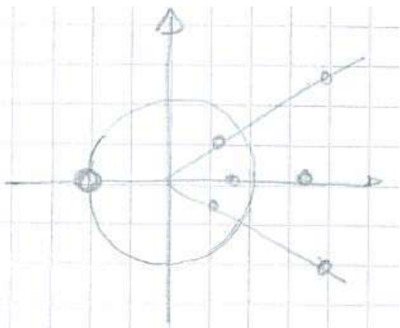
Note that all linear FIR filters with an odd length must have either no zero or an even number of zeros at $+1$ and -1 (because N is even), while all linear FIR filters with an even length must have an odd number of zeros at $+1$ and -1 (since N is odd).
The four cases of linear FIR filters differ in the distribution of zeros at $+1$ and -1. Specifically:

- Type 1 filters: have no zero or an even number of zeros at $+1$ and -1 (N is even).
- Type 2 filters: have no zero or an even number of zeros at $+1$ and an odd number of zeros at -1 (N is odd).
- Type 3 filters: have an odd number of zeros at $+1$ and an odd number of zeros at -1 (N is even).
- Type 4 filters: have an odd number of zeros at $+1$ and no zero or an even number of zeros at -1 (N is odd).

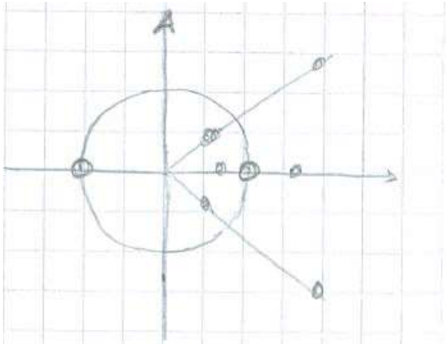
Filters of Type 3 and 4 must have an odd number of zeros at $+1$.This is because the factor associated with a zero at $+1$ is $(1 - z^{-1})$, which imparts antisymmetry to the polynomial. Similarly, Filters of Type 2 and 3 must have an odd number of zeros at -1 (associated with the factor $(1 + z^{-1})$) to achieve the desired value of N – whether odd or even – ensuring the symmetry or antisymmetry of the polynomial. The zeros in the four cases are the following:



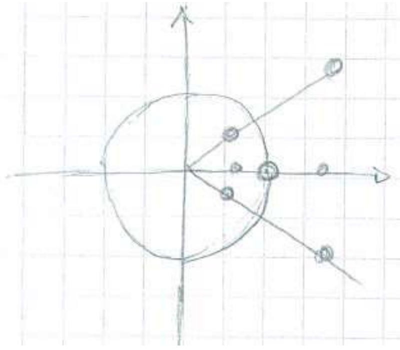
Type 1



Type 2



Type 3



Type 4

Type 1 filters can be used to implement any kind of filters, including lowpass, highpass, passband, and stopband filters. Type 2 filters have $H(e^{j\pi}) = 0$, making them suitable for implementing lowpass and passband filters but not highpass or stopband filters. Type 3 filters with $H(e^{j\pi}) = H(e^{j0}) = 0$ are unsuitable for lowpass, highpass, and stopband filters, but they can be used for passband filters. Type 4 filters with $H(e^{j0}) = 0$ cannot be used for implementing lowpass or stopband filters, but they are suitable for highpass or passband filters.

IIR filters and linear phase

Let us consider the case of a causal IIR filter described by a finite difference equation:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

We have observed that in the case of FIR filters, the linear phase condition manifests as the mirror image symmetry of the zeros. The same criterion could be applied to derive IIR filters with linear phase. If $A(z)$ and $B(z)$ are mirror image polynomials, then $H(z)$ exhibits linear phase. Unfortunately, the zeros of $A(z)$ are the poles of the system and, if the poles satisfy the mirror image symmetry property, the system is unstable. This is because for every pole inside the unit circle, there must be a pole outside the unit circle. IIR filter design cannot overlook the need to ensure filter stability. Therefore, we shall content ourselves only with approximating linear phase in the filter passband.

07.05 Geometric interpretation of frequency response computation

In the following, we will study various examples of FIR and IIR basic filters. For these filters, using the geometric interpretation of frequency response computation, it is very easy to plot amplitude response or phase response diagrams based on the knowledge of pole and zero locations. Let us consider an IIR filter with a transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

where for simplicity we have set $a_0 = 1$.

$$H(z) = b_0 \cdot \frac{(1 - z_1 z^{-1}) \cdot (1 - z_2 z^{-1}) \cdot \ldots \cdot (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \cdot (1 - p_2 z^{-1}) \cdot \ldots \cdot (1 - p_N z^{-1})} =$$

$$= b_0 \cdot z^{N-M} \cdot \frac{(z - z_1) \cdot (z - z_2) \cdot \ldots \cdot (z - z_M)}{(z - p_1) \cdot (z - p_2) \cdot \ldots \cdot (z - p_N)}$$

where $z_1, \ldots, z_M$ are the system zeros and $p_1, \ldots, p_N$ are the system poles.
The frequency response of the system is

$$H(e^{j\omega}) = b_0 \cdot e^{j\omega(N-M)} \frac{(e^{j\omega} - z_1) \cdot (e^{j\omega} - z_2) \cdot \ldots \cdot (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdot (e^{j\omega} - p_2) \cdot \ldots \cdot (e^{j\omega} - p_N)}.$$

The amplitude response and the phase response are given, respectively, by

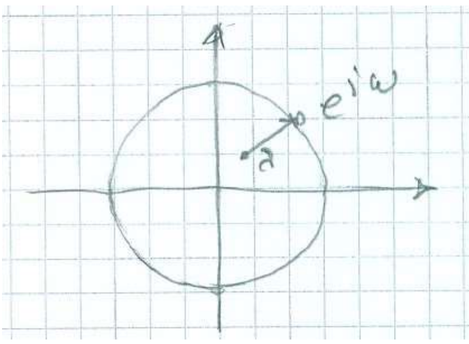
$$|H(e^{j\omega})| = |b_0| \cdot \frac{|e^{j\omega} - z_1| \cdot |e^{j\omega} - z_2| \cdot \ldots \cdot |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdot |e^{j\omega} - p_2| \cdot \ldots \cdot |e^{j\omega} - p_N|},$$

$$\arg H(e^{j\omega}) = \arg b_0 + \omega(N - M) + \arg(e^{j\omega} - z_1) + \arg(e^{j\omega} - z_2) + \ldots + \arg(e^{j\omega} - z_M)$$
$$- \arg(e^{j\omega} - p_1) - \arg(e^{j\omega} - p_2) - \ldots - \arg(e^{j\omega} - p_N).$$

If we examine the frequency response we can notice that the typical factor is

$$(e^{j\omega} - \lambda)$$

with $\lambda = z_i$ or $\lambda = p_i$. If we interpret this factor on the complex plane we have that:
$e^{j\omega}$ is a point on the unit circle,
λ is the zero or pole position,
$e^{j\omega} - \lambda$ is the vector from λ to $e^{j\omega}$.



For ω that goes from 0 to 2π this vector varies in amplitude and phase. From the position of the two points, we can immediately obtain its amplitude and phase behavior. $|e^{j\omega} - \lambda|$ has minimum value when $\omega = \arg\lambda$, and has maximum value when $\omega = \arg\lambda + \pi$.
The amplitude response $|H(e^{j\omega})|$ is given by the product of the modulus of all vectors associated with the zeros, divided by the modulus of all vectors related to the poles, multiplied by the modulus of b_0.
The phase response $\arg H(e^{j\omega})$ is given by the phase of b_0, plus $\omega(N-M)$, plus the phase of all vectors associated with the zeros, minus the phase of all vectors associated with the poles.
When designing a filter that should attenuate a certain frequency range, we shall locate the zeros close to the unit circle around this frequency range. On the contrary, if we have to emphasize certain frequency components of the signal, we shall locate the poles around this frequency range (indeed, the amplitude response maxima correspond to the vectors $z - p_i$ minima).

07.06 Simple digital filters

Lowpass FIR filter

The most simple lowpass filter is the filter that computes the average between samples. Let us consider

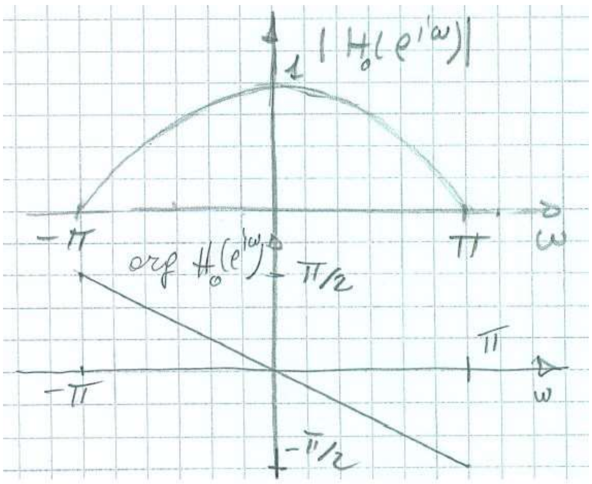
$$y(n) = \frac{1}{2}(x(n) + x(n-1))$$

$$Y(z) = \frac{1}{2}\left(X(z) + z^{-1}X(z)\right)$$

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) = \frac{1}{2}\frac{z+1}{z}$$

This is an FIR filter with a zero at $z = -1$ and a pole at $z = 0$. The vector $e^{j\omega} - \lambda$ related to the pole has always unit amplitude. The vector related to the zero has maximum amplitude for $\omega = 0$ (with amplitude 2) and then its amplitude decreases to 0 as ω goes from 0 to π. The filter is symmetric, and thus its phase is linear.

$$H_0(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) = \frac{1}{2}e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) = e^{-j\omega/2} \cdot \cos(\frac{\omega}{2}).$$



Of particular interest is the frequency ω_c for which

$$|H_0(e^{j\omega_c})| = \frac{1}{\sqrt{2}}|H_0(e^{j\omega})|_{\text{MAX}} = \frac{1}{\sqrt{2}}|H_0(e^{j0})|$$

Let us consider the gain in dB (i.e., the amplitude in dB):

$$G(\omega_c) = 20\log_{10}|H_0(e^{j\omega_c})| =$$

$$= 20\log_{10}|H_0(e^{j0})| - 20\log_{10}(\sqrt{2}) =$$

$$= 0 - 3.0103 \simeq -3\mathrm{dB}.$$

Thus, the frequency ω_c is called the $3\mathrm{dB}$ *cutoff frequency*, because the gain has reduced by $3\mathrm{dB}$ compared with the maximum value.

Imposing,

$$|H_0(e^{j\omega_c})|^2 = \cos^2 \frac{\omega_c}{2} = \frac{1}{2}$$

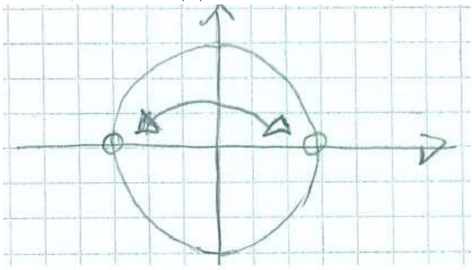
we obtain $\omega_c = \frac{\pi}{2}$.

Highpass FIR filter

The most simple highpass filter can be obtained by replacing z with $-z$ in the previous transfer function:

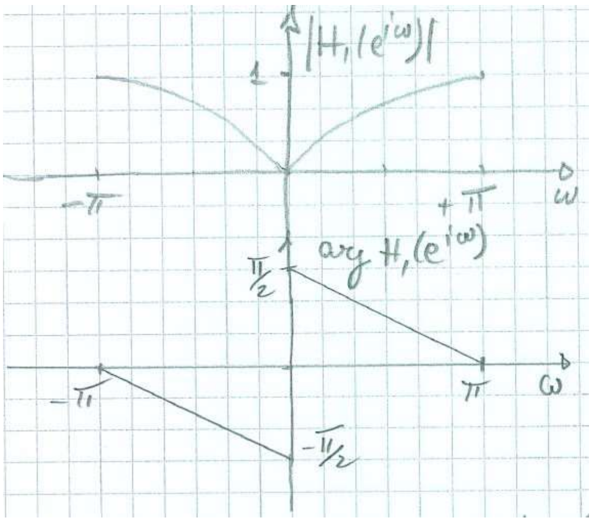
$$H_1(z) = H_0(-z).$$

If we consider the geometric interpretation of the frequency response, we can understand that with this variable change $H_1(z)$ has for $z = 1$ (i.e., for $\omega = 0$) the same behavior of $H_0(z)$ for $z = -1$ (i.e., for $\omega = \pi$), and $H_1(z)$ has for $z = -1$ (i.e., for $\omega = \pi$) the same behavior of $H_0(z)$ for $z = +1$ (i.e., for $\omega = 0$).



$$H_1(z) = \frac{1}{2}(1 - z^{-1})$$

$$H_1(e^{j\omega}) = je^{-j\omega/2} \sin(\frac{\omega}{2})$$



Also in this case the 3dB cutoff frequency falls at $\omega_c = \frac{\pi}{2}$.

We can obtain FIR lowpass or highpass filters with a narrower passband by cascading a certain number of these elementary filters. By considering the cascade of M lowpass filters $H_0(e^{j\omega})$, the resulting frequency response is $H(e^{j\omega}) = H_0^M(e^{j\omega})$ and the 3dB cutoff frequency is given for

$$|H(e^{j\omega_c})| = |H_0(e^{j\omega_c})|^M = \frac{1}{\sqrt{2}}$$

i.e., for

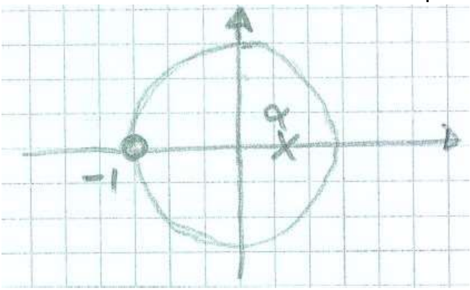
$$|H_0(e^{j\omega_c})| = 2^{-1/(2M)}$$
$$\cos\left(\frac{\omega_c}{2}\right) = 2^{-\frac{1}{2M}}$$
$$\omega_c = 2\arccos\left(2^{-\frac{1}{2M}}\right).$$

IIR Lowpass filter

A lowpass filter of the first order has a transfer function:

$$H_{\mathrm{LP}}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}},$$

where $|\alpha| < 1$ for the stability of the system.
The filter has a zero at $z = -1$ and a pole at $z = \alpha$.



In the geometric interpretation of the frequency response computation, we see that for ω going from 0 to π the vector $e^{j\omega} - (-1)$ decreases from 2 to 0. On the contrary, the vector $e^{j\omega} - \alpha$ for $\alpha > 0$ increases from $1-\alpha$ till $1+\alpha$. The maximum value and the minimum value of the frequency response are obtained for $\omega = 0$ and $\omega = \pi$, respectively.

$$H_{\mathrm{LP}}(e^{j0}) = 1 \qquad H_{\mathrm{LP}}(e^{j\pi}) = 0$$

$$|H_{\mathrm{LP}}(e^{j\omega})|^2 = \frac{(1-\alpha)^2}{4}\cdot\frac{(1+e^{j\omega})(1+e^{-j\omega})}{(1-\alpha e^{j\omega})(1-\alpha e^{-j\omega})} =$$
$$= \frac{(1-\alpha)^2}{4}\cdot\frac{1+e^{j\omega}+e^{-j\omega}+1}{1-\alpha e^{j\omega}-\alpha e^{-j\omega}+\alpha^2} =$$
$$= \frac{(1-\alpha)^2}{2}\cdot\frac{1+\cos(\omega)}{1-2\alpha\cos(\omega)+\alpha^2}$$

$$\frac{\mathrm{d}|H_{\mathrm{LP}}(e^{j\omega})|^2}{\mathrm{d}\omega} = \frac{(1-\alpha)^2}{2}\cdot\frac{-\sin(\omega)\left(1+\alpha^2-2\alpha\cos(\omega)\right)-(1+\cos(\omega))\,(2\alpha\sin(\omega))}{\left(1-2\alpha\cos(\omega)+\alpha^2\right)^2} =$$
$$= \frac{(1-\alpha)^2}{2}\cdot\frac{-\sin(\omega)(1+\alpha)^2}{\left(1-2\alpha\cos(\omega)+\alpha^2\right)^2}.$$

For $0 \le \omega \le \pi$ the derivative is always negative and, thus, the amplitude response decreases monotonically.
The $3\mathrm{dB}$ cutoff frequency is obtained for $|H_{\mathrm{LP}}(e^{j\omega})|^2 = \frac{1}{2}$.

$$\frac{(1-\alpha)^2}{2}\cdot\frac{1+\cos(\omega_c)}{1-2\alpha\cos(\omega_c)+\alpha^2} = \frac{1}{2}$$

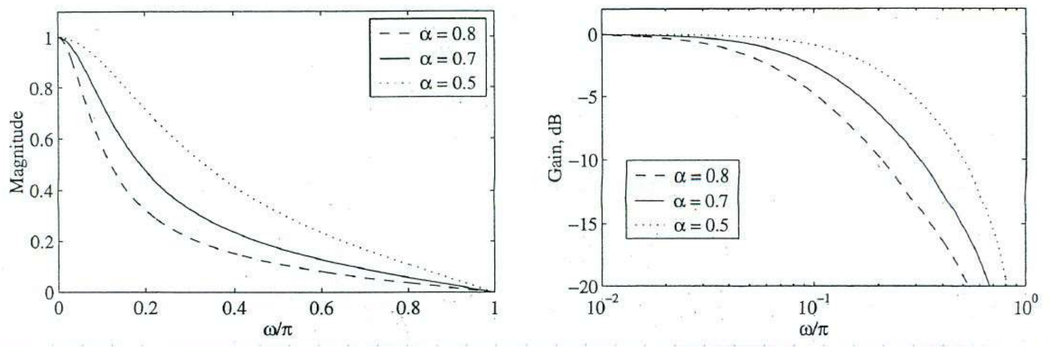
$$(1-\alpha)^2\,(1+\cos(\omega_c)) = 1+\alpha^2 - 2\alpha\cos(\omega_c)$$

$$(1-2\alpha+\alpha^2)\cos(\omega_c) + 2\alpha\cos(\omega_c) = 1+\alpha^2-(1-\alpha)^2$$

$$\cos(\omega_c) = \frac{2\alpha}{1+\alpha^2}$$

If we want a lowpass filter with an assigned cutoff frequency ω_c, we have to solve the previous equation for α. It can be proved that the only stable solution is

$$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$$



(From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

IIR Highpass filter

An order 1 IIR highpass filter is given by

$$H_{\mathrm{HP}}(z) = \frac{1-\alpha}{2}\,\frac{1-z^{-1}}{1+\alpha z^{-1}}$$

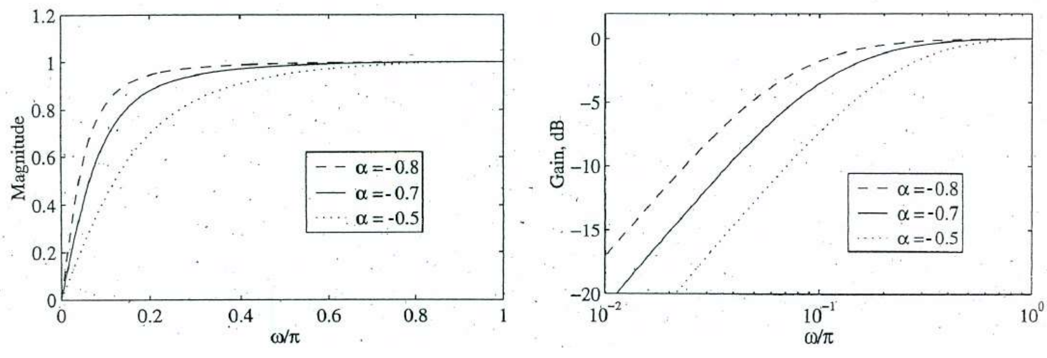
where it must be $|\alpha| < 1$ for stability.

This filter has been obtained from the previous lowpass filter by replacing z with $-z$:

$$H_{\mathrm{HP}}(z) = H_{\mathrm{LP}}(-z)$$

Thus, the same properties of the previous filter hold, apart from a frequency shift of π in the frequency response:

$$H_{\mathrm{HP}}(e^{j\omega}) = H_{\mathrm{LP}}(e^{j(\omega+\pi)}).$$



(From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

IIR Bandpass filter

In order to obtain bandpass or bandstop filters, we must consider at least second-order filters.

A bandpass filter of the second order is given by the following transfer function:

$$H_{\mathrm{BP}}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

The squared amplitude response is

$$|H_{\mathrm{BP}}(e^{j\omega})|^2 = \frac{(1-\alpha)^2\,(1-\cos(2\omega))}{2\,[1+\beta(1+\alpha)^2\cos(\omega)+2\alpha\cos(2\omega)]}$$

which is 0 for $\omega = 0$ and $\omega = \pi$ and assumes the maximum value 1 for $\omega = \omega_0$, called *center frequency* for the bandpass filter, where

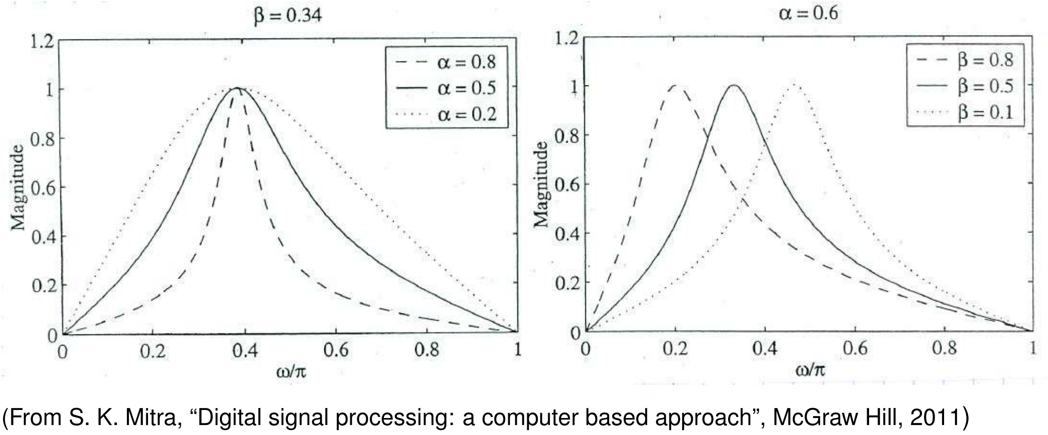
$$\cos(\omega_0) = \beta$$

$$\omega_0 = \arccos(\beta)$$

The frequencies ω_{c1} and ω_{c2} for which $|H_{\mathrm{BP}}(e^{j\omega})|^2 = \frac{1}{2}$ are called $3\mathrm{dB}$ cutoff frequencies and their difference $\omega_{c2} - \omega_{c1}$ is called $3\mathrm{dB}$ *bandwidth.* It can be proved that

$$B_{3\mathrm{dB}} = \omega_{c2} - \omega_{c1} = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right).$$

Thus, β controls the center frequency, while α controls the bandwidth.



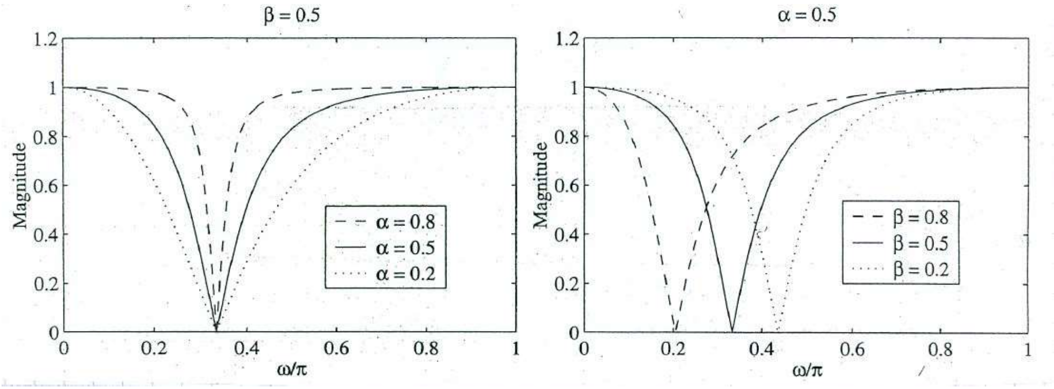
(From S. K. Mitra, “Digital signal processing: a computer based approach”, McGraw Hill, 2011)

IIR Bandstop filter

An second order IIR bandstop filter is given by the following transfer function

$$H_{\mathrm{BP}}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

The amplitude response for different values of α and β is given by:



(From S. K. Mitra, "Digital signal processing: a computer based approach", McGraw Hill, 2011)

This filter is also called a *notch* filter.
Also in this case α and β separately control the stopband bandwidth and the notch position. The notch frequency is

$$\omega_0 = \arccos(\beta)$$

and the stopband bandwidth is

$$B_{I3\text{dB}} = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right).$$

Higher order IIR filters

By cascading a certain number of filters like the ones we have just introduced, it is possible to obtain filters with steeper rising and falling edges in the frequency domain, i.e., filters with smaller transition bands.
For example, consider the cascade of K IIR lowpass filters:

$$H_{\text{LP}}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}},$$

for which we have seen $\cos(\omega_c) = \dfrac{2\alpha}{1+\alpha^2}$.
In the filter cascade, the resulting transfer function is

$$G_{\text{LP}}(z) = \left(\frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)^K$$

$$|G_{\text{LP}}(e^{j\omega})|^2 = \left[\frac{(1-\alpha)^2(1+\cos(\omega))}{2(1+\alpha^2-2\alpha\cos(\omega))}\right]^K$$

The 3dB cutoff frequency can be obtained by setting

$$|G_{\text{LP}}(e^{j\omega})|^2 = \frac{1}{2}.$$

By imposing a certain ω_c and solving this equation for α, we obtain that the only stable solution is

$$\alpha = \frac{1+(1-C)\cos(\omega_c) - \sin(\omega_c)\sqrt{2C-C^2}}{1-C+\cos(\omega_c)}$$

with $C = 2^{\frac{K-1}{K}}$.

Example: Let us design a filter with $3\mathrm{dB}$ cutoff frequency $\omega_c = 0.4\pi$.

For $K = 1$, $C = 1$, it is $\alpha = 0.1584$.

For $K = 4$, $C = 1.6818$, it is $\alpha = -0.251$.

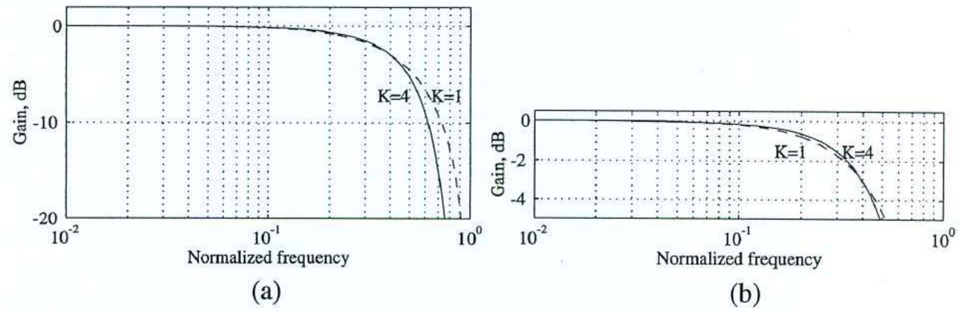


Figure 4.23: (a) Gain responses of a single first-order lowpass filter ($K = 1$) and a cascade of four identical first-order lowpass filters ($K = 4$) with a 3-dB cutoff frequency of $\omega_c = 0.4\pi$. (b) Passband details.

(From S. K. Mitra, “Digital signal processing: a computer based approach”, McGraw Hill, 2011)

07.07 Comb filters

The filters we have seen up to this point have been characterized by a single passband and/or stopband. However, there are several applications where filters with multiple passbands or stopbands are required. Comb filters are an example of such filters.

Comb filters have a periodic frequency response with a period of $\frac{2\pi}{L}$, where L is a positive integer. If $H(z)$ is a transfer function with a single passband and/or stopband, a comb filter can be easily generated by replacing each delay element with L delays, resulting in a transfer function $G(z) = H(z^L)$.

If $|H(e^{j\omega})|$ has a peak at $\omega = \omega_p$, then $|G(e^{j\omega})|$ has L peaks at $\omega = \frac{\omega_p}{L} + \frac{2\pi}{L}k$ where $0 \leq k \leq L-1$, and $0 \leq \omega \leq 2\pi$.

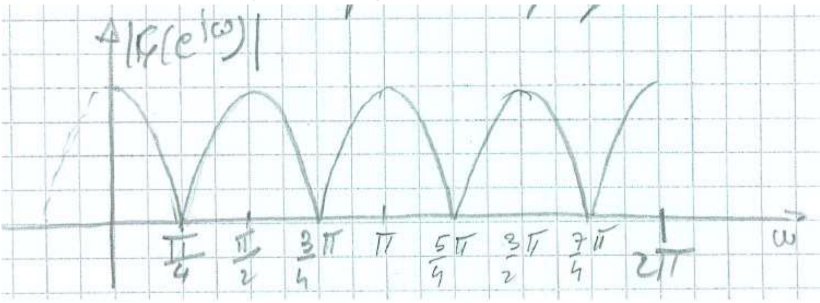
Similarly, if $|H(e^{j\omega})|$ has a notch at $\omega = \omega_0$, then $|G(e^{j\omega})|$ has L notches at $\omega = \frac{\omega_0}{L} + \frac{2\pi}{L}k$ where $0 \leq k \leq L-1$, and $0 \leq \omega \leq 2\pi$.

For example, if we consider

$$H(z) = \frac{1}{2}\left(1 + z^{-1}\right)$$

$$G(z) = \frac{1}{2}\left(1 + z^{-L}\right)$$

and for $L = 4$, the magnitude response is

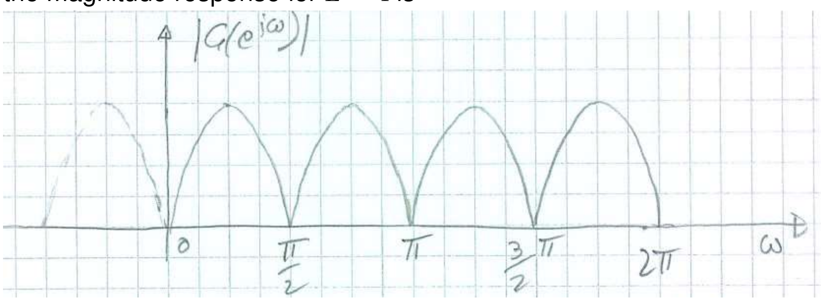


Similarly, if we consider

$$H(z) = \frac{1}{2}\left(1 - z^{-1}\right)$$

$$G(z) = \frac{1}{2}\left(1 - z^{-L}\right)$$

the magnitude response for $L = 4$ is



It is easy to understand the behavior of the frequency response $G(e^{j\omega})$ from the frequency response $H(e^{j\omega})$ because

$$G(e^{j\omega}) = H(e^{j\omega L}).$$

By varying ω from 0 to 2π, $e^{j\omega}$ moves along the unit circle L times, and therefore, the frequency response $G(e^{j\omega})$ coincides with $H(e^{j\omega})$ (which is periodic with period 2π), apart from a frequency axis scaling by a factor of $\frac{1}{L}$.

07.08 All-pass filters

By definition, a transfer function is called *all-pass* if the amplitude response is constant (i.e., one) for all frequencies, that is, if

$$|A(e^{j\omega})| = 1 \qquad \forall\omega$$

An all-pass causal transfer function with real coefficients is given by

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \ldots + d_1 z^{-M+1} + d_0 z^{-M}}{d_0 + d_1 z^{-1} + \ldots + d_{M-1}z^{-M+1} + d_M z^{-M}}$$

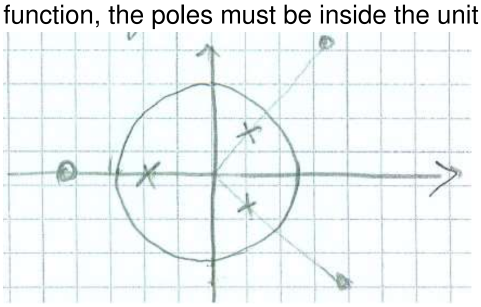
If we call the denominator polynomial $D_M(z)$,

$$D_M(z) = d_0 + d_1 z^{-1} + \ldots + d_{M-1}z^{-M+1} + d_M z^{-M}$$

we have

$$A_M(z) = z^{-M}\frac{D_M(z^{-1})}{D_M(z)}.$$

Note that the denominator polynomial is the *mirror image* polynomial of the numerator, and vice versa. If $z = re^{j\theta}$ is a pole of the transfer function, then $z = \frac{1}{r}e^{-j\theta}$ is a zero. Thus, poles and zeros of an all-pass filter exhibit mirror-image symmetry in the z-plane. By assuming that $A(z)$ is a stable transfer function, the poles must be inside the unit circle and the zeros outside the unit circle.



Let us prove that $A_M(z) = z^{-M}\frac{D_M(z^{-1})}{D_M(z)}$ is an all-pass transfer function. Consider

$$A_M(z^{-1}) = z^{M}\frac{D_M(z)}{D_M(z^{-1})}$$

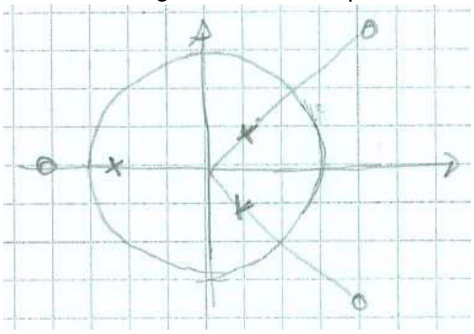
$$A_M(z)\cdot A_M(z^{-1}) = z^{-M}\frac{D_M(z^{-1})}{D_M(z)}\cdot z^{M}\frac{D_M(z)}{D_M(z^{-1})} = 1$$

Then,

$$A_M(e^{j\omega}) \cdot A_M(e^{-j\omega}) = |A_M(e^{j\omega})|^2 = 1$$

Q.E.D.

It is interesting to observe the phase behavior for $0 \leq \omega \leq 2\pi$.



With the geometric interpretation of the phase response, as ω varies from 0 to 2π, the zeros cause phase fluctuations, but the overall phase variation is 0. Conversely, each pole contributes a phase of -2π. Overall, the phase varies from 0 to $-2\pi M$, as ω varies between 0 and 2π. In other words, for ω varying between 0 and π, the phase varies from 0 to $-\pi M$.

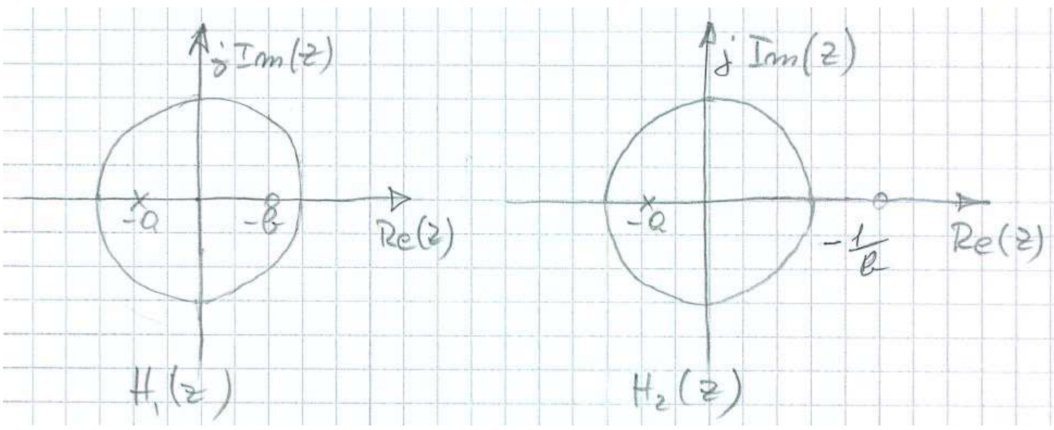
07.09 Minimum-phase and maximum-phase transfer functions

Another useful classification of transfer functions is based on the behavior of the phase response. Let us consider the following two first-order transfer functions with real coefficients:

$$H_1(z) = \frac{z+b}{z+a}$$

$$H_2(z) = \frac{bz+1}{z+a}$$

with $|a| < 1$ and $|b| < 1$.



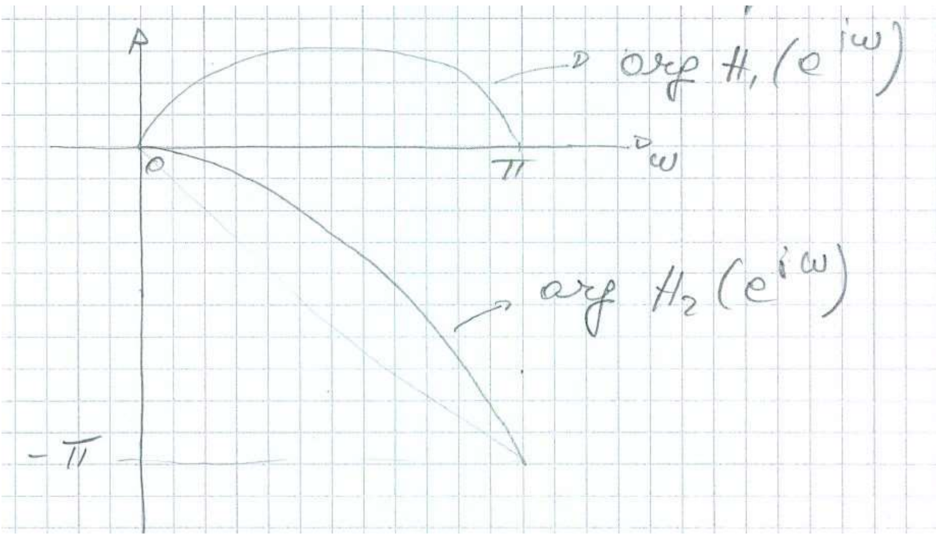
As we can see, both transfer functions have a pole inside the unit circle at $-a$, indicating stability. On the other hand, the zero of $H_1(z)$ falls inside the unit circle (at $z = -b$), while the zero of $H_2(z)$ falls outside the unit circle at $z = -\frac{1}{b}$.
The two transfer functions have the same amplitude response because

$$H_2(z) = H_1(z) \cdot \frac{bz+1}{z+b} = H_1(z) \cdot A(z)$$

with $A(z)$ all-pass. Thus,

$$H_1(e^{j\omega}) \cdot H_1(e^{-j\omega}) = \left|H_1(e^{j\omega})\right|^2 = H_2(e^{j\omega}) \cdot H_2(e^{-j\omega}) = \left|H_2(e^{j\omega})\right|^2$$

but they have different phase responses:



$$\arg\left[H_1(e^{j\omega})\right] = \theta_1(\omega)) = \arctan\frac{\sin(\omega)}{b+\cos(\omega)} - \arctan\frac{\sin(\omega)}{a+\cos(\omega)}$$

$$\arg\left[H_2(e^{j\omega})\right] = \theta_2(\omega)) = \arctan\frac{b\sin(\omega)}{1+b\cos(\omega)} - \arctan\frac{\sin(\omega)}{a+\cos(\omega)}$$

$$\arg\left[H_2(e^{j\omega})\right] = \arg\left[H_1(e^{j\omega})\right] + \arg\left[A(e^{j\omega})\right]$$

For ω ranging from 0 to π, we observe that $H_2(e^{j\omega})$ undergoes a phase variation of $-\pi$, while $H_1(e^{j\omega})$ undergoes no phase variation, i.e., $H_2(e^{j\omega})$ exhibits an excess phase variation compared to $H_1(e^{j\omega})$. In general, for ω ranging from 0 to π, a causal and stable transfer function with all zeros outside the unit circle experiences an excess phase variation compared to a causal and stable transfer function with the same amplitude response but with all zeros inside the unit circle. Consequently, a transfer function with all zeros inside the unit circle is termed a *minimum-phase transfer function*, while if all zeros lie outside the unit circle, it is termed a *maximum-phase transfer function*.

07.10 Inverse system

Two LTI systems with impulse response $h_1(n)$ and $h_2(n)$ are inverses of each other if

$$h_1(n) \circledast h_2(n) = \delta(n),$$

i.e., if their convolution is the unit impulse function, indicating that the cascade of the two systems (in any order) results in the identity system.
Let us characterize the inverse system (or the inverse filter) in the frequency domain. By taking the Z-transform of both sides of the equality, we have

$$H_1(z) \cdot H_2(z) = 1,$$

which implies

$$H_2(z) = \frac{1}{H_1(z)}$$

and if $H_1(z)$ is rational,

$$H_1(z) = \frac{N(z)}{D(z)} \quad \Longrightarrow \quad H_2(z) = \frac{D(z)}{N(z)},$$

thus $H_2(z)$ is also rational, and the poles (zeros) of the inverse filter are the zeros (poles) of $H_1(z)$. Assuming the inverse system to be causal, it will be stable if and only if $H_1(z)$ is minimum-phase.
Note that by relinquishing the assumption of causality for the inverse filter, the inverse filter is not unique. For example, consider the causal system

$$H_1(z) = \frac{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{5}\right)}{\left(z + \frac{1}{8}\right)\left(z - \frac{1}{7}\right)}$$

with R.O.C.: $|z| > \dfrac{1}{7}$
The inverse filter has a transfer function

$$H_2(z) = \frac{\left(z + \frac{1}{8}\right)\left(z - \frac{1}{7}\right)}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{5}\right)}$$

with three possible R.O.C.: (i) $|z| < \dfrac{1}{5}$, (ii) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$, (iii) $|z| > \dfrac{1}{4}$. Each region of convergence corresponds to a different inverse system. Only the last R.O.C. corresponds to a causal system.

07.11 Deconvolution

If a system has a known causal impulse response $h(n)$ and it is excited by a causal input signal $x(n)$, then, from the knowledge of the output signal $y(n)$ for $n \geq 0$, we can estimate the input signal $x(n)$ using a recursive relation without the need to evaluate the inverse system. In fact, it is

$$y(n) = \sum_{m=0}^{n} h(m)x(n-m).$$

Let us assume $h(0) \neq 0$,

$$y(0) = h(0)x(0) \quad \Longrightarrow \quad x(0) = \frac{y(0)}{h(0)}$$
$$y(1) = h(0)x(1) + h(1)x(0) \quad \Longrightarrow \quad x(1) = \frac{y(1) - h(1)x(0)}{h(0)}$$
$$y(2) = h(0)x(2) + h(1)x(1) + h(2)x(0) \quad \Longrightarrow \quad x(2) = \frac{y(2) - h(1)x(1) - h(2)x(0)}{h(0)}$$
$$y(n) = \sum_{m=0}^{n} h(m)x(n-m) \quad \Longrightarrow \quad x(n) = \frac{y(n) - \sum_{m=1}^{n} h(m)x(n-m)}{h(0)}.$$

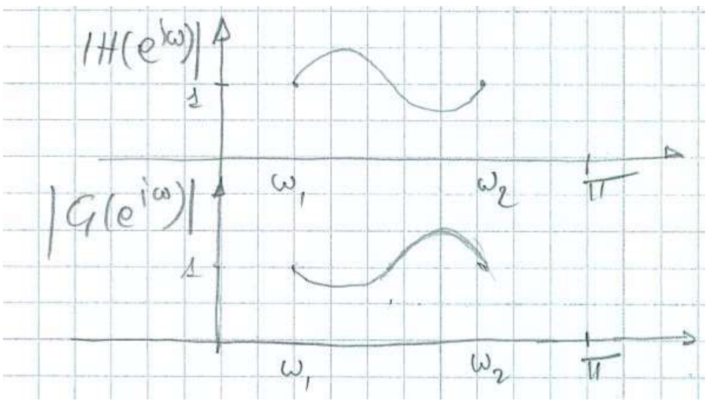
This procedure, which estimates the input signal $x(n)$ from the convolution sum, is called *deconvolution*.

07.12 Amplitude equalizer and phase equalizer

Given a system $H(z)$, a filter $G(z)$ whose amplitude response $|G(e^{j\omega})|$ satisfies the following condition

$$|G(e^{j\omega})| = |H(e^{j\omega})|^{-1} \qquad \forall \omega \in [\omega_1, \omega_2]$$

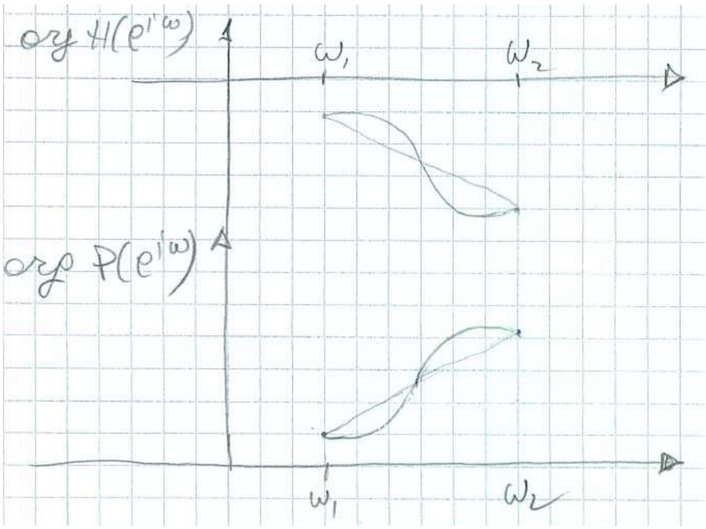
is called an *amplitude equalizer* for the system $H(z)$ in band $[\omega_1, \omega_2]$.



Given a system $H(z)$, a filter $P(z)$ whose phase response $\arg P(e^{j\omega})$ satisfies the following condition

$$\arg P(e^{j\omega}) = -\arg H(e^{j\omega}) - K\omega \qquad \omega \in [\omega_1, \omega_2],$$

with $K \in \mathbb{R}$, is called a *phase equalizer* for the system $H(z)$ in band $[\omega_1, \omega_2]$.



(in this case $K = 0$)

07.13 Stability test for IIR filters

We have seen that a causal IIR filter, described by a finite difference equation or by a rational transfer function, is stable in the BIBO sense if and only if its poles fall inside the unit circle. There exist stability tests that do not require the estimation of the poles.

07.13.1 The stability triangle

For real coefficient IIR filters of second order, we can easily determine if the filter is stable from the coefficients of the denominator. Consider the polynomial

$$D(z) = z^2 + a_1 z + a_2 = (z - z_1)(z - z_2)$$

$$z_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$$

and thus

$$a_2 = z_1 \cdot z_2$$

$$a_1 = -(z_1 + z_2)$$

We can prove that the roots z_1 and z_2 fall inside the unit circle if and only if

$$|a_2| < 1,$$

$$|a_1| < 1 + a_2.$$

In fact, if the zeros are complex conjugate the condition $|a_2| < 1$ is necessary and sufficient for stability. If the zeros are real: $a_1^2 - 4a_2 \geq 0$ and it is still necessary that $|a_2| < 1$.

$$|z_{1,2}|_{\max} = \frac{|a_1|}{2} + \sqrt{\frac{a_1^2 - 4a_2}{4}},$$

from which, if $|a_2| < 1$, the filter is stable if and only if $|a_1| < 1 + a_2$.
In fact, if

$$\frac{|a_1|}{2} + \sqrt{\frac{a_1^2 - 4a_2}{4}} < 1$$

then

$$\sqrt{\frac{a_1^2 - 4a_2}{4}} < 1 - \frac{|a_1|}{2}$$

$$\frac{a_1^2 - 4a_2}{4} < 1 - |a_1| + \frac{a_1^2}{4}$$

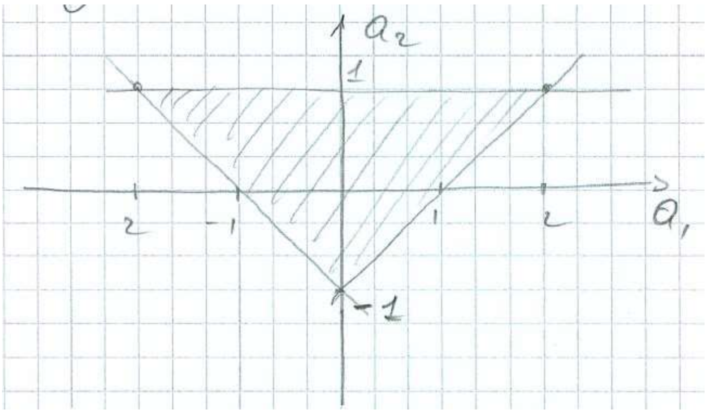
$$-a_2 < 1 - |a_1|$$

$$|a_1| < 1 + a_2.$$

Conversely, if $|a_1| < 1 + a_2$ then

$$|z_{1,2}|_{\max} < \frac{1 + a_2}{2} + \sqrt{\frac{(1 + a_2)^2 - 4a_2}{4}} = 1$$

On the plane a_1, a_2, the region of points for which the filter is stable is a triangle, called the *stability triangle*.



07.13.2 Schur-Cohn stability test

The Schur-Cohn stability test can be applied to polynomials of any order. Let us consider:

$$D_M(z) = \sum_{i=0}^{M} d_i z^{-i} = 1 + d_1 z^{-1} + \ldots + d_M z^{-M}$$

with $d_0 = 1$ for simplicity. Let us take the mirror image polynomial,

$$\widetilde{D}_M(z) = z^{-M} D_M(z^{-1}) = z^{-M} \sum_{i=0}^{M} d_i z^{i} =$$

$$= d_M + d_{M-1} z^{-1} + \ldots + d_1 z^{-M+1} + z^{-M},$$

and let us build the all-pass filter

$$A_M(z) = \frac{\widetilde{D}_M(z)}{D_M(z)}$$

If $D_M(z) = \prod_{i=1}^{M} \left(1 - \lambda_i z^{-1}\right)$, with λ_i the filter poles, then

$$d_M = \prod_{i=1}^{M} \lambda_i (-1)^M.$$

Thus, if we call $K_M = d_M$, a necessary condition for the all-pass filter stability is that

$$|K_M| < 1.$$

Let us assume $|K_M| < 1$ and let us build the all-pass filter

$$\begin{aligned} A_{M-1}(z) &= z \cdot \left[\frac{A_M(z) - K_M}{1 - K_M A_M(z)}\right] \\ &= z \cdot \frac{\widetilde{D}_M(z) - d_M D_M(z)}{D_M(z) - d_M \widetilde{D}_M(z)} = \\ &= z \cdot \frac{(d_M - d_M \cdot 1) + (d_{M-1} - d_M d_1)z^{-1} + \ldots + (1 - d_M^2)z^{-M}}{(1 - d_M^2) + (d_1 - d_M d_{M-1})z^{-1} + \ldots + (d_M - d_M \cdot 1)z^{-M}} = \\ &= \frac{(d_{M-1} - d_M d_1) + \ldots + (1 - d_M^2)z^{-M+1}}{(1 - d_M^2) + (d_1 - d_M d_{M-1})z^{-1} + \ldots + (d_{M-1} - d_M d_1)z^{-M+1}}. \end{aligned}$$

It can be proved that the all-pass filter $A_M(z)$ is BIBO stable if and only if $|K_M| < 1$ and $A_{M-1}(z)$ is BIBO stable.

Proof:

The proof is based on the fact that if the real-coefficients all-pass filter $A_M(z)$ is stable then:

$$|A_M(z)| = 1 \quad \text{for} \quad |z| = 1,$$

$$|A_M(z)| < 1 \quad \text{for} \quad |z| > 1,$$

$$|A_M(z)| > 1 \quad \text{for} \quad |z| < 1.$$

We have already seen that the condition $|K_M| < 1$ is necessary for the stability.

In the hypothesis that $|K_M| < 1$ and that $A_M(z)$ is stable, let us prove that $A_{M-1}(z)$ is stable.

If λ_0 is a pole of $A_{M-1}(z)$ then λ_0 is a root of the equation

$$D_M(z) - K_M \widetilde{D}_M(z) = 0$$

i.e., $A_M(\lambda_0) = \dfrac{\widetilde{D}_M(\lambda_0)}{D_M(\lambda_0)} = \dfrac{1}{K_M}$. Since $|K_M| < 1$, $|A_M(\lambda_0)| > 1$, and λ_0 must fall inside the unit circle because of the stability of $A_M(z)$.

Let us prove that $A_M(z)$ is stable when $|K_M| < 1$ and $A_{M-1}(z)$ is stable.

But if λ_0 is a pole of $A_M(z)$, $D_M(\lambda_0) = 0$,

$$A_{M-1}(\lambda_0) = -\lambda_0 \frac{\widetilde{D}_M(\lambda_0)}{K_M \widetilde{D}_M(\lambda_0)} = -\frac{\lambda_0}{K_M}$$

Since $|K_M| < 1$, $\left|\dfrac{1}{\lambda_0} \cdot A_{M-1}(\lambda_0)\right| > 1$, and

$$|A_{M-1}(\lambda_0)| > |\lambda_0|$$

If for absurd we assume $|\lambda_0| > 1$, we also have $|A_{M-1}(\lambda_0)| > 1$, which contradicts the hypothesis of stability of $A_{M-1}(z)$. Thus, it must be $|\lambda_0| < 1$. Q.E.D.

The test procedure can be repeated. Let us consider:

$$A_{M-1}(z) = \frac{d'_{M-1} + d'_{M-2}z^{-1} + \ldots + d'_1 z^{-M+2} + z^{-M+1}}{1 + d'_1 z^{-1} + \ldots + d'_{M-1} z^{-M+1}}$$

with

$$d'_i = \frac{d_i - d_M d_{M-i}}{1 - d_M^2} = \frac{d_i - K_M d_{M-i}}{1 - K_M^2}$$

Let us set $K_{M-1} = d'_{M-1}$ and build

$$A_{M-2}(z) = z \cdot \frac{A_{M-1}(z) - K_{M-1}}{1 - K_{M-1}A_{M-1}(z)}$$

the filter $A_M(z)$ is stable if and only if $|K_M| < 1$, $|K_{M-1}| < 1$, and $A_{M-2}(z)$ is stable.
By iterating this procedure $M-1$ times, given the coefficients K_M, K_{M-1}, $\ldots$, K_1 associated with the all-pass filters $A_M(z)$, $A_{M-1}(z)$, $\ldots$, $A_1(z)$, the all-pass filter $A_M(z)$ is stable (and the polynomial $D_M(z)$ has all its roots inside the unit circle) if and only if

$$|K_i| < 1 \qquad \forall i.$$

For exercise, let's ascertain whether the polynomial $D_4(z)$,

$$D_4(z) = 1 + \frac{1}{3}z^{-1} - \frac{2}{15}z^{-2} - \frac{1}{3}z^{-3} + \frac{1}{3}z^{-4},$$

has roots inside the unit circle.
We apply the Schur-Cohn stability test. To apply the method it suffices to remember that the denominator of $A_{M-1}(z)$ is given by $[D_M(z) - K_M\widetilde{D}_M(z)]/(1-K_M^2)$.
$K_4 = \frac{1}{3}$, and $1 - K_4^2 = 1 - \frac{1}{9} = \frac{8}{9}$.

$$\begin{aligned}
D_3(z) &= [D_4(z) - K_4\widetilde{D}_4(z)]/(1-K_4^2) = \\
&= \left[1 + \frac{1}{3}z^{-1} - \frac{2}{15}z^{-2} - \frac{1}{3}z^{-3} + \frac{1}{3}z^{-4} - \frac{1}{3}\left(\frac{1}{3} - \frac{1}{3}z^{-1} - \frac{2}{15}z^{-2} + \frac{1}{3}z^{-3} + z^{-4}\right)\right]/\frac{8}{9} = \\
&= \left[\left(1 - \frac{1}{9}\right) + \left(\frac{1}{3} + \frac{1}{9}\right)z^{-1} + \left(-\frac{2}{15} + \frac{2}{45}\right)z^{-2} + \left(-\frac{1}{3} - \frac{1}{9}\right)z^{-3} + \left(\frac{1}{3} - \frac{1}{3}\right)z^{-4}\right]/\frac{8}{9} = \\
&= \left[\frac{8}{9} + \frac{4}{9}z^{-1} - \frac{4}{45}z^{-2} - \frac{4}{9}z^{-3}\right]/\frac{8}{9} = \\
&= 1 + \frac{1}{2}z^{-1} - \frac{1}{10}z^{-2} - \frac{1}{2}z^{-3}.
\end{aligned}$$

$K_3 = -\frac{1}{2}$, $1 - K_3^2 = \frac{3}{4}$

$$\begin{aligned}
D_2(z) &= [D_3(z) - K_3\widetilde{D}_3(z)]/(1-K_3^2) = \\
&= \left[1 + \frac{1}{2}z^{-1} - \frac{1}{10}z^{-2} - \frac{1}{2}z^{-3} + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{10}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}\right)\right]/\frac{3}{4} = \\
&= \left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{20}\right)z^{-1} + \left(-\frac{1}{10} + \frac{1}{4}\right)z^{-2} + \left(-\frac{1}{2} + \frac{1}{2}\right)z^{-3}\right]/34 = \\
&= \left[\frac{3}{4} + \frac{9}{20}z^{-1} + \frac{3}{20}z^{-2}\right]/34 = \\
&= 1 + \frac{3}{5}z^{-1} + \frac{1}{5}z^{-2}.
\end{aligned}$$

$K_2 = \frac{1}{5}$, $1 - K_2^2 = \frac{24}{25}$

$$\begin{aligned}
D_1(z) &= [D_2(z) - K_2\widetilde{D}_2(z)]/(1-K_2^2) \\
&= \left[1 + \frac{3}{5}z^{-1} + \frac{1}{5}z^{-2} - \frac{1}{5}\left(\frac{1}{5} + \frac{3}{5}z^{-1} + z^{-2}\right)\right]/\frac{24}{25} = \\
&= \left[\left(1 - \frac{1}{25}\right) + \left(\frac{3}{5} - \frac{3}{25}\right)z^{-1} + \left(\frac{1}{5} - \frac{1}{5}\right)z^{-2}\right]/\frac{24}{25} = \\
&= \left[\frac{24}{25} + \frac{12}{25}z^{-1}\right]/\frac{24}{25} =
\end{aligned}$$

$$= 1 + \frac{1}{2}z^{-1}.$$

$K_1 = \frac{1}{2}.$

Since K_1, K_2, K_3, and K_4 have absolute value less than 1, the polynomial has roots inside the unit circle.

For exercise, write a Matlab function that, given the vector of the polynomial coefficients, computes the reflection coefficients K_i of the Schur-Cohn method.

For more information study:

S. K. Mitra, ”Digital Signal Processing: a computer based approach,” 4th edition, McGraw-Hill, 2011

Chapter 4.9, pp. 185-188

Chapter 7.1, pp. 333-335

Chapter 7.2.1, pp. 342-344

Chapter 7.3, pp. 349-360

Chapter 6.7.3-6.7.4, pp. 312-315

Chapter 7.2.3, pp. 346-349

Chapter 7.6, pp. 385-388

Chapter 7.9, pp. 394-399