SUPA Gravitational Wave Detection

University of Glasgow

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An Introduction to General Relativity, Gravitational Waves and Detection Principles

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Overview

These lectures present an introduction to General Relativity (GR) and its implications for the basic design properties of gravitational wave detectors.

In Sections 1 to 5 we discuss the foundations of GR and its key physical idea – that gravity manifests itself as a curvature of spacetime. We introduce the mathematical machinery necessary to describe this curvature and then briefly discuss the formulation and solution of Einstein's field equations, which encode the relationship between the curvature of spacetime and its matter and energy content.

In Sections 6 to 9 we then investigate the characteristics of *non-stationary* spacetimes. Although for simplicity we restrict our discussion to the **weak-field approximation**, we nevertheless derive a number of key results for gravitational wave research:

- We show that the free-space solutions for the metric perturbations of a 'nearly flat' spacetime take the form of a wave equation, propagating at the speed of light. This encapsulates the central physical idea of General Relativity: that the instantaneous 'spooky action at a distance' of Newton's gravitational force is replaced in Einstein's theory by spacetime curvature. Moreover, changes in this curvature – the so-called 'ripples in spacetime' beloved of popular accounts of gravitational waves – propagate outward from their source at the speed of light in a vacuum.

- We consider carefully the mathematical characteristics of our metric perturbation, and show that a coordinate system (known as the **Transverse–Traceless gauge**) can be chosen in which its 16 components reduce to only 2 independent components (which correspond to 2 distinct **polarisation states**).

- We investigate the geodesic equations for nearby test particles in a 'nearly flat' spacetime, during the passage of a gravitational wave. We demonstrate the quadrupolar nature of the wave disturbance and the change in the proper distance between test particles which it induces, and explore the implications of our results for the basic design principles of gravitational wave detectors.

- As a 'taster' for the lectures to follow, we derive some basic results on the generation of gravitational waves from a simple astrophysical source: a binary star system. In particular, we estimate the amplitude and frequency from e.g. a binary neutron star system, and demonstrate that any gravitational waves detected at the Earth from such an astrophysical source will be extremely weak.

These lectures are extracted, adapted and extended from a 20 lecture undergraduate course on General Relativity and a short graduate course on gravitational waves – both of which I have taught in recent years at the University of Glasgow. Anyone who wishes to may access the complete lecture notes for the undergraduate course via the following websites:

Part 1: Introduction to General Relativity.

`http://www.astro.gla.ac/users/martin/teaching/gr1/gr1_index.html`

Part 2: Applications of General Relativity.

`http://www.astro.gla.ac.uk/users/martin/teaching/gr2/gr2_index.html`

Both websites are password-protected, with username and password 'honours'.

Suggested Reading

Much of the material covered in these lectures can be found, in much greater depth, in Bernard Schutz' classic textbook '*A first Course in General Relativity*' (CUP; ISBN 0521277035). For an even more comprehensive treatment see also '*Gravitation*', by Misner, Thorne and Wheeler (Freeman; ISBN 0716703440) – although this book is not for the faint-hearted (quite literally, as it has a mass of several kg!)

Notation, Units and Conventions

In these lectures we will follow the convention adopted in Schutz' textbook and use a metric with signature $(-,+,+,+)$. Although we will generally present equations in component form, where appropriate we will write one forms, vectors and tensors in bold face (e.g. $\tilde{\mathbf{p}}$ and $\vec{\mathbf{V}}$ and $\mathbf{T}$ respectively). We will adopt throughout the standard Einstein summation convention that an upper and lower repeated index implies summation over that index. We will normally use Roman indices to denote spatial components (i.e. $i = 1, 2, 3$ etc) and Greek indices to denote 4-dimensional components (i.e. $\mu = 0, 1, 2, 3$ etc).

We will generally use commas to denote partial differentiation and semi-colons to denote covariant differentation, when expressions are given in component form. We will also occasionally follow the notation adopted in Schutz, and denote the covariant derivative of e.g. the vector field $\vec{\mathbf{V}}$ by $\nabla\vec{\mathbf{V}}$. Similarly, we will denote the covariant derivative of a vector $\vec{\mathbf{V}}$ along a curve with tangent vector $\vec{\mathbf{U}}$ by $\nabla_{\vec{\mathbf{U}}}\vec{\mathbf{V}}$

Finally, in most situations we will use so-called **geometrised units** in which $c = 1$ and $G = 1$. Thus we effectively measure time in units of **length** – specifically, the distance travelled by light in that time. Thus

$$1 \text{ second} \equiv 3 \times 10^8 \text{ m} .$$

Recall that the gravitational constant, G, in SI units is

$$G \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$$

but the Newton is a composite SI unit; i.e.

$$1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$$

so that

$$G \simeq 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$$

Replacing our unit of time with the unit of length defined above, gives

$$G \simeq 7.41 \times 10^{-28}\,\mathrm{m\,kg^{-1}}$$

So by setting $G = 1$ we are effectively measuring *mass* also in units of length. It follows that, in these new units

$$1 \text{ kg} = 7.41 \times 10^{-28} \text{ m} .$$

In summary, then, our geometrised units take the form

Unit of length:	1 m		
Unit of time:	1 m	$\equiv$	3.33×10^{-9} s
Unit of mass:	1 m	$\equiv$	1.34×10^{27} kg

1 Foundations of General Relativity

General relativity (GR) is Albert Einstein's theory of gravitation, which he published in 1916, 11 years after publishing his Special Theory of Relativity. While the latter theory probably has had wider cultural impact (*everyone* has heard of "E equals mc squared"!) GR is widely recognised among physicists as Einstein's real masterwork: a truly remarkable achievement, greater than all the feats of his 'Annus Mirabilis' in 1905. It was described by Max Born as "the greatest feat of human thinking about nature, the most amazing combination of philosophical penetration, physical intuition and mathematical skill".

GR explains gravitation as a consequence of the curvature of spacetime, while in turn spacetime curvature is a consequence of the presence of matter. Spacetime curvature affects the movement of matter, which reciprocally determines the geometric properties and evolution of spacetime. We can sum this up neatly (slightly paraphrasing a quotation due to John Wheeler) as follows:

Spacetime tells matter how to move,

and matter tells spacetime how to curve

A useful metaphor for gravity as spacetime curvature is to visualise a stretched sheet of rubber, deformed by the presence of a massive body (see Figure 1).

To better understand how Einstein arrived at this remarkable theory we should first briefly consider his Special Theory of Relativity.

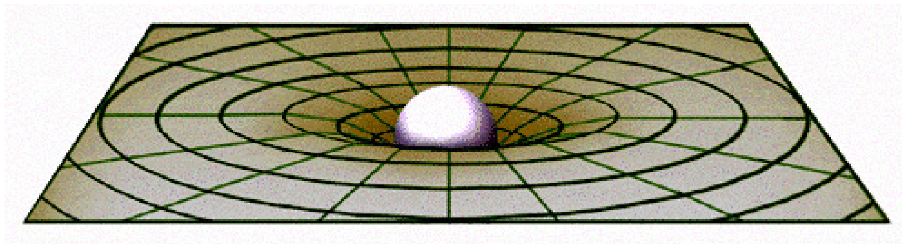

Figure 1: Familiar picture of spacetime as a stretched sheet of rubber, deformed by the presence of a massive body. A particle moving in the gravitational field of the central mass will follow a curved – rather than a straight line – trajectory as it moves across the surface of the sheet and approaches the central mass.

1.1 Special relativity

GR is a generalisation of special relativity (SR), in which Einstein set out to formulate the laws of physics in such a way that they be valid in all **inertial reference frames** – i.e. all frames in which Newton's first and second laws of motion hold – independently of their relative motion. At the heart of SR is the fundamental postulate of relativity: that the speed of light in the vacuum be the same for every inertial observer. Einstein showed that, as a consequence of this postulate, the Newtonian concept of a rigid framework of space and time against which physical phenomena were played out was no longer tenable: measurements of time and space cannot be absolute, but depend on the observer's motion and are related via the **Lorentz transformations**. Space and time as distinct entities gave way to a unified spacetime, and only the **spacetime interval** between events is independent of the observer's reference frame (referred to as **Lorentz frame**). For neighbouring events taking place at spacetime coordinates (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$, the interval is defined by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

Another inertial observer using a different coordinate system (t', x', y', z') (i.e. in a different Lorentz frame) will find the same value for the spacetime interval between the events, despite measuring them to have a different spatial and temporal separation. Moreover:

- Intervals between neighbouring events with $ds^2 < 0$ are **timelike**; there exists a Lorentz frame ($\mathcal{A}$) in which the events occur at the same spatial coordinates, and in frame $\mathcal{A}$ the coordinate time separation of the events is equal to the **proper time** interval between them. Furthermore, the two events could lie on the **worldline** (i.e. the trajectory through spacetime) of a material particle.

- Intervals between neighbouring events with $ds^2 > 0$ are **spacelike**; there exists a Lorentz frame ($\mathcal{B}$) in which the events occur at the same coordinate time, and in frame $\mathcal{B}$ the spatial separation of the events is equal to the **proper distance** between them. The two events could *not* lie on the worldline of a material particle (since in frame $\mathcal{B}$ the particle would appear to be in two places at once!).

- Intervals between neighbouring events with $ds^2 = 0$ are **null** or **lightlike**; the two events could lie on the worldline of a photon

The geometry described by equation (1) differs markedly from that of 'flat' Euclidean space, and is known as Minkowski spacetime. We can re-write equation (1) more generally as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

where summation over the indices μ and ν is implied and both indices range over $0, 1, 2, 3$, corresponding to t, x, y, z. In equation (2) the $g_{\mu\nu}$ are components of the **metric** (we will provide a more formal definition of the metric shortly) which describes how intervals are measured in our spacetime. In general these components may be complicated functions of the spacetime coordinates but for Minkowski spacetime, in Cartesian coordinates and setting $c = 1$, the metric takes a very simple form

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{3}$$

How does Newtonian gravity fit into this SR picture? We see immediately that the answer is “not well”. Newtonian gravity is inherently non-relativistic since it describes the gravitational force between two masses as acting instantaneously and as depending on the distance separating the two masses. Different inertial observers would not agree about either point, and so would not agree about the force of gravity between the two masses. And what about *non-inertial* observers? The elegant geometry of Minkowski spacetime applied only to observers in uniform relative motion, not to accelerated motion. Yet Einstein sought a **fully covariant** theory, capable of describing the laws of physics (including gravity) in *any* coordinate system and for *any* relative motion. The key step towards achieving this came with Einstein's realisation that gravity and accleration are fundamentally equivalent – an idea enshrined in the **principle of equivalence**.

1.2 The equivalence principles

The principle of equivalence is often presented in two distinct forms: the **weak equivalence principle** and the **strong equivalence principle**.

1.2.1 The weak equivalence principle

In Newtonian physics the **inertial mass** of a body is a measure of its resistance to acceleration, and is the quantity that appears on the right hand side of the equation describing Newton's second law: “force equals mass times acceleration”. The gravitational mass of a body, on the other hand, is the quantity that appears in Newton's law of universal gravitation. There is no *a priori* reason, in Newtonian physics, why these two masses should be equal and yet experimentally they are found to be identically so, to extremely high precision.

The **weak equivalence principle** (WEP) takes the equivalence of inertial and gravitational mass as axiomatic, stating that the inertial mass, m_I, and the gravitational mass, m_G, of a body are indeed identically equal.

A profound result follows immediately from the WEP: a massive object 'freely falling' in a uniform gravitational field (e.g. a lift plummeting Earthwards after its cable has been snapped, or the interior of a spacecraft orbiting the Earth) will obey Newton's first and second laws of motion. In other words **the freely falling object inhabits an inertial frame in which all gravitational forces have disappeared**.

We call the reference frame inhabited by our freely-falling object a **local inertial frame** (LIF): the reference frame is only inertial over the region of spacetime for which the gravitational field is uniform. (The effective size of the LIF therefore depends on how rapidly the gravitational field varies as a function of position, and on how accurately we can measure the separation and velocity of freely-falling bodies).

1.2.2 The strong equivalence principle

The strong equivalence principle (SEP) goes further and states that locally (i.e. in a LIF) *all* the laws of physics have their usual special relativistic form – except for gravity, which simply disappears. Moreover, the SEP states that there is *no* experiment that we can carry out to distinguish between a LIF which is freely-falling in a uniform gravitational field and an inertial frame which is in a region of the Universe far from any gravitating masses (and therefore well-described by the SR geometry of Minkowski spacetime).

Conversely, if we are inside a spaceship in our remote region of the cosmos free from gravity, we can *simulate* gravity by giving the rocket an acceleration; this acceleration will be indistinguishable from a uniform gravitational field with equal (but opposite) gravitational acceleration – gravity and acceleration are equivalent.

The equivalence principles have a number of important, and testable, physical consequences:

1. The empirically observed equality of gravitational and inertial mass is explained.

2. The acceleration of a test mass[1] in a gravitational field is entirely independent of its nature, mass and composition.

3. The path of a light ray will be bent by the gravitational field of a massive body.

4. A light ray emitted from the surface of a massive body will be *redshifted* (the effect is referred to as the **gravitational redshift**) when its wavelength is measured by a distant observer.

Experimental verification of the second point has a long and illustrious history (from Galileo to Apollo 15!). Experimental measurement of the gravitational redshift has been carried out from various astrophysical sources, as well as terrestrially via the Pound-Rebka experiment. (See e.g. Wikipedia for a clear and concise overview). We will briefly discuss gravitational light deflection later, in the context of the so-called *Schwarzschild solution* of Einstein's equations.

1.3 From special to general relativity

So the equivalence principles tell us that – provided our gravitational field is uniform (or can be reasonably approximated to be uniform) we can define a LIF: a coordinate system within which gravity has locally been 'transformed away', the laws of physics agree with SR and the metric of spacetime can be reduced to the simple Minkowski form of equation (3). Yet this is only the first step towards a fully covariant theory of gravity. In general gravitational fields are decidedly *not* uniform (the inertial frames defined by two freely-falling lifts in London and Sydney are very different) so we can *only* transform away their effects locally.

[1] i.e. an object with mass that is sufficiently small that it produces no measurable change in the gravitational field of the larger body towards which it is attracted.

In order to be an effective theory of gravity, GR must provide us with the means to 'stitch together' LIFs across extended regions of spacetime, containing non-uniform gravitational fields. This stitching process turns the locally 'flat' geometry of Minkowski spacetime – which can be applied to each LIF – into the curved geometry that characterises the gravitational field of extended regions.

To describe spacetime curvature rigorously will require mathematical tools from the field of differential geometry: essentially extending the familiar description of physical quantities in terms of scalars and vectors to **tensors**. We will discuss tensors and their properties sparingly, in order not to get too bogged down in mathematics and too far removed from the physics of GR, but some discussion of tensors will be essential if we are to fully understand GR and its theoretical importance for the study of gravitational waves.

Before we introduce tensors to our mathematical toolbox, however, we can first gain some further physical insight into the relationship between gravity, acceleration and spacetime curvature by considering in a simplified way the trajectories of freely falling particles – paths known in GR as **geodesics**.

2 Introduction to Geodesic Deviation

2.1 Basic concepts of geodesics

According to Newton's laws the 'natural' trajectory of a particle which is not being acted upon by any external force is a straight line. In GR, since gravity manifests itself as spacetime curvature, these 'natural' straight line trajectories generalise to curved paths known as **geodesics**. These are defined physically as the trajectories followed by freely falling particles – i.e. particles which are not being acted upon by any *non-gravitational* external force.

Geodesics are defined *mathematically* as spacetime curves that parallel transport their own tangent vectors – concepts that we will explain in Section 3. For metric spaces (i.e. spaces on which a metric function can be defined) we can also define geodesics as extremal paths in the sense that – along the geodesic between two events E_1, and E_2, the elapsed proper time is an extremum, i.e.

$$\delta \int_{E_1}^{E_2} d\tau = 0 \tag{4}$$

Mathematically, the curvature of spacetime can be revealed by considering the deviation of neighbouring geodesics. The behaviour of geodesic deviation is represented qualitatively in Figure 2, which shows three 2-dimensional surfaces of different intrinsic curvature on which two ants are moving along neighbouring, initially parallel trajectories. On the leftmost surface, a flat piece of paper with zero intrinsic curvature, the separation of the ants remains constant as they move along their neighbouring geodesics. On the middle surface, a spherical tennis ball with positive intrinsic curvature, the ants' separation *decreases* with time – i.e. the geodesics move towards each other. On the rightmost surface, a 'saddle' shape with negative intrinsic curvature, the ants' separation *increases* with time – i.e. the geodesics move apart.

Specifically it is the **acceleration of the deviation between neighbouring**

geodesics which is a signature of spacetime curvature, or equivalently (as we would describe it in Newtonian physics) the presence of a non-uniform gravitational field. This latter point is important: geodesic deviation cannot distinguish between a *zero* gravitational field and a *uniform* gravitational field; in the latter case the acceleration of the geodesic deviation is also zero. Only for a non-uniform, or **tidal** gravitational field does the geodesic deviation accelerate. We will see later that it is precisely these tidal variations to which gravitational wave detectors are sensitive.

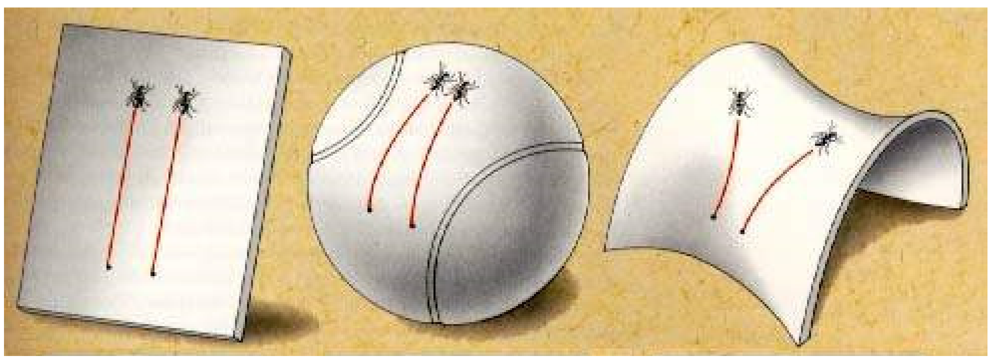

Figure 2: Geodesic deviation on surfaces of different intrinsic curvature. On a flat surface, with zero curvature, the separation of the ants remains constant – i.e. neighbouring geodesics remain parallel. On the surface of a sphere, with positive curvature, the separation of the ants decreases – i.e. neighbouring geodesics converge. On the surface of a saddle, with negative curvature, the separation of the ants increases – i.e. neighbouring geodesics diverge.

Before we consider a full GR description of the relationship between geodesic deviation and spacetime curvature, it is instructive to consider a rather simpler illustration: a Newtonian description of the behaviour of neighbouring free-falling particles in a non-uniform gravitational field.

2.2 Geodesic deviation in Newtonian gravity

Figure 3 is a (hugely exaggerated) cartoon illustration of two test particles that are initially suspended at the same height above the Earth's surface (assumed to be spherical) and are released from rest. According to Newtonian physics, the separation of these test particles will reduce as they freely fall towards the Earth, because they are falling in a non-uniform gravitational field. (The gravitational force on each particle is directed towards the centre of the Earth, which means that their acceleration vectors are not parallel).

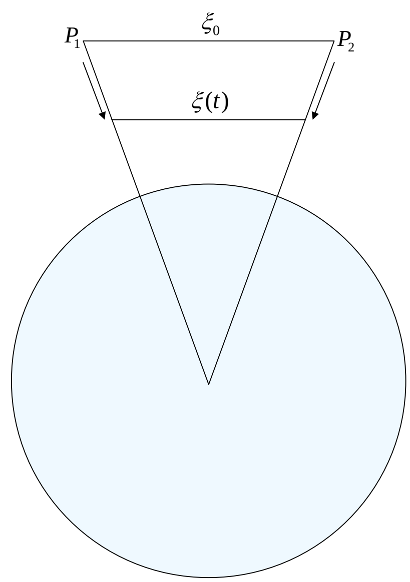

Figure 3: Cartoon illustrating how in Newtonian physics the separation of test particles will change in time if they are falling freely in a non-uniform gravitational field.

Suppose the initial separation of the test particles is ξ_0 and their distance from the centre of the Earth is r_0, while after some time t their separation is $\xi(t)$ and their distance from the centre of the Earth is $r(t)$. From similar triangles we can see that

$$\frac{\xi(t)}{r(t)} = \frac{\xi_0}{r_0} = k \tag{5}$$

where k is a constant. Taking derivatives with respect to time gives

$$\ddot{\xi} = k\ddot{r} = -\frac{kGM}{r^2} \tag{6}$$

where M is the mass of the Earth. Substituting for $k = \xi/r$ gives

$$\ddot{\xi} = -\frac{\xi}{r}\frac{GM}{r^2} = -\frac{GM\xi}{r^3} \tag{7}$$

If the test particles are released close to the Earth's surface then $r \approx R$, where R is the radius of the Earth, so $\ddot{\xi} = -GM\xi/R^3$. We can re-define our time coordinate and express this equation as

$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM}{R^3c^2}\,\xi \tag{8}$$

Notice that in equation (8) the coefficient of ξ on the right hand side has dimensions $[\text{length}]^{-2}$. Evaluated at the Earth's surface this quantity equals about $2\times10^{-23}\ \text{m}^{-2}$.

2.3 Intrinsic curvature and the gravitational field

We can begin to understand the physical significance of equation (8) by making use of a 2-dimensional analogy. Suppose P_1 and P_2 are on the equator of a sphere of radius a (see Figure 4). Consider two geodesics – 'great circles' of constant longitude perpendicular to the equator, passing through P_1 and P_2, and separated by a distance ξ_0 at the equator. The arc distance along each geodesic is denoted by s and the separation of the geodesics at s is $\xi(s)$.

Evidently the geodesic separation is not constant as we change s and move towards the north pole N. We can write down the differential equation which governs the change in this geodesic separation. If the (small) difference in longitude between the two geodesics is $d\phi$ (in radians) then $\xi_0 = a\,d\phi$. At latitude θ (again, in radians) corresponding to arc length s, on the other hand, the geodesic separation is

$$\xi(s) = a\cos\theta\,d\phi = \xi_0\cos\theta = \xi_0\cos s/a \tag{9}$$

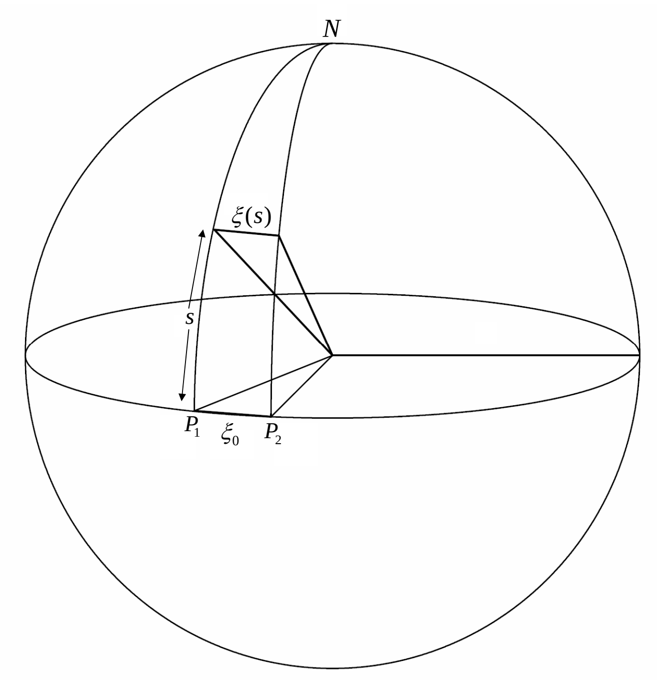

Figure 4: Illustration of the change in geodesic separation as we move along great circles of constant longitude on the surface of a sphere.

Differentiating $\xi(s)$ twice with respect to s yields

$$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi \tag{10}$$

Comparing equations (8) and (9) we see that in some sense the quantity

$$\mathcal{R} = \left\{ \frac{GM}{R^3c^2} \right\}^{-\frac{1}{2}} \tag{11}$$

represents the *radius of curvature of spacetime* at the surface of the Earth. Evaluating this radius for the Earth we find that $\mathcal{R} \sim 2 \times 10^{11}$ m. The fact that this value is so much larger than the physical radius of the Earth tells us that spacetime is 'nearly' flat in the vicinity of the Earth – i.e. the Earth's gravitational field is rather weak. (By contrast, if we evaluate $\mathcal{R}$ for e.g. a white dwarf or neutron star then we see evidence that their gravitational fields are much stronger).

We will soon return to the subject of spacetime curvature and consider how it is expressed in GR. Before we can do that, however, we need to expand the mathematical armoury at our disposal.

3 A mathematical toolbox for GR

To deal rigorously with the geometrical properties of curved spacetimes we need to introduce a number of mathematical concepts and tools. We begin with the concept of a **manifold**.

3.1 Manifolds

A manifold is a continuous space which is locally flat. More generally we can regard a manifold as any set which can be continuously parametrised: the number of independent parameters is the *dimension* of the manifold, and the parameters themselves are the *coordinates* of the manifold. A *differentiable manifold* is one which is both continuous and differentiable. This means that we can define a scalar function (or *scalar field*) – ϕ, say – at each point of the manifold, and that ϕ is differentiable.

> **Mathematical Aside:** The formal mathematics of defining coordinates need not concern us in these lectures, but the interested reader can find useful discussions in any introductory textbook on differential geometry. Loosely speaking, it involves covering the points of the manifold by a collection of *open sets*, $\mathbf{U^i}$, each of which is mapped onto $\mathbf{R^n}$ by a one-to-one mapping, ϕ^i. The pair $(\mathbf{U^i}, \phi^i)$ is called a *chart*, and the collection of charts an *atlas*. One can think of each chart as defining a different coordinate system.

In GR we are concerned with a particular class of differentiable manifolds known as **Riemannian manifolds**. A Riemannian manifold is a differentiable manifold on which a *distance*, or *metric*, has been defined.

We can see why the mathematics of Riemannian manifolds are appropriate for GR. According to the WEP spacetime is locally flat and the interval between spacetime events is described by a metric, following equation (2). Moreover, any dynamical theory of gravity will involve space and time derivatives; hence our description of spacetime should be differentiable. Riemannanian manifolds meet all three criteria.

3.2 Scalar functions on a manifold

One can define a function, f, on a manifold, $\mathbf{M}$. At any point, P, of the manifold the function takes a real value

$$f : \mathbf{M} \rightarrow \mathbf{R} \tag{12}$$

In a particular coordinate representation, P has coordinates $\{x^1, x^2, ..., x^n\}$. We may then write simply

$$f_P \quad = \quad f(x^1, x^2, ..., x^n) \tag{13}$$

Here f is called a *scalar* function; this means that its numerical value at each point of the manifold is the same real number, no matter which coordinate representation is used.

3.3 Vectors in curved spaces

The intuitive picture of a vector which we have learned in elementary maths and physics courses is based on the simple idea of an arrow representing a *displacement* between two points in space. Moreover a vector, $\vec{a}$, exists independently of our choice of coordinate system, but the *components* of $\vec{a}$ take different values in different coordinate systems, and we can define a transformation law for the components of the vector.

Consider, for example, the displacement vector, $\Delta\vec{x}$, with components Δx^μ and $\Delta x'^\mu$ in an unprimed and primed coordinate system respectively. Then, by the chain rule for differentiation

$$\Delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} \Delta x^\alpha \tag{14}$$

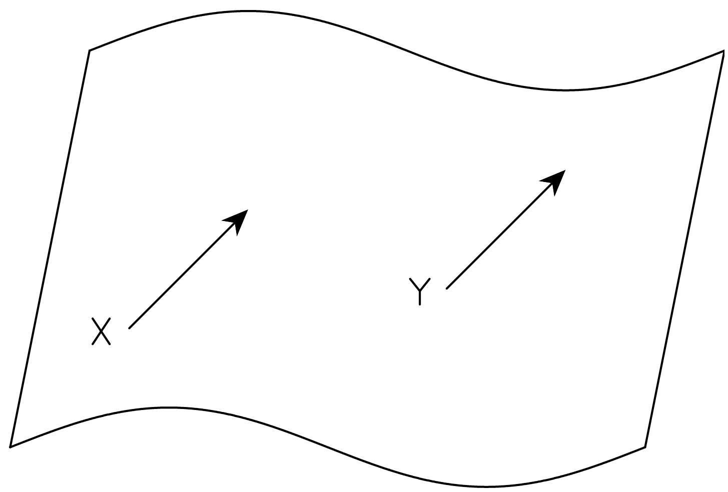

Figure 5: Vectors on a curved manifold. In general the components of $\Delta\vec{x}$ at X and of $\Delta\vec{y}$ at Y will be different, because they are defined at different points in the manifold.

Consider now two displacement vectors, $\Delta\vec{x}$ and $\Delta\vec{y}$. How can we decide if $\Delta\vec{x}$ and $\Delta\vec{y}$ are equal when – as shown in Figure 5 – they are defined at different points on our manifold? For vectors in flat space with Cartesian coordinates, for example, we can simply 'translate' $\Delta\vec{y}$ to X and compare the components of $\Delta\vec{y}$ with those of $\Delta\vec{x}$. This will **not** be valid for a general curved manifold, however, because the coefficients of the transformation law in equation (14) are in general functions of position. In other words, the transformation law between the primed and unprimed coordinate systems is in general *different* at different points of the manifold. Thus, it is *not* enough to define the components of a vector; we also need to specify the point of the manifold at which the vector (and its components) are defined.

The fact that the transformation law coefficients of equation (14) are in general functions of position also means that we have no 'universal' set of coordinate basis vectors on a curved manifold, as is the case for Euclidean space. There is, however, a means of defining a natural set of basis vectors for each point of the manifold which allows us to develop a more general picture of what we mean by a vector – and one which is equally valid in a curved spacetime.

3.4 Tangent vectors

Suppose we have a scalar function, ϕ, defined at a point, P, of a Riemannian manifold, where P has coordinates $\{x^1, x^2, ..., x^n\}$ in some coordinate system. Since our manifold is differentiable we can evaluate the derivative of ϕ with respect to each of the coordinates, x^i, for $i = 1, ..., n$. In fact, since ϕ is completely arbitrary, we can think of the derivatives as a set of n 'operators', denoted by

$$\frac{\partial}{\partial x^i}$$

These operators can act on any scalar function, ϕ, and yield the rate of change of the function with respect to the x^i.

We can now define a **tangent vector** at point, P, as a linear operator of the form

$$a^\mu \frac{\partial}{\partial x^\mu} \equiv a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + ... + a^n \frac{\partial}{\partial x^n} \tag{15}$$

(Note the use of the summation convention). This tangent vector operates on any function, ϕ, and essentially gives the rate of change of the function – or the *directional derivative* – in a direction which is defined by the numbers $(a^1, a^2, ..., a^n)$. We can define the addition of two tangent vectors in the obvious way

$$a^\mu \frac{\partial}{\partial x^\mu} + b^\mu \frac{\partial}{\partial x^\mu} = (a^\mu + b^\mu) \frac{\partial}{\partial x^\mu} \tag{16}$$

> **Mathematical Aside:** With this straightforward definition of addition, a little formal mathematics easily shows that the set of all tangent vectors form a *vector space*

Thus, the operator,

$$a^\mu \frac{\partial}{\partial x^\mu}$$

behaves like a vector, the components of which are $(a^1, a^2, ..., a^n)$. We can therefore write

$$\vec{a} = a^\mu \frac{\partial}{\partial x^\mu} \tag{17}$$

The n operators $\frac{\partial}{\partial x^\mu}$ can be thought of as forming a set of basis vectors, $\{\vec{e_\mu}\}$, spanning the vector space of tangent vectors at P.

What exactly do these basis vectors represent? We can find a simple geometrical picture for the $\vec{e_\mu}$ by first crystallising the notion of a *curve*, C, defined on our manifold. Our intuitive notion of a curve is simply of a connected series of points on the manifold; in the mathematical literature, however, we call this a *path*, and the term *curve* is instead reserved for the particular case of a path which has been *parametrised*.

Thus, a curve is a function which maps an interval of the real line into the manifold. Putting this more simply, a curve is a path with a real number (s, say) associated with each point of the path; we call s the parameter of the curve. Note also that once we choose a coordinate system each point on the curve has coordinates, $\{x^\mu\}$, which may also be expressed as functions of the parameter, s, i.e.

$$x^\mu = x^\mu(s) \quad \mu = 1, ..., n \tag{18}$$

Once we specify our coordinate system, we can consider a particular set of curves which use the *coordinates themselves* as their parameter. For example, point P with coordinates $\{x^1, x^2, ..., x^n\}$ lies on the n curves which we obtain by allowing only the value of x^i to vary along the i^{th} curve $(i = 1, ..., n)$ and fixing all other coordinate values to be equal to their values at P. (To visualise a simple example, think of circles of equal latitude and longitude on the 2-sphere manifold). The basis vector, $\vec{e_i} \equiv \frac{\partial}{\partial x^i}$ can be thought of simply as the *tangent* to the i^{th} curve. This geometrical

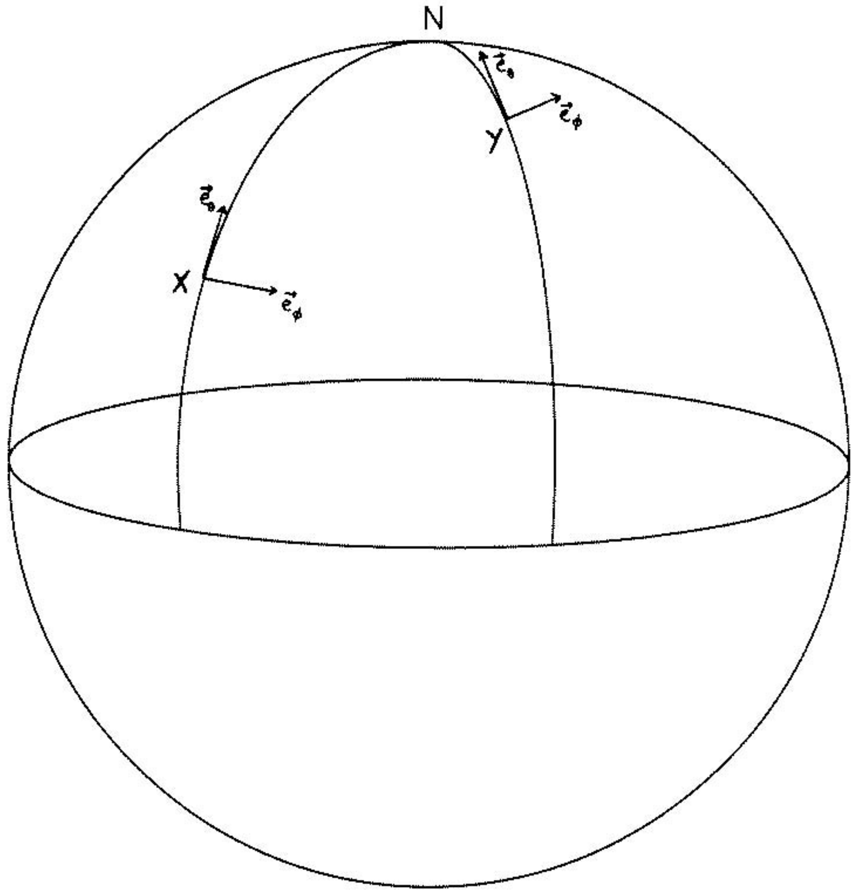

Figure 6: Illustration of the coordinate basis vectors defined at two points X and Y on the surface of a sphere. Note that the basis vectors $\vec{e_\phi}$ and $\vec{e_\theta}$ are *different* at points X and Y.

picture is illustrated in Figure 6, again for the straightforward example of the 2-sphere. Note that the basis vectors $\vec{e_\phi}$ and $\vec{e_\theta}$ are *different* at points X and Y of the manifold.

And what of a more general curve in the manifold? Here we simply connect the notion, introduced above, of a tangent vector as a directional derivative to our straightforward geometrical picture of a tangent to a curve. Figure 7 (adapted from Schutz) shows a curve, with parameter s, and with tangent vectors drawn at points with different parameter values. Suppose the coordinates of the points on the curve are $\{x^\mu(s)\}$, for $\mu = 1, ..., n$. Then the components, T^μ, of the tangent vector with respect to the basis $\{\vec{e_\mu}\} \equiv \{\frac{\partial}{\partial x^\mu}\}$ are simply given by

$$T^\mu = \frac{dx^\mu}{ds} \tag{19}$$

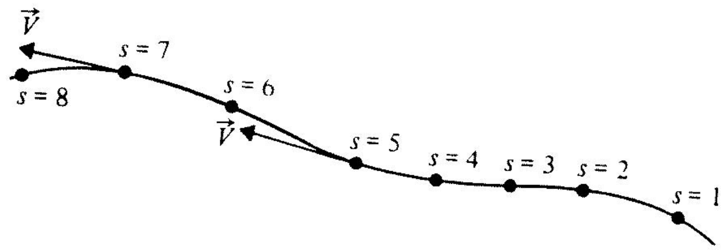

Figure 7: Schematic representation of a curve, parametrised by s, and showing tangent vectors drawn at $s = 5$ and $s = 7$.

To sum up, we can represent vectors as tangent vectors of curves in our manifold. Once we have specified our coordinate system, we can write down the components of a vector defined at any point of the manifold with respect to the natural basis generated by the derivative operators $\{\frac{\partial}{\partial x^\mu}\}$ at that point. A vector **field** can then be defined by assigning a tangent vector at *every* point of the manifold.

3.5 Transformation law for vectors

Suppose we change to a new coordinate system $\{x'^1, x'^2, ..., x'^n\}$. Our basis vectors are now

$$\vec{e'_\mu} \equiv \frac{\partial}{\partial x'^\mu}. \tag{20}$$

How do the components, $\{a^1, a^2, ..., a^n\}$, transform in our new coordinate system? To see how the law arises within the framework of our tangent vector description, let the vector $\vec{a}$ operate on an arbitrary scalar function, ϕ. Then

$$\vec{a}(\phi) = a^\nu \frac{\partial \phi}{\partial x^\nu} \tag{21}$$

By the chain rule for differentiation we may write this as

$$\vec{a}(\phi) = a^\nu \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial \phi}{\partial x'^\mu} \tag{22}$$

However, if we write $\vec{a}$ directly in terms of coordinate basis $\{\vec{e'_\mu}\} = \{\frac{\partial}{\partial x'^\mu}\}$, we have

$$\vec{a}(\phi) = a'^\mu \frac{\partial \phi}{\partial x'^\mu} \tag{23}$$

Comparing equation (22) with (23) we see that

$$a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} a^\nu \tag{24}$$

Thus the components of the tangent vector transform according to equation (24). We call this equation the transformation law for a **contravariant vector**, and say that the components of $\vec{a}$ transform **contravariantly**. (The term 'contravariant' is used to distinguish these vectors from another type of geometrical object – covariant vectors or 'covectors' – which we will meet in the next subsection. The more modern name for covariant vectors, however, is 'one-forms', and we will generally adopt that name in order to avoid this source of ambiguity). We denote the components of a contravariant vector using superscripts.

Equation (24) is the prototype transformation law for *any* contravariant vector. Any set of n components, A^μ, which can be evaluated in any coordinate system, and which transform according to the transformation law of equation (24), we call a contravariant vector.

3.6 Transformation law for one-forms

What is the relationship between the basis vectors $\vec{e'_\mu}$ and $\vec{e_\mu}$ in the primed and unprimed coordinate systems? From equation (20) we have

$$\vec{e'_\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \vec{e_\nu} \tag{25}$$

Thus we see that the basis vectors do *not* transform in the same way as the components of a contravariant vector. This should not be too surprising, since the transformation of a basis and the transformation of components are different things: the former is the expression of *new* vectors in terms of *old* vectors; the latter is the expression of the *same* vector in terms of a new basis.

In fact, the form of the transformation in equation (25) is the same as the transformation law for another type of geometrical object, which we call a **covariant vector**, **covector**, or (in more modern literature) a **one-form**. Any set of n components, A_μ, which can be evaluated in any coordinate system, is said to be a one-form if the components transform according to the equation

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \tag{26}$$

One can simply regard equation (26) as *defining* a one-form. Many modern textbooks on differential geometry, however, begin by defining a one-form as a linear mapping which acts on a vector to give a real number. Starting from this definition one can then arrive at equation (26).

One-forms are usually denoted by a tilde above a symbol, just as vectors are denoted by an arrow above a symbol. Thus e.g. $\tilde{p}(\vec{a})$ is a real number.

3.7 Transformation law for tensors

Having defined what we mean by vectors and one-forms, in terms of how their components transform under a general coordinate transformation, we can now extend our definition to the more general class of geometrical object which we call *tensors*.

A tensor of type (l, m), defined on an n dimensional manifold, is a linear operator which maps l one-forms and m (contravariant) vectors into a real number (i.e.

scalar). Such a tensor has a total of n^{l+m} components.

The transformation law for a general (l, m) tensor follows from its linearity, and from the transformation laws for a vector and one-form, in order that the scalar quantity obtained when the tensor operates on l one-forms and m vectors is independent of one's choice of coordinate system. We can write this general transformation law as follows

$$A'^{u_1\,u_2\,\ldots\,u_l}_{r_1\,r_2\,\ldots\,r_m} = \frac{\partial x'^{u_1}}{\partial x^{t_1}} \ldots \frac{\partial x'^{u_l}}{\partial x^{t_l}} \frac{\partial x^{q_1}}{\partial x'^{r_1}} \ldots \frac{\partial x^{q_m}}{\partial x'^{r_m}} A^{t_1\,t_2\,\ldots\,t_l}_{q_1\,q_2\,\ldots\,q_m} \tag{27}$$

This rather intimidating equation appears much more straightforward for some specific cases. First note that a contravariant vector is in fact a $(1, 0)$ tensor (since it operates on a one-form to give a scalar). Similarly a one-form is a $(0, 1)$ tensor (and more trivially a *scalar* is a $(0, 0)$ tensor).

A $(2, 0)$ tensor, say T^{ij}, is called a *contravariant* tensor of rank 2 and transforms according to the transformation law

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} T^{kl} \tag{28}$$

A $(0, 2)$ tensor, say B_{ij}, is called a *covariant* tensor of rank 2, and transforms according to the law

$$B'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} B_{kl} \tag{29}$$

An important example of a $(0, 2)$ tensor is the *metric tensor*, $g_{\mu\nu}$, which we already met in Section 1. This is a symmetric tensor, i.e.

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{for all}\, \mu, \nu \tag{30}$$

We can now see that the form of equation (2) makes sense: the small displacements dx^μ and dx^ν transform as contravariant vectors; the metric tensor operates on these vectors to give a scalar (the interval) which is invariant.

A tensor which has both upper and lower indices, which means that it has both contravariant and covariant terms in its transformation law, is known as a *mixed tensor*. The simplest example (after the trivial case of a $(0,0)$ tensor) is a $(1,1)$ tensor, D^i_j, say. Its transformation law is

$$D'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} D'^k_l \tag{31}$$

An important example of a $(1,1)$ tensor is the Kronecker delta, δ^i_j, which has the property (in *any* coordinate system)

$$\delta^l_j = 1 \text{ when } j = l, \text{ and 0 otherwise} \tag{32}$$

3.8 Contraction of tensors

We can take the **inner product**, or **contraction** of a vector and one-form; i.e. we form the quantity $A^i B_i$ (where, as usual, the summation convention is implied). This quantity is invariant in the sense that

$$A'^j B'_j = A^i B_i \tag{33}$$

We can generalise the operation of contraction to the case of any two tensors, and over an arbitrary number of indices, provided that an equal number of upper and lower indices are selected. In general, contraction over k indices will produce from a tensor of type (l, m) a new tensor of type $(l-k, m-k)$. For example, the contraction of the two tensors G^{ijk}_{lm} and R^s_{tu} over the indices i and t, j and u and l and s will give the $(1,1)$ tensor $G^{ijk}_{lm} R^l_{ij}$, where now only the indices k and m are free indices.

3.9 Raising and lowering indices

Given any contravariant vector A^μ it is possible to define, via the metric tensor, an associated one-form, which we denote as A_ν and which is defined by

$$A_\nu = g_{\mu\nu} A^\mu \tag{34}$$

This operation is often called *lowering the index*.

Similarly by using g^{ij} we can raise the index of a covariant quantity B_i to obtain a contravariant quantity B^i, viz.

$$B^i = g^{ij} B_j \tag{35}$$

The process of raising or lowering indices can be carried out with tensors of any rank and type. For example

$$D^{ijk..}_{lm} = g_{lp} g_{mq} D^{ijkpq} \tag{36}$$

Some care must be taken in positioning the indices. The dots have been placed here to indicate the indices over which contraction has taken place, although in general we shall omit the dots and just write D^{ijk}_{lm}. Note that $D^{..ijk}_{lm}$ defined by

$$D^{..ijk}_{lm} = g_{lp} g_{mq} D^{pqijk} \tag{37}$$

is not the same as $D^{ijk..}_{lm}$ unless D^{ijkpq} possesses some symmetry.

The *magnitude* of a vector with components A^μ is $g_{\mu\nu} A^\mu A^\nu$, which is of course invariant, since $g_{\mu\nu}$ is a $(0,2)$ tensor and A^μ and A^ν are both $(1,0)$ tensors. Notice that

$$g_{\mu\nu} A^\mu A^\nu = A_\mu A^\nu = g^{\mu\nu} A_\nu A_\nu \tag{38}$$

The quantity $g_{\mu\nu} A^\mu B^\nu$ is the scalar product of the two vectors.

3.10 Covariant differentiation and parallel transport

Any dynamical physical theory must deal in time varying quantities and, if this theory is also to be relativistic, spatially varying quantities too. Since GR is a covariant theory, we are confronted with the problem of constructing quantities that represent rates of change, but which can be defined in any coordinate system. In other words, we need to define a *derivative* which transforms covariantly under a general coordinate transformation: we call this a **covariant derivative**.

For any scalar function, say ϕ, defined on the manifold, the partial derivative

$$\phi_{,\nu} \equiv \frac{\partial \phi}{\partial x^{\nu}}$$

transforms as a $(0,1)$ tensor, i.e.

$$\phi'_{,\nu} \equiv \frac{\partial \phi'(x')}{\partial x'^{\nu}} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \frac{\partial \phi(x)}{\partial x^{\mu}} \equiv \frac{\partial x^{\mu}}{\partial x'^{\nu}} \phi_{,\mu} \tag{39}$$

On the other hand, partial derivatives of the components of a contravariant vector transform as:

$$A'^{i}_{,j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} A^{k}_{,l} + \frac{\partial x^{l}}{\partial x'^{j}} \frac{\partial^2 x'^{i}}{\partial x^{l} \partial x^{k}} A^{k} \tag{40}$$

The presence of the second term of equation (40),

$$\frac{\partial x^{l}}{\partial x'^{j}} \frac{\partial^2 x'^{i}}{\partial x^{l} \partial x^{k}} A^{k},$$

is the reason why $A^{i}_{,j}$ does not transform as a tensor. The root of the problem is that computing the derivative involves subtracting vectors at two neighbouring points, at each of which the transformation law will in general be different. To overcome this problem we need to transport one of the vectors to the neighbouring point, so that they are subtracted at the *same* point in the manifold. This can be achieved by the so-called **parallel transport** procedure, which transports the vector components along a curve in the manifold in a manner which preserves the angle between vectors.

Suppose we have a point with coordinates x^i while its neighbour has coordinates $x^i + dx^i$. It is easiest to consider first the parallel transport of the components of a *covariant* vector, or one-form, from $x^i + dx^i$ to x^i. Suppose that the change, δB_k, in the components will be a linear function of the original components, B_j, and of the displacement dx^i, so we can write

$$\delta B_k = -\Gamma^j_{ik} B_j dx^i \tag{41}$$

The coefficients Γ^j_{ik} are called the **Christoffel symbols** and they describe how the basis vectors at different points in the manifold change as one moves across the manifold, i.e.

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta} \, \vec{e}_\mu \tag{42}$$

It then follows that the covariant derivative of a one-form may be written as

$$B_{i;k} = B_{i,k} - \Gamma^j_{ik} B_j \tag{43}$$

i.e. $B_{i,k} - \Gamma^j_{ik} B_j$ transforms as a $(0, 2)$ tensor. Given that the covariant and partial derivatives of a scalar are identical, it is then straightforward to show that the covariant derivative of a vector takes the form

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{jk} A^j \tag{44}$$

which transforms as a $(1, 1)$ tensor. The generalisation of equations (43) and (44) to tensors of arbitrary rank is straightforward.

It is also straightforward to show that the covariant derivative of the metric tensor is equal to zero, i.e. for all α, β, γ

$$g_{\alpha\beta;\gamma} = 0 \quad \text{and} \quad g^{\alpha\beta}_{;\gamma} = 0 \tag{45}$$

and this result can be used to obtain an expression for the Christoffel symbols in terms of the metric and its partial derivatives

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) \tag{46}$$

3.11 The geodesic equation

As we discussed in Section 2, material particles not acted on by forces other than gravitational forces have worldlines that are **geodesics**. Similarly photons also follow geodesics. One can define a geodesic as an extremal path between two events, in the sense that the proper time along the path joining the two events is an extremum. Equivalently, one can define a geodesic as a **curve along which the tangent vector to the curve is parallel-transported**. In other words, if one parallel transports a tangent vector along a geodesic, it remains a tangent vector.

3.11.1 Geodesics of material particles

The worldline of a material particle may be written with the proper time, τ, as parameter along the worldline. The **four velocity** of the particle is the tangent vector to the worldline. One may show that the geodesic equation for the particle is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{47}$$

3.11.2 Geodesics of photons

For photons, the proper time τ cannot be used to parametrise their worldline, since $d\tau$ for a photon is zero. We need to find some other way to parametrise the worldline of the photon (e.g. an angular coordinate along the trajectory) so that, with respect to this parameter (λ, say) the geodesic equation is satisfied, i.e.

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \tag{48}$$

The parameter λ is known as an **affine parameter**.

4 Spacetime Curvature in GR

4.1 The Riemann Christoffel tensor

In our simple example of Section 2 the curvature of our spherical surface depended on only a single parameter: the radius a of the sphere. More generally, the curvature of spacetime can be described by the **Riemann Christoffel tensor**, **R** (often also referred to simply as the **Riemann tensor**), which depends on the metric and its first and second order partial derivatives. The functional form of the Riemann Christoffel tensor can be derived in several different ways:

1. by parallel transporting of a vector around a closed loop in our manifold
2. by considering the commutator of the second order covariant derivative of a vector field
3. by computing the deviation of two neighbouring geodesics in our manifold

In view of our previous discussion we will focus on the third method for deriving **R**, although there are close mathematical similarities between all three methods. Also, the first method has a particularly simple pictorial representation. Figure 8 (adapted from Schutz) shows the result of parallel transporting a vector around a closed triangle. Panel (a) shows a flat surface of zero curvature; when the vector is parallel transported from A to B to C and then back to point A, the final vector is parallel to the original one. Panel (b) on the other hand shows a spherical surface of positive curvature; when we parallel transport a vector from A to B to C and back to A, the final vector is *not* parallel to the original one. We can express the net change in the components of the vector, after transport around the closed loop, in terms of the Riemann Christoffel tensor.

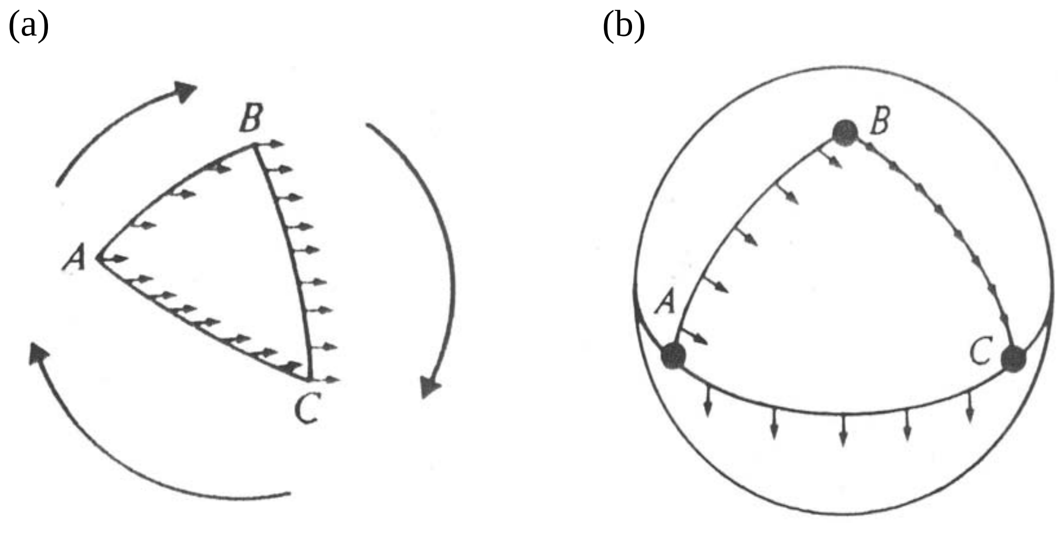

Figure 8: Parallel transport of a vector around a closed curve in a flat space (panel a) and a curved space (panel b).

4.2 Acceleration of the geodesic deviation

To see how we can derive the form of the Riemann Christoffel tensor via method 3, consider two test particles (labelled 1 and 2) moving along nearby geodesics. Let $\xi^\mu(\tau)$ denote the (infinitesimal) separation of the particles at proper time τ, so that

$$x_2^\mu(\tau) = x_1^\mu(\tau) + \xi^\mu(\tau) \tag{49}$$

Now the worldline of each particle is described by the geodesic equation, i.e.

$$\frac{d^2 x_1^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x_1)\frac{dx_1^\alpha}{d\tau}\frac{dx_1^\beta}{d\tau} = 0 \tag{50}$$

and

$$\frac{d^2 x_2^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x_2)\frac{dx_2^\alpha}{d\tau}\frac{dx_2^\beta}{d\tau} = 0 \tag{51}$$

Note also that, by Taylor expanding the Christoffel symbols at x_1 in terms of ξ, we may write

$$\Gamma^\mu_{\alpha\beta}(x_2) = \Gamma^\mu_{\alpha\beta}(x_1 + \xi) = \Gamma^\mu_{\alpha\beta}(x_1) + \Gamma^\mu_{\alpha\beta,\gamma}\xi^\gamma \tag{52}$$

Subtracting equation (50) from equation (51), substituting from equation (52) and keeping only terms up to first order in ξ yields the following equation for the acceleration of ξ^μ (dropping the subscript 1)

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} v^\alpha \frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta} v^\beta \frac{d\xi^\alpha}{d\tau} + \Gamma^\mu_{\alpha\beta,\gamma} \xi^\gamma v^\alpha v^\beta = 0 \tag{53}$$

where we have used the fact that $v^\alpha \equiv dx^\alpha/d\tau$.

Equation (53) is not a tensor equation, since the Christoffel symbols and their derivatives do not transform as a tensor. We can develop the corresponding covariant expression, however, by taking *covariant* derivatives of the geodesic deviation. To this end, consider the covariant derivative of a general vector field $\vec{\mathbf{A}}$ along a geodesic with tangent vector $\vec{\mathbf{v}} \equiv d\vec{\mathbf{x}}/d\tau$. We write this covariant derivative as $\nabla_{\vec{\mathbf{v}}}\vec{\mathbf{A}}$, or in component form, introducing the covariant operator $D/D\tau$

$$\frac{DA^\mu}{D\tau} = v^\beta A^\mu_{;\beta} = \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} A^\alpha \frac{dx^\beta}{d\tau} \tag{54}$$

Now consider the second covariant derivative of the geodesic deviation, evaluated along the geodesic followed by test particle 1. From equation (54) in component form

$$\frac{D\xi^\mu}{D\tau} = \frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} \xi^\alpha \frac{dx^\beta}{d\tau} \tag{55}$$

It then follows that

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{D}{D\tau}\left(\frac{D\xi^\mu}{D\tau}\right) = \frac{d}{d\tau}\left(\frac{D\xi^\mu}{D\tau}\right) + \Gamma^\mu_{\sigma\delta} \frac{D\xi^\sigma}{D\tau} v^\delta \tag{56}$$

Substituting again for $D\xi^\mu/D\tau$ we obtain

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{d}{d\tau}(\frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\xi^\alpha v^\beta) + \Gamma^\mu_{\sigma\delta}(\frac{d\xi^\sigma}{d\tau} + \Gamma^\sigma_{\alpha\beta}\ \xi^\alpha v^\beta)v^\delta \tag{57}$$

Now, applying the product rule for differentiation

$$\frac{d}{d\tau}(\Gamma^\mu_{\alpha\beta}\xi^\alpha v^\beta) = \Gamma^\mu_{\alpha\beta,\gamma}\frac{dx^\gamma}{d\tau}\xi^\alpha v^\beta + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\tau}v^\beta + \Gamma^\mu_{\alpha\beta}\xi^\alpha\frac{dv^\beta}{d\tau} \tag{58}$$

Since each particle's worldline is a geodesic we also know that

$$\frac{dv^\beta}{d\tau} = \frac{d^2x^\beta}{d\tau^2} = -\Gamma^\beta_{\sigma\delta}v^\sigma v^\delta \tag{59}$$

where again we have written $v^\beta = dx^\beta/d\tau$.

Substituting equations (58) and (59) into (57) and permuting some repeated indices, we obtain

$$\begin{aligned}\frac{D^2\xi^\mu}{D\tau^2} = \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta,\gamma}v^\gamma\xi^\alpha v^\beta + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\tau}v^\beta + \Gamma^\mu_{\sigma\delta}\frac{d\xi^\sigma}{d\tau}v^\delta \\ +(\Gamma^\mu_{\beta\delta}\Gamma^\beta_{\alpha\sigma} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\alpha\delta})v^\sigma v^\delta\xi^\alpha\end{aligned} \tag{60}$$

However we can now use the result we obtained in equation (53) i.e.

$$\frac{d^2\xi^\mu}{d\tau^2} = -(\Gamma^\mu_{\alpha\beta}v^\alpha\frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}v^\beta\frac{d\xi^\alpha}{d\tau} + \Gamma^\mu_{\alpha\beta,\gamma}\xi^\gamma v^\alpha v^\beta) \tag{61}$$

so that finally we obtain the compact expression

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\gamma}\, v^\alpha\, v^\beta\, \xi^\gamma \tag{62}$$

where

$$R^\mu{}_{\alpha\beta\gamma} = \Gamma^\sigma_{\alpha\gamma}\Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\gamma} + \Gamma^\mu_{\alpha\gamma,\beta} - \Gamma^\mu_{\alpha\beta,\gamma} \tag{63}$$

One also frequently encounters the notation (see e.g. Schutz, Chapter 6)

$$\nabla_{\vec{\mathbf{v}}}\,\nabla_{\vec{\mathbf{v}}}\,\xi^\mu \;=\; R^\mu{}_{\alpha\beta\gamma}\, v^\alpha\, v^\beta\, \xi^\gamma \tag{64}$$

The $(1,3)$ tensor, $\mathbf{R}$, in equations (62) – (64) is the Riemann Christoffel tensor referred to at the beginning of this section. It is a tensor of rank 4, with 256 components – although the symmetries inherent in any astrophysical example greatly reduce the number of independent components.

If spacetime is flat then

$$R^{\mu}{}_{\alpha\beta\gamma} = 0 \tag{65}$$

i.e. all components of the Riemann Christoffel tensor are identically zero. We then see from equation (64) that the acceleration of the geodesic deviation is identically zero – i.e. the separation between neighbouring geodesics remains constant. Conversely, however, if the spacetime is curved then the geodesic separation changes along the worldline of neighbouring particles.

Equations (62) and (64) are of fundamental importance in GR. In a sense they are the properly covariant mathematical expression of the physical idea embodied in Wheeler's phrase *"spacetime tells matter how to move"*. Although we derived them by considering the geodesics of neighbouring material particles, we can equally apply them to determine how changes in the spacetime curvature will influence the geodesic deviation of photons. In that case in equation (62) we simply need to replace the proper time τ along the geodesic by another suitable affine parameter λ (say), and we also replace v^{α} by $dx^{\alpha}/d\lambda$. Otherwise, the form of equation (62) is unchanged.

The second part of Wheeler's statement – *"matter tells spacetime how to curve"* – has its mathematical embodiment in Einstein's equations, which relate the **Einstein tensor** (and thus, as we will see, implicitly the Riemann Christoffel tensor) to the **energy momentum tensor**. We consider Einstein's equations in the next section.

5 Einstein's equations

5.1 The energy momentum tensor

The energy momentum tensor (also known as the stress energy tensor) is the *source* of spacetime curvature. It describes the presence and motion of gravitating matter. In these lectures we will discuss the energy momentum tensor for the particular case of a **perfect fluid**, which is a mathematical idealisation but one which is a good approximate description of the gravitating matter in many astrophysical situations.

5.1.1 Perfect fluids

Many Newtonian gravitational problems can be considered simply as the interaction of a small number of point-like massive particles. Even in Newtonian theory, however, there are many contexts (e.g. the motion of stars in the Galaxy) where the number of gravitating 'particles' is too large to follow their individual trajectories. Instead we treat the system as a smooth continuum, or **fluid**, and describe its behaviour in terms of the locally averaged properties (e.g. the *density*, *velocity* or *temperature*) of the particles in each **fluid element** – by which we mean a small region of the fluid surrounding some point in the continuum within which the behaviour of the particles is fairly homogeneous.

This fluid description is also useful for many-particle systems in special relativity, although we must be careful about defining quantities such as density and velocity which are frame-dependent – i.e. we need to find a *covariant* description of the fluid (which, we will see, is why we require a *tensor* to describe the gravitating matter).

The simplest type of relativistic fluid is known as **dust**. To a physicist, a fluid element of dust means a collection of particles which are all at rest with respect to some Lorentz frame. Many textbooks (including Schutz) refer to this Lorentz frame as the **momentarily comoving rest frame** (MCRF) of the fluid element.

This name helps to reinforce the point that the fluid element as a whole may possess a bulk motion with respect to the rest of the fluid, and indeed this relative motion may not be uniform – i.e. the fluid element may be accelerating. At any moment, however, the instantaneous velocity of the fluid element allows us to define its MCRF, although the MCRF of neighbouring elements will in general be different at that instant, and the MCRF of the fluid element will also in general be different at different times. If the fluid element is dust, however, then at any instant in the MCRF of the fluid element the individual particles are taken to possess no random motions of their own.

Generally, however, the particles within a fluid element *will* have random motions, and these will give rise to **pressure** in the fluid (c.f. motions of the molecules in an ideal gas). A fluid element may also be able to exchange energy with its neighbours via **heat conduction**, and there may be **viscous forces** present between neighbouring fluid elements. When viscous forces exist they are directed parallel to the interface between neighbouring fluid elements, and result in a **shearing** of the fluid.

A relativistic fluid element is said to be a **perfect fluid** if, in its MCRF, the fluid element has *no* heat conduction or viscous forces. It follows from this definition that dust is the special case of a pressure-free perfect fluid.

5.1.2 Definition of the energy momentum tensor

We can define the energy momentum tensor, $\mathbf{T}$, in terms of its components in some coordinate system, $\{x^1, x^2, ..., x^n\}$, for each fluid element. Thus we define $T^{\alpha\beta}$ for a fluid element to be equal to the **flux of the α component of four momentum of all gravitating matter**[2] **across a surface of constant x^β**.

[2]By 'gravitating matter' we mean here all material particles, plus (from the equivalence of matter and energy) any electromagnetic fields and particle fields which may be present

Thus, the change, Δp^α, in the α component of the four momentum due to the flux through a surface element, ΔS_ν, at constant x^ν, is given by

$$\Delta p^\alpha = T^{\alpha\nu} \Delta S_\nu \tag{66}$$

(Note the use of the summation convention).

5.1.3 Components of T in the MCRF for dust

In this case the energy momentum tensor takes a very simple form. Since the particles in the fluid element are at rest, there is no momentum transfer. (For a general fluid, even if the particles are at rest there can be a flux of energy and momentum through heat conduction, but not for dust, which is a perfect fluid). Also there is no momentum flux, which means that $T^{ij} = 0$, $(i, j = 1, 2, 3)$. In fact the only non-zero component is $T^{00} = \rho$, the energy density of the fluid element.

5.1.4 Components of T in the MCRF for a general perfect fluid

This case is only slightly less straightforward than that of dust. Again the T^{00} component is equal to the energy density, ρ. Since there is no bulk motion of the fluid element and there is no heat conduction for a perfect fluid, the energy flux $T^{0i} = 0$ for $i = 1, 2, 3$. Moreover, from the symmetry of $\mathbf{T}$ we also have that the momentum density, $T^{i0} = 0$, for $i = 1, 2, 3$. For the spatial components, $T^{ij} = 0$ if $i \neq j$, since these terms correspond to viscous forces parallel to the interface between fluid elements and these forces are zero for a perfect fluid. Thus T^{ij} is a diagonal matrix. But T^{ij} must be diagonal in all reference frames – e.g. under all possible rotations. This is possible only if T^{ij} is a scalar multiple of the identity matrix, i.e. $T^{11} = T^{22} = T^{33}$.

Thus, T^{ii} is the flux of the i^{th} component of momentum in the x^i direction, perpendicular to the fluid element interface. Equivalently, it is the *force per unit area*, perpendicular to the interface. This is just the **pressure**, P, exerted by the random motions of the particles in the fluid element. Hence we can write $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{67}$$

5.1.5 Components of T in a general Lorentz frame

Extending our expression for $T^{\alpha\beta}$ from the MCRF to a general Lorentz frame is fairly straightforward, but the interested reader is referred e.g. to Schutz for the details and here we just state the result. If $\vec{u} = \{u^\alpha\}$ is the *four* velocity of a fluid element in some Lorentz frame, then

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P\eta^{\alpha\beta}, \tag{68}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric of SR, as given by equation (3) but with raised indices.

Conservation of energy and momentum requires that

$$T^{\alpha\beta}_{\ \ ,\beta} = 0. \tag{69}$$

i.e. the **divergence** of the energy momentum tensor is equal to zero.

5.1.6 Extending to GR

In Section 1 we introduced the strong principle of equivalence which stated that, in a LIF, all physical phenomena are in agreement with special relativity. In the light of our discussion of tensors, we can write down an immediate consequence of the strong principle of equivalence as follows

Any physical law which can be expressed as a tensor equation in SR has exactly the same form in a local inertial frame of a curved spacetime

This statement holds since, in the LIF, physics – and hence the form of physical laws – is indistinguishable from the physics of SR. This is a very important result because it allows us to generalise the form of physical laws involving tensors which are valid in SR to the case of GR, with semi-colons (denoting covariant derivatives) replacing commas (denoting partial derivatives) where appropriate.

How is this extension justified? From the principle of covariance a tensorial description of physical laws must be equally valid in any reference frame. Thus, if a tensor equation holds in one frame it must hold in any frame. In particular, a tensor equation derived in a LIF (i.e. assuming SR) remains valid in an arbitrary reference frame (i.e. assuming GR).

Hence, the energy momentum tensor for a perfect fluid in GR takes the form

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu}, \tag{70}$$

where $g^{\mu\nu}$ denotes the contravariant metric tensor for a general curved spacetime (which of course reduces locally to $\eta^{\mu\nu}$, according to the WEP).

We can extend to GR in this way the result of equation (69), on the conservation of energy and momentum. Thus, for a fluid element in a general curved spacetime

$$T^{\mu\nu}_{;\nu} = 0. \tag{71}$$

If this were *not* the case – i.e. if there existed some point, P, at which $T^{\mu\nu}_{;\nu} \neq 0$ – then we could construct a LIF at P in which all Christoffel symbols are zero. In this new frame covariant derivatives reduce to partial derivatives, implying that $T^{\mu\nu}_{,\nu} \neq 0$, which contradicts equation (69).

The general technique of using the principles of covariance and equivalence to extend the validity of tensor equations from SR to GR, usually by evaluating their components in the LIF where Christoffel symbols vanish, is a very powerful one and is commonly met in the relativity literature. It is sometimes referred to informally as the 'comma goes to semi colon rule'.

5.2 The Einstein tensor and Einstein's equations

We have seen that the Riemann Christoffel tensor, $R^{\mu}{}_{\alpha\beta\gamma}$, describes the curvature of spacetime. Given Wheeler's phrase *"matter tells spacetime how to curve"*, we expect the Riemann Christoffel tensor and energy momentum tensor to be related, and indeed they are via Einstein's equations. But we saw in Section 5.1 that **T** is a $(2,0)$ tensor. Thus, Einstein's equations must involve various *contractions* of the Riemann Christoffel tensor.

First we can contract the Riemann Christoffel tensor to form a $(0,2)$ tensor, which we call the **Ricci tensor** defined by

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma} \tag{72}$$

i.e. contracting on the *second* of the lower indices. (N.B. some authors choose to define $R_{\alpha\gamma}$ as minus this value). We can also write the components of the Ricci tensor as

$$R_{\alpha\gamma} = g^{\sigma\delta} R_{\sigma\alpha\delta\gamma} \tag{73}$$

It is easy to show that $R_{\alpha\beta} = R_{\beta\alpha}$, i.e. the Ricci tensor is symmetric.

By further contracting the Ricci tensor with the contravariant components of the metric, one obtains the **curvature scalar**, viz:

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{74}$$

One may also use the metric to raise the indices of the Ricci tensor, and thus express it in contravariant form, viz:

$$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} \tag{75}$$

$R^{\mu\nu}$ is also symmetric.

Using the contravariant form of the Ricci tensor, we define the **Einstein tensor**, **G** as follows

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{76}$$

where R is the curvature scalar. Note that since $R^{\mu\nu}$ is symmetric, so too is $G^{\mu\nu}$.

The Einstein tensor is of crucial physical significance in GR, since it can be shown from the Bianchi identities (see Appendix 1) that

$$G^{\mu\nu}_{;\nu} = 0 \tag{77}$$

Thus we have immediately **Einstein's equations** which state that

$$T^{\mu\nu}_{;\nu} = G^{\mu\nu}_{;\nu} \tag{78}$$

Einstein took the solution of these equations to be of the form

$$G^{\mu\nu} = kT^{\mu\nu} \tag{79}$$

where we can determine the constant k by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion. In fact k turns out to equal $8\pi G/c^4$.

5.3 Solving Einstein's equations

Solving Einstein's equations is in general a highly non-trivial task. Provided that the metric tensor is known, or assumed, for the system in question, then we can directly compute the Christoffel symbols, Riemann Christoffel tensor and Einstein tensor, and use these to determine the geodesics of material particles and photons.

Given the Einstein tensor we can also compute the components of the energy momentum tensor, and thus determine the spatial and temporal dependence of physical characteristics such as the density and pressure of the system.

Proceeding in the other direction, however, is considerably more difficult. Given, or assuming, a form for the energy momentum tensor, Einstein's equations immediately yield the Einstein tensor, but from that point solving for the Riemann Christoffel tensor and the metric is in general intractable.

Certain exact analytic solutions exist (e.g. the Schwarzschild, Kerr and Robertson-Walker metrics) and these are applicable to a range of astrophysical problems. The Schwarzschild metric, for example, can be applied to derive some of the classical predictions of GR, including the advance of pericentre of planetary orbits and the gravitational deflection of light. (Appendix 2 derives the form of the Schwarzschild metric and briefly considers these classical GR predictions in more detail.)

However, as we mentioned on Page 2, the description of sources of gravitational waves requires a *non-stationary* metric (see below for a precise definition) for which we must resort to approximate methods if we are to make progress analytically.

Fortunately we will see that we can describe very well at least the *detection* of gravitational waves within the so-called **weak field approximation**, in which we assume that the gravitational wave results from deviations from the flat spacetime of Special Relativity which are small.

As we will see in the following sections, the weak field approximation greatly simplifies our analysis and permits gravitational waves detected at the Earth to be described as a linear perturbation to the Minkowski metric of Special Relativity.

6 Wave Equation for Gravitational Radiation

We show that the free-space solutions for the metric perturbations of a 'nearly flat' spacetime take the form of a wave equation, propagating at the speed of light.

6.1 Non-stationarity

The Schwarzschild solution for the spacetime exterior to a point mass is an example of a **static metric**, defined as a metric for which we can find a time coordinate, t, satisfying

1. all metric components are independent of t

2. the metric is unchanged if we apply the transformation $t \mapsto -t$

A metric which satisfies property (1) but not property (2) is known as **stationary**. An example is the metric of a spherically symmetric star which is *rotating*: reversing the time coordinate changes the sense of the rotation, even though one can find a coordinate system in which the metric components are all independent of time. The Kerr metric, which can be used to describe the exterior spacetime around a rotating black hole, is an example of a stationary metric.

In these lectures we explore some consequences of also relaxing the assumption of property (1), by considering spacetimes in which the metric components are time dependent – as can happen when the source of the gravitational field is varying. Such a metric is known as **non-stationary**.

6.2 Weak gravitational fields

6.2.1 'Nearly' flat spacetimes

Since spacetime is flat in the absence of a gravitational field, a weak gravitational field can be defined as one in which spacetime is 'nearly' flat. What we mean by 'nearly' here is that we can find a coordinate system in which the metric has components

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{80}$$

where

$$\eta_{\alpha\beta} = \text{diag}\ (-1, 1, 1, 1) \tag{81}$$

is the Minkowski metric of Special Relativity, and $|h_{\alpha\beta}| << 1$ for all α and β.

A coordinate system which satisfies equations (80) and (81) is referred to as a **nearly Lorentz** coordinate system. Notice that we say that we can find *a* coordinate system satisfying these equations. It certainly does *not* follow that for *any* choice of coordinate system we can write the metric components of the nearly flat spacetime in the form of equations (80) and (81). Indeed, even if the spacetime is precisely Minkowski, we could adopt (somewhat unwisely perhaps!) a coordinate system in which the metric components were very far from the simple form of equation (81).

In some coordinate systems, therefore, the components may be enormously more complicated than in others. The secret to *solving* tensor equations in General Relativity is, often, to first choose a coordinate system in which the components are as simple as possible. In that sense, equations (80) and (81) represent a 'good' choice of coordinate system; just as equation (81) represents the simplest form we can find for the metric components in flat spacetime, so equation (80) represents the metric components of a nearly flat spacetime in their simplest possible form.

The coordinate system in which one may express the metric components of a nearly flat spacetime in the form of equations (80) and (81) is certainly not unique. If we have identified such a coordinate system then we can find an infinite family of others by carrying out particular coordinate transformations. We next consider two types of coordinate transformations which preserve the properties of equations (80) and (81). These are known respectively as **Background Lorentz transformations** and **Gauge transformations**.

6.2.2 Background Lorentz transformations

Suppose we are in the Minkowski spacetime of Special Relativity, and we define the inertial frame, S, with coordinates (t, x, y, z). Suppose we then transform to another inertial frame, S', corresponding to a **Lorentz boost** of speed v in the direction of the positive x-axis. Under the Lorentz transformation S' has coordinates given by, in matrix form

$$(t', x', y', z')^T = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t, x, y, z)^T \tag{82}$$

where $\gamma = (1 - v^2)^{-1/2}$. (Remember that we are taking $c = 1$). We can write this in more compact notation as

$$x'^{\alpha} = \Lambda^{\alpha'}_{\beta} x^{\beta} \quad \equiv \quad \frac{\partial x'^{\alpha}}{\partial x^{\beta}} x^{\beta} \tag{83}$$

The Lorentz matrix has inverse, corresponding to a boost of speed v along the *negative* x-axis, given by

$$(t, x, y, z)^T = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t', x', y', z')^T \tag{84}$$

or

$$x^{\alpha} = \Lambda^{\alpha}_{\beta'} x'^{\beta} \quad \equiv \quad \frac{\partial x^{\alpha}}{\partial x'^{\beta}} x'^{\beta} \tag{85}$$

Now suppose we are in a nearly flat spacetime in which we have identified nearly Lorentz coordinates (t, x, y, z) satisfying equations (80) and (81). We then transform to a new coordinate system (t', x', y', z') defined such that

$$x'^{\alpha} = \Lambda^{\alpha'}_{\beta} x^{\beta} \tag{86}$$

i.e. where the transformation matrix is identical in form to equation (82) for some constant v. In this new coordinate system the metric components take the form

$$g'_{\alpha\beta} \;=\; \Lambda^{\mu}_{\alpha'}\Lambda^{\nu}_{\beta'}g_{\mu\nu} \;=\; \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}g_{\mu\nu} \tag{87}$$

Substituting from equation (80) we can write this as

$$g'_{\alpha\beta} \;=\; \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}\eta_{\mu\nu} + \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}h_{\mu\nu} \tag{88}$$

Because of the particular form of the coordinate transformation in this case, it follows that

$$g'_{\alpha\beta} \;=\; \eta'_{\alpha\beta} + \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}h_{\mu\nu} \;=\; \eta_{\alpha\beta} + h'_{\alpha\beta} \tag{89}$$

(The last equation follows because the components of the Minkowski metric are the same in any Lorentz frame).

Thus, provided we consider **only** transformations of the form of equation (82) the components of $h_{\mu\nu}$ transform **as if** they are the components of a $(0,2)$ tensor defined on a background flat spacetime. Moreover, provided $v << 1$, then if $|h_{\alpha\beta}| << 1$ for all α and β, then $|h'_{\alpha\beta}| << 1$ also.

Hence, our original nearly Lorentz coordinate system remains nearly Lorentz in the new coordinate system. In other words, a spacetime which looks nearly flat to one observer still looks nearly flat to any other observer in uniform relative motion with respect to the first observer.

6.2.3 Gauge transformations

Suppose now we make a very small change in our coordinate system by applying a coordinate transformation of the form

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta}) \tag{90}$$

i.e. where the components ξ^{α} are functions of the coordinates $\{x^{\beta}\}$. It then follows that

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \xi^{\alpha}_{,\beta} \tag{91}$$

From equation (90) we can also write

$$x^{\alpha} = x'^{\alpha} - \xi^{\alpha}(x^{\beta}) \tag{92}$$

If we now demand that the ξ^{α} are small, in the sense that

$$|\xi^{\alpha}{}_{,\beta}| << 1 \quad \text{for all} \quad \alpha, \beta \tag{93}$$

then it follows by the chain rule that

$$\frac{\partial x^{\alpha}}{\partial x'^{\gamma}} = \delta^{\alpha}_{\gamma} - \frac{\partial x^{\beta}}{\partial x'^{\gamma}} \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} \simeq \delta^{\alpha}_{\gamma} - \xi^{\alpha}_{,\gamma} \tag{94}$$

where we have neglected terms higher than first order in small quantities. We have also used the fact that the components of the Kronecker delta are the same in any coordinate system.

Suppose now that the unprimed coordinate system is nearly Lorentz – i.e. the metric components satisfy equations (80) and (81). What about the metric components in the primed coordinate system?

Since the metric is a tensor, we know that

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu} \tag{95}$$

Substituting from equations (80) and (94) this becomes, to first order

$$g'_{\alpha\beta} = \left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \xi^{\mu}_{,\alpha}\delta^{\nu}_{\beta} - \xi^{\nu}_{,\beta}\delta^{\mu}_{\alpha}\right)\eta_{\mu\nu} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}h_{\mu\nu} \tag{96}$$

This further simplifies to

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{97}$$

Note that in equation (97) we have defined

$$\xi_{\alpha} = \eta_{\alpha\nu}\xi^{\nu} \tag{98}$$

i.e. we have used the Minkowski metric, rather than the full metric $g_{\alpha\nu}$ to lower the index on the vector components ξ^{ν}. This is permitted because we are working to first order, and both the metric perturbation $h_{\alpha\nu}$ and the components ξ^{ν} are small. We have also used the fact that all the partial derivatives of $\eta_{\alpha\nu}$ are zero.

Thus, equation (98) has the same form as equation (80) provided that we write

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{99}$$

Note that if $|\xi^{\alpha}{}_{,\beta}|$ are small, then so too are $|\xi_{\alpha,\beta}|$, and hence $h'_{\alpha\beta}$. Thus, our new primed coordinate system is *still* nearly Lorentz.

The above results tell us that – once we have identified a coordinate system which is nearly Lorentz – we can add an arbitrary small vector ξ^{α} to the coordinates x^{α} without altering the validity of our assumption that spacetime is nearly flat. We can, therefore, choose the components ξ^{α} to make Einstein's equations as simple as possible. We call this step choosing a **gauge** for the problem – a name which has resonance with a similar procedure in electromagnetism – and coordinate transformations of the type given by equation (99) are known as **gauge transformation**. We will consider below specific choices of gauge which are particularly useful.

6.3 Einstein's equations for a weak gravitational field

If we can work in a nearly Lorentz coordinate system for a nearly flat spacetime this simplifies Einstein's equations considerably, and will eventually lead us to spot that the deviations from the metric of Minkowski spacetime – the components $h_{\alpha\beta}$ in equation (80) – obey a **wave equation**.

Before we arrive at this key result, however, we have some algebraic work to do first. We begin by deriving an expression for the Riemann Christoffel tensor in a weak gravitational field.

6.3.1 Riemann Christoffel tensor for a weak gravitational field

In its fully covariant form the Riemann Christoffel tensor is given by

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^{\mu}_{\beta\gamma\delta} = g_{\alpha\mu} \left[\Gamma^{\sigma}_{\beta\delta} \Gamma^{\mu}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma} \Gamma^{\mu}_{\sigma\delta} + \Gamma^{\mu}_{\beta\delta,\gamma} - \Gamma^{\mu}_{\beta\gamma,\delta} \right] \tag{100}$$

Recall from the previous section that, if we are considering Background Lorentz transformations – i.e. if we restrict our attention only to the class of coordinate transformations which obey equation (86) – then the metric perturbations $h_{\alpha\beta}$ transform as if they are the components of a $(0, 2)$ tensor defined on flat, Minkowski spacetime. In this case the Christoffel symbols of the first two bracketed terms on the right hand side of equation (100) are equal to zero. It is then straightforward to show that, to first order in small quantities, the Riemann Christoffel tensor reduces to

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma} \right) \tag{101}$$

Moreover, it is also quite easy to show that equation (101) is invariant under gauge transformations – i.e. the components of the Riemann Christoffel tensor are independent of the choice of gauge. This follows from the form of equation (99) and the fact that partial derivatives are commutative – i.e.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Hence, our choice of gauge will **not** fundamentally change our determination of the curvature of spacetime or the behaviour of neighbouring geodesics.

6.3.2 Einstein tensor for a weak gravitational field

From equations (100) and (101) we can contract the Riemann Christoffel tensor and thus obtain an expression for the Ricci tensor in linearised form. This can be shown (see Appendix 3) to take the form

$$R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu\alpha} + h^{\alpha}_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right) \tag{102}$$

where we have written

$$h \equiv h^{\alpha}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta} \tag{103}$$

Recalling equation (81) we see that h is essentially the *trace* of the perturbation $h_{\alpha\beta}$. Note also that again we have raised the indices of the components $h_{\alpha\beta}$ using $\eta^{\alpha\beta}$ and not $g^{\alpha\beta}$. This is justified since $h_{\alpha\beta}$ behaves like a $(0,2)$ tensor defined on a flat spacetime, for which the metric is $\eta^{\alpha\beta}$. The derivation of equation (102) also uses the fact that all partial derivatives of $\eta^{\alpha\nu}$ are zero.

Note further that we have introduced the notation, generalising the definition of equation (98)

$$f^{\alpha} = \eta^{\alpha\nu} f_{\nu} \tag{104}$$

where f^{α} are the components of a vector. We can also extend this notation for raising and lowering indices to the components of more general geometrical objects, and to their partial derivatives. For example, in equation (102)

$$h_{\mu\nu,\alpha}{}^{,\alpha} \;=\; \eta^{\alpha\sigma}\left(h_{\mu\nu,\alpha}\right)_{,\sigma} \;=\; \eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma} \tag{105}$$

After a further contraction of the Ricci tensor, to obtain the curvature scalar, R, where

$$R = \eta^{\alpha\beta} R_{\alpha\beta} \tag{106}$$

and substitution into the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \tag{107}$$

we obtain, after further algebraic manipulation (see Appendix 3 for the details) an expression for the Einstein tensor, $G_{\mu\nu}$, in linearised, fully covariant form

$$G_{\mu\nu} = \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right] \tag{108}$$

This rather messy expression for the Einstein tensor can be simplified a little by introducing a modified form for the metric perturbation defined by

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{109}$$

after which (see Appendix 3) equation (108) becomes

$$G_{\mu\nu} = -\frac{1}{2}\left[\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} + \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} - \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} - \overline{h}_{\nu\alpha,\mu}{}^{,\alpha}\right] \tag{110}$$

6.3.3 Linearised Einstein equations

Having ploughed our way through all of the above algebra, we can now write down Einstein's equations in their linearised, fully covariant form for a weak gravitational field, in terms of the (re-scaled) metric perturbations $\overline{h}_{\mu\nu}$. Since

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{111}$$

it follows that

$$-\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} - \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} + \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} + \overline{h}_{\nu\alpha,\mu}{}^{,\alpha} = 16\pi T_{\mu\nu} \qquad (112)$$

Can we simplify equation (112) any further? Fortunately the answer is 'yes'.

We saw in Sections 6.2.3 and 6.3.1 that we can carry out a gauge transformation in a nearly Lorentz coordinate system and the new coordinate system is still nearly Lorentz, with (to first order) identical curvature. It would be useful, therefore, to find a gauge transformation which eliminated the last three terms on the left hand side of equation (112). In Appendix 4 we show that a transformation with this property always exists, and in fact is equivalent to finding a coordinate system in which

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0 \qquad (113)$$

We call this gauge transformation the **Lorentz gauge**, and it plays an important role in simplifying Einstein's equations for a weak gravitational field:

- Suppose we begin with arbitrary metric perturbation components $h^{(\mathrm{old})}_{\mu\nu}$ (defined on a background Minkowski spacetime).
- We transform $h^{(\mathrm{old})}_{\mu\nu}$ to the Lorentz gauge by finding vector components ξ^{μ} which satisfy the **Lorentz gauge condition**, explained in detail in Appendix 4. The new metric perturbation components $h^{(\mathrm{LG})}_{\mu\nu}$, in the Lorentz gauge, are given by

$$h^{(\mathrm{LG})}_{\mu\nu} = h^{(\mathrm{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \qquad (114)$$

- We convert $h^{(\mathrm{LG})}_{\mu\nu}$ to $\overline{h}^{(\mathrm{LG})}_{\mu\nu}$ using equation (109), i.e.

$$\overline{h}^{(\mathrm{LG})}_{\mu\nu} = h^{(\mathrm{LG})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h^{(\mathrm{LG})} \qquad (115)$$

- Provided the ξ^μ satisfy the Lorentz gauge condition, then the $\overline{h}_{\mu\nu}^{(\mathrm{LG})}$ components will satisfy equation (113). This in turn means that – when our metric perturbation is expressed in terms of $\overline{h}_{\mu\nu}^{(\mathrm{LG})}$ – the last three terms on the left hand side of equation (112) are all zero.

Thus, in the Lorentz gauge, the linearised Einstein field equations reduce to the somewhat simpler form (dropping the label '(LG)' for clarity)

$$-\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 16\pi T_{\mu\nu} \tag{116}$$

6.3.4 Solution to Einstein's equations in free space

In free space we can take the energy momentum tensor to be identically zero. The free space solutions of equation (116) are, therefore, solutions of the equation

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 0 \tag{117}$$

or, using equation (105)

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} \quad \equiv \quad \eta^{\alpha\alpha}\overline{h}_{\mu\nu,\alpha\alpha} \tag{118}$$

In fact, when we write out equation (118) explicitly, it takes the form

$$\left(-\frac{\partial^2}{\partial t^2} \; + \; \nabla^2 \right) \overline{h}_{\mu\nu} \quad = \quad 0 \tag{119}$$

which is often also written as

$$\Box \, \overline{h}_{\mu\nu} \quad = \quad 0 \tag{120}$$

where the operator $\Box$ is known as the *D'Alembertian*.

Remembering that we are taking $c = 1$, if instead we write

$$\eta^{00} = -\frac{1}{c^2} \tag{121}$$

then equation (119) can be re-written as

$$\left(-\frac{\partial^2}{\partial t^2} + c^2 \nabla^2 \right) \overline{h}_{\mu\nu} = 0 \tag{122}$$

This is a key result. Equation (122) has the mathematical form of a wave equation, propagating with speed c. Thus, we have shown that the metric perturbations – the 'ripples' in spacetime produced by disturbing the metric – propagate at the speed of light as waves in free space.

7 The Transverse – Traceless Gauge

We show that a coordinate system can be chosen in which the 16 components of a linear metric perturbation reduce to only 2 independent components.

We now explore further the properties of solutions to equation (119). The simplest solutions are **plane waves**

$$\overline{h}_{\mu\nu} = \mathrm{Re}\left[A_{\mu\nu}\, \exp\left(ik_\alpha x^\alpha\right)\right] \tag{123}$$

where 'Re' denotes the real part, and the constant components $A_{\mu\nu}$ and k_α are known as the **wave amplitude** and **wave vector** respectively. (Note that, as they appear in equation (123), the k_α are the components of a one-form. However, since we are considering the weak field limit of a background Minkowski spacetime, converting between covariant and contravariant components is very straightforward).

Equation (123) may appear to restrict the metric perturbations to a particular mathematical form, but *any* $\overline{h}_{\mu\nu}$ can be Fourier-expanded as a superposition of plane waves.

The wave amplitude and wave vector components are not arbitrary. Firstly, $A_{\mu\nu}$ is symmetric, since $\overline{h}_{\mu\nu}$ is symmetric. This immediately reduces the number of independent components from 16 to 10. Next, given that

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\mu\nu,\alpha\sigma} = 0 \tag{124}$$

it is easy to show that

$$k_\alpha\, k^\alpha = 0 \tag{125}$$

i.e. the wave vector is a **null vector**.

Thus, equation (123) describes a plane wave of frequency

$$\omega = k^t = \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2} \tag{126}$$

propagating in direction $(1/k^t)\,(k_x, k_y, k_z)$.

Also, it follows from the Lorentz gauge condition

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{127}$$

that

$$\left(\overline{h}^{\alpha}_{\mu}\right)_{,\alpha} = 0 \tag{128}$$

from which it then follows that

$$A_{\mu\alpha}\, k^\alpha = 0 \tag{129}$$

i.e. the wave amplitude components must be orthogonal to the wave vector **k**.

Equation (129) is, in fact, four linear equations – one for each value of the free coordinate index μ. This means that we have sufficient freedom to fix the values of four components of $A_{\mu\nu}$, thus reducing from 10 to 6 the number of independent components $A_{\mu\nu}$.

Can we restrict the components of the wave amplitude further still? The answer is again 'yes', since still we have some additional freedom remaining in our choice of gauge transformation.

Note that the transformation defined by equation (134) does **not** determine ξ^μ uniquely. We saw in Appendix 3 that the Lorentz gauge condition requires that the ξ^μ satisfy

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \xi^\mu = \overline{h}^{(\text{old})\mu\nu}_{\quad\ ,\nu} \tag{130}$$

However, to any set of components ξ^μ which satisfy equation (130), we could add the components ψ^μ to define a new transformation

$$x'^\mu \mapsto x^\mu + \zeta^\mu = x^\mu + \xi^\mu + \psi^\mu \tag{131}$$

and provided the ψ^μ satisfy

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \psi^\mu = 0 \tag{132}$$

then ζ^μ will *still* satisfy

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \zeta^\mu = \overline{h}^{(\text{old})\mu\nu}_{\quad\ ,\nu} \tag{133}$$

so that the modified metric perturbation

$$h^{(\text{TT})}_{\mu\nu} = h^{(\text{old})}_{\mu\nu} - \zeta_{\mu,\nu} - \zeta_{\nu,\mu} \tag{134}$$

still express the Einstein tensor in the simplified, Lorentz gauge form of equation (122).

The label '(TT)' stands for **Transverse Traceless**, and the gauge transformation ψ^μ (or equivalently ζ^μ) defines the **Transverse Traceless gauge**. The reason for this name, and the importance of the Transverse Traceless gauge for describing gravitational waves will become clear shortly.

Equation (132) gives us four additional equations with which we can adjust the components of our gauge transformation, in order to choose a coordinate system which makes $\overline{h}_{\mu\nu}$ – and hence $A_{\mu\nu}$ – as simple as possible. In fact, it can be shown that the freedom we retain in our choice of ψ^{μ}, while still satisfying the Lorentz gauge conditions, allows us to restrict further $A_{\mu\nu}$ by fixing the values of four more of its components, thus reducing it to having only two independent components.

Specifically, if $\mathbf{U}$ is some arbitrarily chosen four vector with components U^{β}, then we have sufficient freedom in choosing the components of ψ to ensure that the wave amplitude tensor satisfies

$$A_{\alpha\beta}\, U^{\beta} = 0 \tag{135}$$

Moreover, we can also choose the components of ψ so that

$$A^{\mu}_{\mu} = \eta^{\mu\nu}\, A_{\mu\nu} = 0 \tag{136}$$

i.e. we can set the trace of $\mathbf{A}$ to be equal to zero. (This is the origin of the 'Traceless' part of the name Transverse Traceless gauge).

To fix our ideas and to see explicitly the form of $A_{\alpha\beta}$ which emerges from our adoption of the Transverse Traceless gauge, consider a test particle experiencing the passage of a gravitational wave in a nearly flat region of spacetime. Suppose we now transform to the background Lorentz frame in which the test particle is at rest – i.e. its four-velocity U^{β} has components $(1, 0, 0, 0)$, which we may also write as

$$U^{\beta} = \delta^{\beta}_{t} \tag{137}$$

Equations (135) and (137) then imply that

$$A_{\alpha t} = 0 \quad \text{for all} \quad \alpha \tag{138}$$

Next suppose we orient our spatial coordinate axes so that the wave is travelling in the positive z-direction, i.e.

$$k^t = \omega\,, \quad k^x = k^y = 0\,, \quad k^z = \omega \tag{139}$$

and

$$k_t = -\omega\,, \quad k_x = k_y = 0\,, \quad k_z = \omega \tag{140}$$

It then follows from equation (129) that

$$A_{\alpha z} = 0 \quad \text{for all } \alpha \tag{141}$$

i.e. there is no component of the metric perturbation in the direction of propagation of the wave. This explains the origin of the 'Transverse' part of the Transverse Traceless gauge; in this gauge the metric perturbation is entirely transverse to the direction of propagation of the gravitational wave.

To summarise, in the Transverse Traceless gauge equation (123) simplifies to become

$$\overline{h}^{(\mathrm{TT})}_{\mu\nu} = A^{(\mathrm{TT})}_{\mu\nu} \cos\left[\omega(t-z)\right] \tag{142}$$

Equations (138) and (141), combined with the symmetry of $A_{\mu\nu}$, imply that the only non-zero components of $A_{\mu\nu}$ are A_{xx}, A_{yy} and $A_{xy} = A_{yx}$. Moreover, the traceless condition, equation (136), implies that $A_{xx} = -A_{yy}$. Hence, the components of $A_{\mu\nu}$ in the Transverse Traceless gauge are

$$A^{(\mathrm{TT})}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{(\mathrm{TT})}_{xx} & A^{(\mathrm{TT})}_{xy} & 0 \\ 0 & A^{(\mathrm{TT})}_{xy} & -A^{(\mathrm{TT})}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{143}$$

It then follows trivially that

$$\overline{h}^{(\mathrm{TT})}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{h}^{(\mathrm{TT})}_{xx} & \overline{h}^{(\mathrm{TT})}_{xy} & 0 \\ 0 & \overline{h}^{(\mathrm{TT})}_{xy} & -\overline{h}^{(\mathrm{TT})}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{144}$$

where

$$\overline{h}^{(\mathrm{TT})}_{xx} = A^{(\mathrm{TT})}_{xx} \cos\left[\omega(t-z)\right] \tag{145}$$

and

$$\overline{h}^{(\mathrm{TT})}_{xy} = A^{(\mathrm{TT})}_{xy} \cos\left[\omega(t-z)\right] \tag{146}$$

Finally, we should note that in the Transverse Traceless gauge it is trivial to relate the components $\overline{h}_{\mu\nu}$ to the original metric perturbation $h_{\mu\nu}$. Following equation (103) and substituting from equation (109) we define

$$\overline{h} = \eta^{\alpha\beta}\,\overline{h}_{\alpha\beta} = \eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\,h\right) = h - 2h = -h \tag{147}$$

Clearly, then, for the particular case of the Transverse Traceless gauge, both $\overline{h}$ and h are identically zero. (We should not be surprised by this, because of how we constructed h and $\overline{h}$ in the first place). It then follows trivially that for all α, β

$$\overline{h}^{(\mathrm{TT})}_{\alpha\beta} = h^{(\mathrm{TT})}_{\alpha\beta} \tag{148}$$

8 Effect of Gravitational Waves on Free Particles

We investigate the geodesic equations for the trajectories of material particles and photons in a nearly flat spacetime, during to the passage of a gravitational wave.

There is a danger that the mathematical details of the previous section might appear rather abstract, and detached from practical issues relating to the design and operation of gravitational wave detectors. However, nothing could be further from the truth. The simplifications introduced by the adoption of the Transverse Traceless gauge have important practical consequences for detector design, as we will now illustrate.

8.1 Proper distance between test particles

We saw from equations (144) – (146) that the amplitude of the metric perturbation is described by just two independent constants, A_{xx} and A_{xy}. We can understand the physical significance of these constants by examining the effect of the gravitational wave on a free particle, in an initially wave-free region of spacetime.

We choose a background Lorentz frame in which the particle is initially at rest – i.e. the initial four-velocity of the particle is given by equation (137) – and we set up our coordinate system according to the Transverse Traceless Lorentz gauge.

The free particle's trajectory satisfies the geodesic equation

$$\frac{dU^\beta}{d\tau} + \Gamma^\beta_{\mu\nu} U^\mu U^\nu = 0 \tag{149}$$

where τ is the proper time. The particle is initially at rest, i.e. initially $U^\beta = \delta^\beta_t$.

Thus, the initial acceleration of the particle is

$$\left(\frac{dU^\beta}{d\tau} \right)_0 = -\Gamma^\beta_{tt} = -\frac{1}{2}\eta^{\alpha\beta} \left(h_{\alpha t,t} + h_{t\alpha,t} - h_{tt,\alpha} \right) \tag{150}$$

However, from equation (138)

$$A_{\alpha t} = 0 \quad \Rightarrow \quad \overline{h}_{\alpha t} = 0 \tag{151}$$

Also, recall that $h = \overline{h} = 0$. Therefore it follows that

$$h_{\alpha t} = 0 \quad \text{for all } \alpha \tag{152}$$

which in turn implies that

$$\left(\frac{dU^\beta}{d\tau} \right)_0 = 0 \tag{153}$$

Hence a free particle, initially at rest, will remain at rest indefinitely. However, 'being at rest' in this context simply means that the *coordinates* of the particle do *not* change. This is simply a consequence of our judicious choice of coordinate system, via the adoption of the Transverse Traceless Lorentz gauge. As the gravitational wave passes, this coordinate system adjusts itself to the ripples in the spacetime, so that any particles remain 'attached' to their initial coordinate positions. Coordinates are merely frame-dependent labels, however, and do not directly convey any invariant geometrical information about the spacetime.

Suppose instead we consider the **proper distance** between two nearby particles, both initially at rest, in this coordinate system: one at the origin and the other at spatial coordinates $x = \epsilon$, $y = z = 0$. The proper distance between the particles is then given by

$$\Delta\ell = \int \left| g_{\alpha\beta} dx^\alpha dx^\beta \right|^{1/2} \tag{154}$$

i.e.

$$\Delta\ell = \int_0^\epsilon |g_{xx}|^{1/2} \quad \simeq \quad \sqrt{g_{xx}(x=0)}\ \epsilon \tag{155}$$

Now

$$g_{xx}(x=0) = \eta_{xx} + h_{xx}^{(\mathrm{TT})}(x=0) \tag{156}$$

so

$$\Delta\ell \simeq \left[1 + \frac{1}{2} h_{xx}^{(\mathrm{TT})}(x=0)\right] \epsilon \tag{157}$$

Since $h_{xx}^{(\mathrm{TT})}(x=0)$ in general is **not** constant, it follows that the proper distance between the particles will change as the gravitational wave passes. It is essentially this change in the proper distance between test particles which gravitational wave detectors attempt to measure.

8.2 Geodesic deviation of test particles

We can study the behaviour of test particles more formally using the idea of **geodesic deviation**, first introduced in Section 2. We define the vector ξ^α which connects the two particles introduced above. Then, for a weak gravitational field, from equation (62) and taking proper time approximately equal to coordinate time

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = R^\alpha_{\mu\nu\beta}\, U^\mu\, U^\nu\, \xi^\beta \tag{158}$$

where U^μ are the components of the four-velocity of the two particles. Since the particles are initially at rest, then

$$U^\mu \quad = \quad (1,0,0,0)^T \tag{159}$$

and

$$\xi^\beta \quad = \quad (0,\epsilon,0,0)^T \tag{160}$$

Equation (158) then simplifies to

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = \epsilon R^\alpha_{ttx} = -\epsilon R^\alpha_{txt} \tag{161}$$

Substituting from equation (101) for a weak gravitational field, we can write down the relevant components of the Riemann Christoffel tensor in terms of the non-zero components of the metric perturbation (remembering always that we are working in the Transverse Traceless gauge)

$$R^x_{txt} = \eta^{xx} R_{xtxt} = -\frac{1}{2} h^{(\mathrm{TT})}_{xx,tt} \tag{162}$$

$$R^y_{txt} = \eta^{yy} R_{ytxt} = -\frac{1}{2} h^{(\mathrm{TT})}_{xy,tt} \tag{163}$$

Hence, two particles initially separated by ϵ in the x-direction, have a geodesic deviation vector which obeys the differential equations[3]

$$\frac{\partial^2}{\partial t^2}\,\xi^x = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx} \tag{164}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy} \tag{165}$$

Similarly, it is straightforward to show that two particles initially separated by ϵ in the y-direction, have a geodesic deviation vector which obeys the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy} \tag{166}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y = -\frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx} \tag{167}$$

[3] We are being a little sloppy in our notation here, as we have defined ξ as a $(1,0)$ tensor and $\mathbf{h}$ as a $(0,2)$ tensor. However, since we are in a background Lorentz spacetime, the x and y components of vectors and one-forms are identical.

8.3 Ring of test particles: polarisation of gravitational waves

We can further generalise equations (164) – (167) to consider the geodesic deviation of two particles – one at the origin and the other initially at coordinates $x = \epsilon\cos\theta$, $y = \epsilon\sin\theta$ and $z = 0$, i.e. in the x-y plane – as a gravitational wave propagates in the z-direction. We can show that ξ^x and ξ^y obey the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x \;=\; \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h_{xx}^{(\mathrm{TT})} + \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h_{xy}^{(\mathrm{TT})} \tag{168}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y \;=\; \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h_{xy}^{(\mathrm{TT})} - \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h_{xx}^{(\mathrm{TT})} \tag{169}$$

Substituting from equations (146) and (148), we can identify the solution

$$\xi^x \;=\; \epsilon\cos\theta + \frac{1}{2}\epsilon\cos\theta\,A_{xx}^{(\mathrm{TT})}\cos\omega t + \frac{1}{2}\epsilon\sin\theta\,A_{xy}^{(\mathrm{TT})}\cos\omega t \tag{170}$$

and

$$\xi^y \;=\; \epsilon\sin\theta + \frac{1}{2}\epsilon\cos\theta\,A_{xy}^{(\mathrm{TT})}\cos\omega t - \frac{1}{2}\epsilon\sin\theta\,A_{xx}^{(\mathrm{TT})}\cos\omega t \tag{171}$$

Suppose we now vary θ between 0 and 2π, so that we are considering an initially circular ring of test particles in the x-y plane, initially equidistant from the origin. Figure 9 shows the effect of the passage of a plane gravitational wave, propagating along the z-axis, on this ring of test particles.

The upper panel shows the case where the metric perturbation has $A_{xx}^{(\mathrm{TT})} \neq 0$ and $A_{xy}^{(\mathrm{TT})} = 0$. In this case the solutions for ξ^x and ξ^y reduce to

$$\xi^x \;=\; \epsilon\cos\theta\left(1 + \frac{1}{2}A_{xx}^{(\mathrm{TT})}\cos\omega t\right) \tag{172}$$

and

$$\xi^y \;=\; \epsilon\sin\theta\left(1 - \frac{1}{2}A_{xx}^{(\mathrm{TT})}\cos\omega t\right) \tag{173}$$

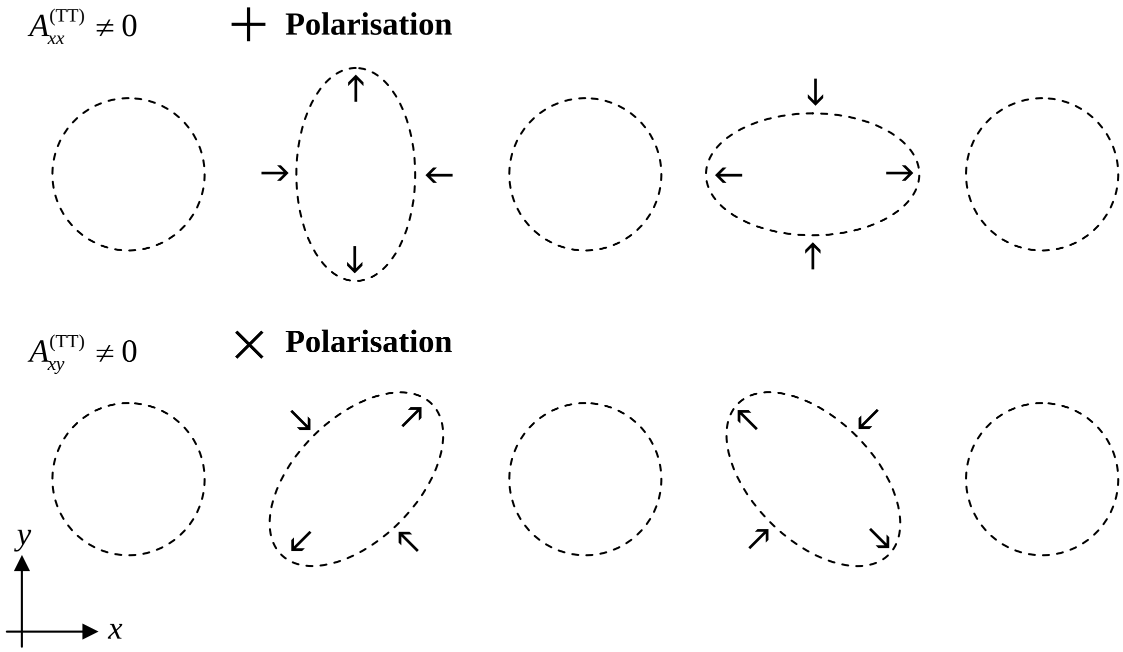

Figure 9: Cartoon illustrating the effect of a gravitational wave on a ring of test particles. The upper panel shows a wave for which $A_{xx}^{(\mathrm{TT})} \neq 0$ and $A_{xy}^{(\mathrm{TT})} = 0$, which we denote as the '+' polarisation. The lower panel shows a wave for which $A_{xy}^{(\mathrm{TT})} \neq 0$ and $A_{xx}^{(\mathrm{TT})} = 0$, which we denote as the '×' polarisation.

Each of the five rings across the upper panel of Figure 9 corresponds to a different phase (i.e. different value of ωt) in the oscillation of the wave: the first, third and fifth phases shown are all odd multiples of $\pi/2$, so that the $\cos \omega t$ terms in equations (172) and (173) vanish. The second and fourth rings, on the other hand, correspond to a phase of π and 2π respectively. At phase π we can see from equations (172) and (173) that the effect of the wave will be to move test particles on the x-axis inwards – i.e. the gravitational wave *reduces* their proper distance from the centre of the ring – while test particles on the y-axis are moved outwards – i.e. the gravitational wave *increases* their proper distance from the centre of the ring. At phase 2π, on the other hand, the wave will produce an opposite effect, increasing the proper distance from the ring centre of particles on the x-axis and reducing the proper distance of particles on the y-axis.

The lower panel of Figure 9 shows the contrasting case where the metric perturbation has $A_{xy}^{(\mathrm{TT})} \neq 0$ and $A_{xx}^{(\mathrm{TT})} = 0$. Again, the ring of test particles is shown for five different phases in the oscillation of the gravitational wave: $\pi/2$, π, $3\pi/2$, 2π and $5\pi/2$ respectively. In this case the solutions for ξ^x and ξ^y reduce to

$$\xi^x \;=\; \epsilon \cos\theta \;+\; \frac{1}{2}\epsilon \sin\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t \tag{174}$$

and

$$\xi^y \;=\; \epsilon \sin\theta \;+\; \frac{1}{2}\epsilon \cos\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t \tag{175}$$

To understand the relationship between these solutions and those for $A_{xx}^{(\mathrm{TT})} \neq 0$, we define new coordinate axes x' and y' by rotating the x and y axes through an angle of $-\pi/4$, so that

$$x' = \frac{1}{\sqrt{2}}\,(x - y) \tag{176}$$

and

$$y' = \frac{1}{\sqrt{2}}\,(x + y) \tag{177}$$

If we write the solutions for $A_{xx}^{(\mathrm{TT})} \neq 0$ in terms of the new coordinates x' and y', after some algebra we find that

$$\xi'^x \;=\; \epsilon \cos\left(\theta + \frac{\pi}{4}\right) \;+\; \frac{1}{2}\epsilon \sin\left(\theta + \frac{\pi}{4}\right) \, A_{xy}^{(\mathrm{TT})} \cos\omega t \tag{178}$$

and

$$\xi'^y \;=\; \epsilon \sin\left(\theta + \frac{\pi}{4}\right) \;+\; \frac{1}{2}\epsilon \cos\left(\theta + \frac{\pi}{4}\right) \, A_{xy}^{(\mathrm{TT})} \cos\omega t \tag{179}$$

Comparing equations (178) and (179) with equations (174) and (175) we see that our solutions with $A_{xy}^{(\mathrm{TT})} \neq 0$ are identical to the solutions with $A_{xx}^{(\mathrm{TT})} \neq 0$ apart from the rotation of $\pi/4$ – as can be seen from the lower panel of Figure 9.

We note some important features of the results of this section.

- The two solutions, for $A_{xx}^{(\mathrm{TT})} \neq 0$ and $A_{xy}^{(\mathrm{TT})} \neq 0$ represent two independent gravitational wave **polarisation states**, and these states are usually denoted by '+' and '×' respectively. In general any gravitational wave propagating along the z-axis can be expressed as a linear combination of the '+' and '×' polarisations, i.e. we can write the wave as

$$\mathbf{h} \;=\; a\,\mathbf{e}_+ \;+\; b\,\mathbf{e}_\times \tag{180}$$

where a and b are scalar constants and the *polarisation tensors* $\mathbf{e}_+$ and $\mathbf{e}_\times$ are

$$\mathbf{e}_+ \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{181}$$

and

$$\mathbf{e}_\times \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{182}$$

- We can see from the panels in Figure 9 that the distortion produced by a gravitational wave is **quadrupolar**. This is a direct consequence of the fact that gravitational waves are produced by changes in the curvature of spacetime, the signature of which is acceleration of the deviation between neighbouring geodesics. Recall our comment in Section 2.1 that geodesic deviation cannot distinguish between a zero gravitational field and a uniform gravitational field: only for a non-uniform, **tidal** gravitational field does the geodesic deviation accelerate. Such tidal variations are quadrupolar in nature. In Section 9.1 below we discuss briefly a useful analogy with electromagnetic radiation that helps to explain why gravitational radiation is (at lowest order) quadrupolar.

- We can also see from Figure 9 that, at any instant, a gravitational wave is invariant under a rotation of 180° about its direction of propagation (in this case, the z-axis). By contrast, an electromagnetic wave is invariant under a rotation of 360°, and a *neutrino* wave is invariant under a rotation of 720°. This behaviour can be understood in terms of the *spin states* of the corresponding **gauge bosons**: the particles associated with the quantum mechanical versions of these waves.

  In general, the classical radiation field of a particle of spin, S, is invariant under a rotation of $360°/S$. Moreover, a radiation field of spin S has precisely two independent polarisation states, which are inclined to each other at an angle of $90°/S$. Thus, for an electromagnetic wave, corresponding to a photon of spin $S = 1$, the independent polarisation modes are inclined at 90° to each other.

  We can, therefore, deduce from the inclination of the gravitational wave polarisation states, that the **graviton** (which is, as yet undiscovered, since we do not yet have a fully developed theory of quantum gravity!) must be a spin $S = 2$ particle. The fact that electromagnetic waves correspond to a spin $S = 1$ field and gravitational waves correspond to a spin $S = 2$ field is also intimately connected to their mathematical description in terms of geometrical objects: spin $S = 1$ fields are **vector fields**, which is why we require only a vector description for the electromagnetic field; spin $S = 2$ fields, on the other hand, are **tensor fields**, which is why we required to introduce tensors to describe the properties of the gravitational field.

8.4 The design of gravitational wave detectors: basic considerations

As we stated in Section 8.1, a change in the proper separation of test particles during the passage of a gravitational wave is the physical quantity which gravitational wave detectors are designed to measure. As will be discussed in detail in the subsequent lectures, in most of the gravitational wave detectors currently operational or planned for the future (e.g. LIGO, GEO600, VIRGO, LISA) these changes in the proper separation are monitored via measurement of the *light travel time* of a laser beam travelling back and forth along the arms of a Michelson Interferometer. Differences in the light travel time along perpendicular arms will produce interference fringes in the laser output of the interferometer.

We illustrate this in Figure 10, which shows in cartoon form the basic design of a Michelson interferometer gravitational wave detector. Here we suppose that gravitational wave with the '+' polarisation is propagating along the z-axis. Laser light of wavelength λ enters the apparatus at A, and is split into two perpendicular beams which are bounced off test mass mirrors M_1 and M_2 at the end of the arms (each of proper length L in the absence of any gravitational wave). The two beams are then re-combined and exit the system at B.

The three panels of Figure 10 denote three different phases of the wave as it passes through the system. In the left panel the wave causes no change in the proper length of the arms; in the middle panel the horizontal arm is shortened by ΔL while the vertical arm is lengthened by the same proper distance. In the right hand panel

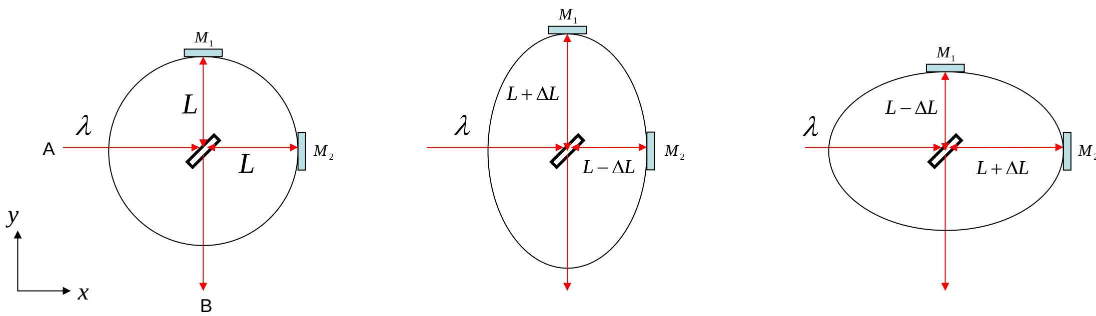

Figure 10: Cartoon illustration of the basic design features of a Michelson interferometer gravitational wave detector. See text for details.

we see the opposite: the horizontal and vertical arms are lengthened and shortened respectively by ΔL.

How are the physical dimensions of the interferometer related to the amplitude of the gravitational wave? Consider, for example, a gravitational wave $\mathbf{h} = h\mathbf{e}_+$ propagating along the z-axis. If we place two test masses along the x-axis, initially separated by proper distance L, we can see from equation (172) that the minimum and maximum proper distance between the test masses, as the gravitational wave passes, is $L - h/2$ and $L + h/2$ respectively. Thus, the fractional change $\Delta L/L$ in the proper separation of the test masses satisfies

$$\frac{\Delta L}{L} = \frac{h}{2} \tag{183}$$

Of course in general the arms of a gravitational wave detector will *not* be optimally aligned with the polarisation and direction of propagation of an incoming wave. Figure 11 sketches the orientation of one axis of a gravitational wave detector with respect to an incoming wave propagating along the z-axis. The detector axis is defined by standard spherical polar angles θ and ϕ. If the incoming wave has '+'

polarisation, i.e. $\mathbf{h} = h\,\mathbf{e}_+$, then the detector 'sees' an effective amplitude of

$$h_+ \;=\; h \,\sin^2\theta\, \cos 2\phi \tag{184}$$

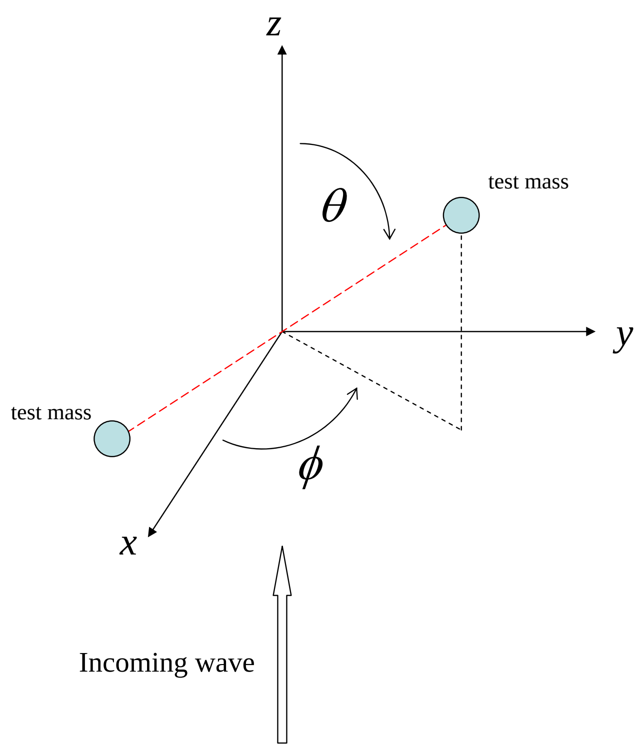

Figure 11: Cartoon illustration showing the relative orientation of a detector arm and the direction of propagation of a gravitational wave.

So we see that the wave produces a maximum response in the detector arm if $\theta = \pi/2$ and $\phi = 0$, and produces a *null* response for $\theta = 0$ or $\phi = \pi/4$. This makes sense when we consider Figure 11; we already commented previously that a metric perturbation produces no disturbance along its direction of propagation.

If, on the other hand, the incoming wave has the '$\times$' polarisation, then in this case the detector 'sees' an effective amplitude of

$$h_\times \;=\; h\,\sin^2\theta\,\sin 2\phi \tag{185}$$

Now the wave produces a maximum response for $\theta = \pi/2$ and $\phi = \pi/4$, while the response is null for $\theta = 0$ or $\phi = 0$.

How large do we expect h to be? We have developed our understanding of the physics and mathematics of gravitational waves within the framework of the weak field approximation, which requires that $h << 1$. But how small is small? Much of the remainder of this School will focus on the enormous technical challenge which the detection of gravitational waves presents because h is indeed a tiny quantity: unless we are extraordinarily lucky, then from even the most cataclysmic astrophysical sources we expect h to be no larger than one part in 10^{20}.

To end these lectures, therefore, and as a pre-cursor to the more detailed treatments which will follow in later lectures, in the final section below we briefly consider the expected magnitude and character of gravitational waves from astrophysical sources.

9 The Production of Gravitational Waves

We discuss qualitatively the reasons why gravitational radiation is quadrupolar to lowest order, and we estimate the amplitude of the gravitational wave signal from a binary neutron star system.

9.1 The quadrupolar nature of gravitational waves

We can understand something important about the nature of gravitational radiation by drawing analogies with the formulae that describe electromagnetic radiation. This approach is crude at best since the electromagnetic field is a vector field while the gravitational field is a tensor field, but it is good enough for our present purposes. Essentially, we will take familiar electromagnetic radiation formulae and simply replace the terms which involve the Coulomb force by their gravitational analogues from Newtonian theory.

9.1.1 Electric and magnetic dipoles

In electromagnetic theory, the dominant form of radiation from a moving charge or charges is **electric dipole radiation**. For a single particle (e.g. an electron) of charge, e, with acceleration, $\mathbf{a}$, and dipole moment changing as $\ddot{\mathbf{d}} = e\,\ddot{\mathbf{x}} = e\,\mathbf{a}$, the power output, or luminosity, is given by

$$L_{\text{electric dipole}} \propto e^2\,\mathbf{a}^2 \tag{186}$$

For a general distribution of charges, with net dipole moment, $\mathbf{d}$, the luminosity is

$$L_{\text{electric dipole}} \propto e^2\,\ddot{\mathbf{d}}^2 \tag{187}$$

The next strongest types of electromagnetic radiation are **magnetic dipole** and **electric quadrupole radiation**. For a general distribution of charges, the luminosity arising from magnetic dipole radiation is proportional to the second time derivative of the magnetic dipole moment, i.e.

$$L_{\text{magnetic dipole}} \propto \ddot{\mu} \tag{188}$$

where μ is given by a sum (or integral) over a distribution of charges:-

$$\mu \;=\; \sum_{q_i} \,(\text{position of } q_i) \times (\text{current due to } q_i) \tag{189}$$

9.1.2 Gravitational analogues

The gravitational analogue of the electric dipole moment is the **mass dipole moment**, $\mathbf{d}$, summed over a distribution of particles, $\{A_i\}$

$$\mathbf{d} \;=\; \sum_{A_i} m_i \mathbf{x}_i \tag{190}$$

where m_i is the rest mass and $\mathbf{x}_i$ is the position of particle A_i.

By analogy with equation (188), the luminosity of gravitational 'mass dipole' radiation should be proportional to the second time derivative of $\mathbf{d}$. However, the *first* time derivative of $\mathbf{d}$ is

$$\dot{\mathbf{d}} \;=\; \sum_{A_i} m_i \dot{\mathbf{x}}_i \quad \equiv \quad \mathbf{p} \tag{191}$$

where p is the **total linear momentum** of the system. Since the total momentum is conserved, it follows that the gravitational 'mass dipole' luminosity is zero – i.e **there can be no mass dipole radiation from any source**.

Similarly, the gravitational analogue of the magnetic dipole moment is

$$\mu \;=\; \sum_{A_i} \,(\mathbf{x}_i) \times (m_i \mathbf{v}_i) \quad \equiv \quad \mathbf{J} \tag{192}$$

where $\mathbf{J}$ is the **total angular momentum** of the system. Since the total angular momentum is *also* conserved, again it follows that the gravitational analogue of magnetic dipole radiation must have zero luminosity. Hence **there can be no dipole radiation of any sort from a gravitational source**.

The simplest form of gravitational radiation which has non-zero luminosity is, therefore, quadrupolar. We do not consider the mathematical details of quadrupolar radiation any further, save to point out that it can be shown that the quadrupole from a **spherically symmetric mass distribution** is identically zero. This suggests an important result: that, at least up to quadrupole order, **metric perturbations which are spherically symmetric do not produce gravitational radiation**. Thus, if e.g. the collapse of a massive star is spherically symmetric, it will generate *no* gravitational waves.

In fact, it is possible to prove that this result is *also* true for higher order radiation (e.g. octupole etc.), although the proof is very technical and is not discussed further. Interested readers are referred to Chapters 9 and 10 of Schutz '*A First Course in General Relativity*'.

9.2 Example: a binary neutron star system

Finally, we consider the example of the gravitational wave signature of a particular astrophysical system: a binary neutron star.

In general it can be shown (see, e.g. Schutz' textbook) that in the so-called *slow motion approximation* for a weak metric perturbation $h_{\mu\nu} << 1$ then for a source at distance r

$$h_{\mu\nu} = \frac{2G}{c^4 r} \ddot{I}_{\mu\nu} \tag{193}$$

where $I_{\mu\nu}$ is the **reduced quadrupole moment** defined as

$$I_{\mu\nu} = \int \rho(\vec{r}) \left(x_\mu x_\nu - \frac{1}{3}\delta_{\mu\nu} r^2 \right) dV \tag{194}$$

Consider a binary neutron star system consisting of two stars both of Schwarzschild mass M, in a circular orbit of coordinate radius R and orbital frequency f. For simplicity we define our coordinate system so that the orbital plane of the pulsars lies in the $x - y$ plane, and at coordinate time $t = 0$ the two pulsars lie along the $x-$axis. Substituting into equation (194)[4] it is then straightforward to show that

$$I_{xx} = 2MR^2 \left[\cos^2(2\pi f t) - \frac{1}{3}\right] \tag{195}$$

$$I_{yy} = 2MR^2 \left[\sin^2(2\pi f t) - \frac{1}{3}\right] \tag{196}$$

$$I_{xy} = I_{yx} = 2MR^2 \left[\cos(2\pi f t)\sin(2\pi f t)\right] \tag{197}$$

From equations (193) and (195) – (197) it then follows that

$$h_{xx} = -h_{yy} = h\cos\left(4\pi f t\right) \tag{198}$$

and

$$h_{xy} = h_{yx} = -h\sin\left(4\pi f t\right) \tag{199}$$

where the amplitude term h is given by

$$h = \frac{32\pi^2 G M R^2 f^2}{c^4 r} \tag{200}$$

We see from equations (198) and (199) that the binary system emits gravitational waves at twice the orbital frequency of the neutron stars.

[4]taking the mass density distribution to be a sum of dirac delta functions – i.e. treating the pulsars as point masses

How large is h for a typical source? Suppose we take M equal to the Chandrasekhar mass, $M \sim 1.4 M_{\rm solar} = 2.78 \times 10^{30}$kg. We can then evaluate the constants in equation (200) and express h in more convenient units as

$$h = 2.3 \times 10^{-28} \, \frac{R^2[\mathrm{km}] f^2[\mathrm{Hz}]}{r[\mathrm{Mpc}]} \tag{201}$$

If we take $R = 20$km, say, $f = 1000$Hz (which is approximately the orbital frequency that Newtonian gravity would predict) and $r = 15$Mpc (corresponding to a binary system in e.g. the Virgo cluster), then we find that $h \sim 6 \times 10^{-21}$.

Thus we see that the signal produced by a typical gravitational wave source places extreme demands upon detector technology. In the lectures which follow we will explore how these technological challenges are being met and overcome.

Appendix 1: Proof that $G^{\mu\nu}{}_{;\nu} = 0$

The key to proving this result is the Bianchi identities, which state that

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

Hence, our first step will be to prove that the Bianchi identities are true. The left hand side of the above equation is a $(0,5)$ tensor, so if we can show that its components are zero at an arbitrary point on the manifold in a given coordinate system, then they must be zero in all coordinate systems. We choose a LIF at arbitrary point, P and set up a coordinate system in which locally the Christoffel symbols are identically zero. From the definition of the Riemann Christoffel tensor

$$R^{\mu}{}_{\beta\gamma\delta} = \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\mu}_{\sigma\delta} - \Gamma^{\sigma}_{\beta\delta}\Gamma^{\mu}_{\sigma\gamma} + \Gamma^{\mu}_{\beta\gamma,\delta} - \Gamma^{\mu}_{\beta\delta,\gamma}$$

Now, since $g_{\alpha\mu;\lambda} = 0$, we can write

$$R_{\alpha\beta\gamma\delta;\lambda} = (g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta})_{;\lambda} = g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta;\lambda}$$

Now all the Christoffel symbols are zero at P in our chosen coordinate system, so from the above two equations (changing covariant back to partial derivatives, which are interchangeable at P)

$$R_{\alpha\beta\gamma\delta;\lambda} = g_{\alpha\mu}(\Gamma^{\mu}_{\beta\gamma,\delta\lambda} - \Gamma^{\mu}_{\beta\delta,\gamma\lambda})$$

Substituting for the Christoffel symbols using

$$\Gamma^{\sigma}_{\alpha\beta} = g^{\sigma\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

and also using the fact that in our LIF $g_{\alpha\beta,\gamma} = g^{\alpha\beta}_{,\gamma} = 0$ one obtains

$$R_{\alpha\beta\gamma\delta;\lambda} = \frac{1}{2}(g_{\alpha\gamma,\beta\delta\lambda} + g_{\alpha\delta,\beta\gamma\lambda} - g_{\beta\gamma,\alpha\delta\lambda} - g_{\beta\delta,\alpha\gamma\lambda})$$

Writing out the corresponding expressions for the other two terms on the left hand side of the Bianchi identities and adding gives the required result, i.e.

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0$$

in geodesic coordinates at P. Hence the Bianchi identities must hold in all coordinate systems because of their tensorial nature.

Now, from the symmetry properties of the Riemann Christoffel tensor, we have

$$R_{\alpha\beta\lambda\mu;\nu} = -R_{\alpha\beta\mu\lambda;\nu}$$

so that the Bianchi identities can be re-written as

$$R_{\alpha\beta\mu\nu;\lambda} - R_{\alpha\beta\mu\lambda;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

Contracting this equation with $g^{\alpha\mu}$ we obtain

$$g^{\alpha\mu}R_{\alpha\beta\mu\nu;\lambda} - g^{\alpha\mu}R_{\alpha\beta\mu\lambda;\nu} + g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = 0$$

Using the product rule for covariant differentiation and the fact that $g^{\alpha\mu}_{\ \ ;\beta} = 0$, we obtain

$$(g^{\alpha\mu}R_{\alpha\beta\mu\nu})_{;\lambda} - (g^{\alpha\mu}R_{\alpha\beta\mu\lambda})_{;\nu} + (g^{\alpha\mu}R_{\alpha\beta\nu\lambda})_{;\mu} = 0$$

This further simplifies to

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\ \ \beta\nu\lambda;\mu} = 0$$

We can further contract this equation by multiplying by $g^{\beta\nu}$, using the product rule and the fact that $g^{\alpha\mu}_{\ \ ;\beta} = 0$, to obtain (after contracting the first two terms and writing the third in terms of the fully covariant form of the Riemann Christoffel tensor)

$$R_{;\lambda} - R^{\nu}_{\lambda;\nu} + g^{\beta\nu}g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = 0$$

Using the fact that

$$R_{\alpha\beta\nu\lambda} = -R_{\beta\alpha\nu\lambda}$$

the third term on the left hand side can therefore be re-written as

$$g^{\beta\nu}g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = -(g^{\beta\nu}g^{\alpha\mu}R_{\beta\alpha\nu\lambda})_{;\mu} = -(g^{\alpha\mu}R_{\alpha\lambda})_{;\mu} = -R^{\mu}_{\ \ \lambda;\mu} \equiv -R^{\nu}_{\ \ \lambda;\nu}$$

Thus, we have

$$R_{;\lambda} - 2R^{\nu}_{\ \ \lambda;\nu} = 0$$

Multiplying each term by $-\frac{1}{2}g^{\mu\lambda}$ we obtain

$$g^{\mu\lambda}R^{\nu}_{\ \ \lambda;\nu} - \frac{1}{2}g^{\mu\lambda}R_{;\lambda} = 0$$

Once again using the product rule and the fact that the first covariant derivatives of the metric vanish this yields

$$(g^{\mu\lambda}R^{\nu}{}_{\lambda})_{;\nu} - (\frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$$

which of course is equivalent to

$$G^{\mu\nu}_{;\nu} = 0$$

Appendix 2: The Schwarzschild solution

The Schwarzschild metric is a solution of Einstein's equations which describes the space-time exterior to a spherically symmetric, static mass, M.

A2.1: Derivation of the Schwarzschild metric

In most astrophysical situations, we can work with the metric tensor in **orthogonal** form. This means that one can identify a coordinate system in which the components, $g_{\alpha\beta}$, of the metric tensor satisfy

$$g_{\alpha\beta} = 0 \quad \text{for all } \alpha \neq \beta$$

We will derive the Schwarzschild metric in orthogonal form, so we begin by stating some general results which are valid for orthogonal metrics.

We can show that the Christoffel symbols take a simple form for an orthogonal metric

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} &= 0 \quad \text{for } \lambda,\ \mu,\ \nu \text{ all different} \\
\Gamma^{\lambda}_{\lambda\mu} &= \Gamma^{\lambda}_{\mu\lambda} = g_{\lambda\lambda,\mu}/2g_{\lambda\lambda} \\
\Gamma^{\lambda}_{\mu\mu} &= -g_{\mu\mu,\lambda}/2g_{\lambda\lambda} \\
\Gamma^{\lambda}_{\lambda\lambda} &= g_{\lambda\lambda,\lambda}/2g_{\lambda\lambda}
\end{aligned}$$

(Note: the summation convention does not apply in these equations).

For affine parameter, p, the geodesic equation can be re-written for an orthogonal metric in the form

$$\frac{d}{dp}\left(g_{\lambda\lambda}\frac{dx^\lambda}{dp}\right) - \frac{1}{2}\frac{\partial g_{\mu\mu}}{\partial x^\lambda}\left(\frac{dx^\mu}{dp}\right)^2 = 0$$

The Schwarzschild solution is also an example of a spherically symmetric static spacetime (which we henceforth refer to as S^4 for short). We can show that the interval for S^4 takes the general form

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

which, of course, has an orthogonal metric. Note that we have introduced the functions $\nu(r)$ and $\lambda(r)$ in place of g_{tt} and g_{rr}. Since the exponential function is strictly positive for all r, this replacement is legitimate provided that $g_{tt} < 0$ and $g_{rr} > 0$ for all points in our spacetime.

The Christoffel symbols for S^4 are given by

$$\begin{aligned}
\Gamma^t_{rt} = \Gamma^t_{tr} &= \tfrac{1}{2}\nu' & \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \tfrac{1}{r} \\
\Gamma^r_{tt} &= \tfrac{1}{2}\nu' e^{\nu-\lambda} & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^r_{rr} &= \tfrac{1}{2}\lambda' & \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \tfrac{1}{r} \\
\Gamma^r_{\theta\theta} &= -re^{-\lambda} & \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta \\
\Gamma^r_{\phi\phi} &= -re^{-\lambda}\sin^2\theta & &
\end{aligned}$$

All others zero

Further, we can write the Ricci tensor for S^4 as

$$R_{\lambda\nu} = \Gamma^\tau_{\lambda\nu}\Gamma^\sigma_{\tau\sigma} - \Gamma^\tau_{\lambda\sigma}\Gamma^\sigma_{\tau\nu} + \Gamma^\sigma_{\lambda\nu,\sigma} - \Gamma^\sigma_{\lambda\sigma,\nu}$$

i.e.

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}\nu'\right)$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{r}\lambda'\right)$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left[1 + \frac{r}{2}\left(\nu' - \lambda'\right)\right]$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$$

and all other terms of the Ricci tensor are identically zero.

We are now in a position to derive the **Schwarzschild solution**. If the star is in an isolated region of space, then we can assume that all components of the Ricci tensor exterior to the star are identically zero. Thus

$$e^{\lambda-\nu}R_{tt} + R_{rr} = \frac{\nu' + \lambda'}{r} = 0$$

which in turn implies that

$$\nu + \lambda = \text{constant}$$

At large distances from the star we want the Schwarzschild metric to reduce to SR. Thus, as

$$r \to \infty\ , \quad \nu \to 0 \quad \text{and} \quad \lambda \to 0$$

which implies that

$$\nu + \lambda = 0$$

so that

$$e^{\nu} = e^{-\lambda}$$

This allows us to eliminate ν, giving

$$e^{-\lambda}\left(1 - \lambda' r\right) = 1$$

i.e.

$$\frac{d}{dr}\left(re^{-\lambda}\right) = 1$$

which we can integrate to give

$$e^{\nu} = e^{-\lambda} = 1 + \frac{\alpha}{r}$$

where α is a constant.

To evaluate α, suppose we release a material 'test' particle from rest. Thus, initially

$$\frac{dx^j}{d\tau} = 0 \quad \text{for } j = 1, 2, 3$$

where τ is **proper time**, and

$$\frac{dx^0}{d\tau} \equiv \frac{dt}{d\tau} \neq 0$$

Using the fact that

$$g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -1$$

and after some reduction we see that

$$\frac{dt}{d\tau} = e^{-\nu/2}$$

We now make use of the geodesic differential equation for radial coordinate, r. At the instant when the particle is released this reduces to

$$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0$$

After some further substitution we obtain finally

$$\frac{d^2r}{d\tau^2} = \frac{\alpha}{2r^2}$$

In the limit of a weak gravitational field this result must reduce to the prediction of Newtonian gravity, which is

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

where M is the mass of the star. If we adopt convenient units such that the gravitational constant, $G = 1$, this means that

$$\alpha = -2M$$

Finally, then, we can write down the invariant interval for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

A2.2: Geodesics for the Schwarzschild metric

The geodesics for a material 'test' particle in the Schwarzschild metric satisfy, with the proper time, τ, as affine parameter:

$$\frac{d}{d\tau}\left(g_{\lambda\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Notice that the Schwarzschild metric coefficients are independent of both t and ϕ; hence if we set $\lambda = 0$ and $\lambda = 3$ then the second term on the left hand side of the geodesic equation vanishes. Given also that the Schwarzschild metric is *orthogonal*, it follows that

$$\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) = \frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) = 0$$

Integrating this gives us

$$g_{tt}\frac{dt}{d\tau} = \text{ constant}$$

and

$$g_{\phi\phi}\frac{d\phi}{d\tau} = \text{ constant}$$

The geodesic equation for θ (i.e. $\lambda = 2$) is then

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) - \frac{1}{2}\frac{\partial}{\partial\theta}\left(r^2\sin^2\theta\right)\left[\frac{d\phi}{d\tau}\right]^2 = 0$$

which reduces to

$$r^2\frac{d^2\theta}{d\tau^2} + 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

This equation has a particular solution $\theta = \pi/2$; adopting this solution is equivalent to choosing the plane of the orbit of our material particle (e.g. a planet) to lie in the equatorial plane of our coordinate system. Making use of $\theta = \pi/2$ to simplify our solution, it follows that

$$\frac{dt}{d\tau} = \frac{k}{1 - \frac{2M}{r}}$$

and

$$\frac{d\phi}{d\tau} = \frac{h}{r^2}$$

where h and k are constants.

We can now obtain the geodesic differential equation for r, using

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2$$

which in turn reduces to

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right)$$

A2.3: The advance of pericentre in GR

Changing the dependent variable from r to u and the independent variable from τ to ϕ, our radial geodesic equation reduces to

$$h^2\left(\frac{du}{d\phi}\right)^2 = \left(k^2-1\right) - h^2u^2 + 2Mu\left(1+h^2u^2\right)$$

Differentiating, and cancelling the common factor of $\mathrm{d}u/\mathrm{d}\phi$ gives

$$\frac{d^2u}{d\phi^2} = -u + \frac{M}{h^2} + 3Mu^2$$

The effect of GR is to add the extra term $3Mu^2$ on the right hand side. For typical planetary orbits in the Solar System this extra term is tiny compared with the second term; e.g. for the Earth's orbit the ratio

$$\frac{3Mu^2}{M/h^2} \simeq 3\times 10^{-8}$$

Hence, because the extra GR term is very small anyway, we can obtain a very good approximation to the solution by replacing u in the u^2 term on the right hand side by the solution to the *Newtonian* version of this equation. Doing this we obtain

$$\frac{d^2u}{d\phi^2} = -u + \frac{M}{h^2} + 3\frac{M^3}{h^4}\left(1+2e\cos\phi + e^2\cos^2\phi\right)$$

We can write u as the sum of a 'Newtonian' and 'GR' part, so that u_{GR} describes the correction to the Newtonian orbit. Subtracting off the Newtonian solution gives

$$\frac{d^2u_{\mathrm{GR}}}{d\phi^2} = -u_{\mathrm{GR}} + 3\frac{M^3}{h^4}\left(1+2e\cos\phi + e^2\cos^2\phi\right)$$

which we can rewrite as

$$\frac{d^2u_{\mathrm{GR}}}{d\phi^2} + u_{\mathrm{GR}} = 3\frac{M^3}{h^4}\left(1+\frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right)$$

The right hand side of this equation takes the form $A + B\cos\phi + C\cos 2\phi$, where A, B and C are constants. It is easy to verify that particular integrals for each of these terms are, respectively

$$u_{\mathrm{GR}} = A$$

$$u_{\text{GR}} = \frac{1}{2}B\phi \sin\phi$$

$$u_{\text{GR}} = -\frac{1}{3}C\cos 2\phi$$

and the correction to the Newtonian orbit is given by the sum of these three particular integrals. Since each of the constants, A, B and C is of order the tiny constant M^3/h^4, we see that the first and third terms add to the Newtonian solution respectively a completely negligible constant and an equally negligible constant plus a tiny "wiggle".

The second term, on the other hand, is of a different form. Although the constant, B, is negligibly small, the presence of the ϕ means that this term produces a continually increasing – and thus ultimately non-negligible – effect. In fact

$$u = \frac{M}{h^2}\left(1 + e\cos\phi + \frac{3M^2}{h^2}e\phi\sin\phi\right)$$

Now, given that $3M^2/h^2$ is very small, and then using the approximations $\cos\beta \simeq 1$ and $\sin\beta \simeq \beta$ for small angle β, and the cosine addition formula, we can re-cast this last equation as

$$u = \frac{M}{h^2}\left[1 + e\cos\left(1 - \frac{3M^2}{h^2}\right)\phi\right]$$

Comparing this solution with its Newtonian analogue, we see that again the solution is elliptical in form and that u (and hence r) is a periodic function of ϕ. Notice, however, that the period, P, is given by

$$P = \frac{2\pi}{1 - 3M^2/h^2} > 2\pi$$

This means that the values of r trace out an approximate ellipse, but do not begin to repeat until *after* the radius vector has made a complete revolution. In other words the orbit can be regarded as an ellipse that 'precesses', so that the pericentre line advances each orbit by an amount, Δ, given by

$$\Delta = 2\pi\left(1 - \frac{3M^2}{h^2}\right)^{-1} - 2\pi \simeq \frac{6\pi M^2}{h^2} = \frac{6\pi M}{a(1-e^2)}$$

If we apply this equation to the orbit of Mercury, we obtain a perihelion advance which builds up to about 43 seconds of arc per century.

A2.4: Gravitational light deflection in GR

The geodesics for a photon in the Schwarzschild metric may be derived in a similar manner to Appendix 2.2, but we now must introduce a new affine parameter, λ (say), since the proper time for a photon is zero.[5]

For the 't' and 'ϕ' geodesic equations it is straightforward to see that we again obtain equations of the form

$$\frac{dt}{d\lambda} = \frac{k}{1 - 2M/r}$$

$$\frac{d\phi}{d\lambda} = \frac{h}{r^2}$$

since, for the 'θ' equation, we can again spot the particular solution $\theta = \pi/2$. It obviously then also follows that

$$\frac{d\theta}{d\lambda} = 0$$

We can then obtain

$$\left(\frac{dr}{d\lambda}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{2Mh^2}{r^3}$$

We now proceed as in Appendix 2.3, replacing the dependent variable, r by $u = 1/r$, and the independent variable λ by ϕ. This leads to

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2$$

If we ignore the term on the right hand side we can see that a particular integral is

$$u = \frac{\cos\phi}{r_{\text{min}}}$$

Following the same approach as in Appendix 2.3, we can obtain a very good approximation to the solution by replacing u on the right hand side by the solution to the corresponding Newtonian equation[6]. This gives the equation

$$\frac{d^2u}{d\phi^2} + u = \frac{3M}{r_{\text{min}}^2}\cos^2\phi = \frac{3M}{2r_{\text{min}}^2}\left(1 + \cos 2\phi\right)$$

[5] In fact, the initial choice of affine parameter will not be important, since we will determine the trajectory of the photon with the coordinate ϕ as the independent variable

[6] If we regard photons (as, indeed, modern physics holds true) as particles with zero rest mass, then formally they should be 'immune' to Newton's gravitational force. If, on the other hand, we regard photons as having a negligible but non-zero mass then – even within a purely Newtonian

It is straightforward to verify that a particular integral of this approximation is

$$u = \frac{3M}{2r_{\rm min}^2}\left(1 - \frac{1}{3}\cos 2\phi\right)$$

from which it follows that the general solution is

$$u = \frac{\cos\phi}{r_{\rm min}} + \frac{3M}{2r_{\rm min}^2}\left(1 - \frac{1}{3}\cos 2\phi\right)$$

We can rewrite this, for e.g. the outgoing photon trajectory, as

$$u = \frac{\cos\left(\frac{\pi}{2} + \frac{\Delta\phi}{2}\right)}{r_{\rm min}} + \frac{3M}{2r_{\rm min}^2}\left[1 - \frac{1}{3}\cos\left(\pi + \Delta\phi\right)\right]$$

or

$$u = -\frac{\sin(\Delta\phi/2)}{r_{\rm min}} + \frac{3M}{2r_{\rm min}^2}\left[1 + \frac{1}{3}\cos\Delta\phi\right]$$

which further simplifies, since $\Delta\phi << 1$, to

$$u = -\frac{\Delta\phi}{2r_{\rm min}} + \frac{2M}{r_{\rm min}^2}$$

Setting $u = 0$ (i.e. $r \to \infty$) this finally gives us the General Relativistic result

$$\Delta\phi = \frac{4M}{r_{\rm min}} \equiv \frac{4GM}{c^2 r_{\rm min}} = \frac{2R_{\rm S}}{r_{\rm min}}$$

This is exactly twice the deflection angle predicted by a Newtonian treatment. If we take $r_{\rm min}$ to be the radius of the Sun (which would correspond to a light ray grazing the limb of the Sun from a background star observed during a total solar eclipse) then we find that

$$\Delta\phi = \frac{4 \times 1.5 \times 10^3}{6.95 \times 10^8} = 8.62 \times 10^{-6} \text{ radians} \quad = \quad 1.77 \text{ arcsec}$$

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framework – we can calculate the predicted deflection angle as light passes close to a massive object. In fact, this calculation was first carried out in 1801 by Söldner. This is the Newtonian solution we are referring to here.

Appendix 3: Einstein's tensor in the weak field approximation

A3.1: The linearised Riemann Christoffel tensor

In Minkowski spacetime the Christoffel symbols are all identically zero. This reduces the Riemann Christoffel tensor to

$$R_{\alpha\beta\gamma\delta} \;=\; g_{\alpha\mu}\, R^{\mu}_{\beta\gamma\delta} \;=\; g_{\alpha\mu}\Gamma^{\mu}_{\beta\delta,\gamma} \;-\; g_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma,\delta}$$

Substituting for the Christoffel symbols in terms of the metric and its derivatives

$$R_{\alpha\beta\gamma\delta} \;=\; g_{\alpha\mu}\,\frac{g^{\mu\sigma}}{2}(g_{\sigma\beta,\delta\gamma} + g_{\sigma\delta,\beta\gamma} - g_{\beta\delta,\sigma\gamma}) \;-\; g_{\alpha\mu}\,\frac{g^{\mu\sigma}}{2}(g_{\sigma\beta,\gamma\delta} + g_{\sigma\gamma,\beta\delta} - g_{\beta\gamma,\sigma\delta})$$

This reduces to

$$R_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma})$$

If $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ then

$$R_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})$$

A3.2: The linearised Ricci tensor and curvature scalar

Contracting the Riemann Christoffel tensor it then follows that

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu}$$

To first order this reduces to

$$R_{\mu\nu} = \frac{1}{2}\eta^{\sigma\alpha}(h_{\alpha\nu,\mu\sigma} + h_{\mu\sigma,\alpha\nu} - h_{\mu\nu,\alpha\sigma} - h_{\alpha\sigma,\mu\nu})$$

Since the partial derivatives of $\eta^{\sigma\alpha}$ are zero, we can write this as

$$R_{\mu\nu} = \frac{1}{2}\left[(\eta^{\sigma\alpha}\, h_{\alpha\nu})_{,\mu\sigma} + (\eta^{\sigma\alpha}\, h_{\mu\sigma})_{,\alpha\nu} - \eta^{\sigma\alpha} h_{\mu\nu,\alpha,\sigma} - (\eta^{\sigma\alpha}\, h_{\alpha\sigma})_{,\mu\nu}\right]$$

which further reduces to

$$R_{\mu\nu} = \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\alpha}_{\mu}\right)_{,\nu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right]$$

Thus the curvature scalar R is given by

$$R = \eta^{\alpha\beta}R_{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma} - h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{,\alpha\beta}\right]$$

A3.3: The linearised Einstein tensor

Combining our results for the Ricci tensor and curvature scalar we find

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\alpha}_{\mu}\right)_{,\nu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right] \\ &\quad -\frac{1}{4}\eta_{\mu\nu}\,\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma} - h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{,\alpha\beta}\right] \end{aligned}$$

To see that this expression reduces to equation (108) of Section 6.3.2 it is easiest to work backwards from that equation. We have, from equation (108)

$$\begin{aligned} \text{LHS} &= \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right] \\ &= \frac{1}{2}\left[\eta^{\alpha\sigma}\,h_{\mu\alpha,\nu\sigma} + \eta^{\alpha\sigma}\,h_{\nu\alpha,\mu\sigma} - \eta^{\alpha\sigma}\,h_{\mu\nu,\alpha\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\eta^{\beta\sigma}\,h_{\alpha\beta,\gamma\sigma} + \eta_{\mu\nu}\eta^{\beta\alpha}\,h_{,\beta\alpha}\right] \\ &= \frac{1}{2}\left[(\eta^{\alpha\sigma}h_{\mu\alpha})_{,\nu\sigma} + (\eta^{\alpha\sigma}h_{\nu\alpha})_{,\mu\sigma} - h_{\mu\nu,\sigma}{}^{,\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\left(\eta^{\beta\sigma}h_{\alpha\beta}\right)_{,\gamma\sigma} + \eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta}\right] \\ &= \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\sigma}_{\mu}\right)_{,\nu\sigma} - h_{\mu\nu,\sigma}{}^{,\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\,(h^{\sigma}_{\alpha})_{,\gamma\sigma} + \eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta}\right] \end{aligned}$$

The first four bracketed terms match the first four terms in the expression for the Einstein tensor given above. If we now consider the remaining four terms in the above expression for the Einstein tensor, then since $\eta^{\alpha\beta} = \eta^{\beta\alpha}$

$$-\frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma}\right] = -\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\,(h^{\sigma}_{\alpha})_{,\beta\sigma}$$

Also

$$\begin{aligned} \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[h_{\alpha\beta,\sigma}{}^{,\sigma} + h_{,\alpha\beta}\right] &= \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[\eta^{\sigma\gamma}h_{\alpha\beta,\sigma\gamma} + h_{,\alpha\beta}\right] \\ &= \frac{1}{4}\eta_{\mu\nu}\eta^{\sigma\gamma}\left(\eta^{\alpha\beta}h_{\alpha\beta}\right)_{,\sigma\gamma} + \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta} \\ &= \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta} \end{aligned}$$

Thus, apart from some permutation of repeated indices, we see that the remaining four terms of our expression for the Einstein tensor match exactly the final two terms of equation (108). This establishes that equation (108) is indeed the correct expression for $G_{\mu\nu}$.

A3.4: Linearised Einstein tensor in barred form

Equation (110) is also easiest to establish in reverse – i.e. we start with equation (110) and show that its terms can be rewritten in a manner that reduces to equation (108). Consider in turn each of the four bracketed terms on the right hand side of equation (110).

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = \eta^{\alpha\sigma}\overline{h}_{\mu\nu,\alpha\sigma} = \eta^{\alpha\sigma}\left[h_{\mu\nu,\alpha\sigma} - \frac{1}{2}\eta_{\nu\mu}\,h_{,\alpha\sigma}\right]$$

$$\eta_{\mu\nu}\,\overline{h}_{\alpha\beta}{}^{,\alpha\beta} = \eta_{\mu\nu}\,\eta^{\alpha\sigma}\,\eta^{\beta\sigma}\,\overline{h}_{\alpha\beta,\gamma\sigma} = \eta_{\mu\nu}\,\eta^{\alpha\sigma}\,\eta^{\beta\sigma}\left[h_{\alpha\beta,\gamma\sigma} - \frac{1}{2}\eta_{\alpha\beta}\,h_{,\gamma\sigma}\right]$$

$$\overline{h}_{\mu\alpha,\nu}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\mu\alpha,\nu\sigma} = \eta^{\alpha\sigma}\left[h_{\mu\alpha,\nu\sigma} - \frac{1}{2}\eta_{\mu\alpha}\,h_{,\nu\sigma}\right]$$

$$\overline{h}_{\nu\alpha,\mu}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\nu\alpha,\mu\sigma} = \eta^{\alpha\sigma}\left[h_{\nu\alpha,\mu\sigma} - \frac{1}{2}\eta_{\nu\alpha}\,h_{,\mu\sigma}\right]$$

Hence we can write equation (110) as

$$\begin{aligned}
G_{\mu\nu} \;=\; & -\frac{1}{2}\eta^{\alpha\sigma}\,h_{\mu\nu,\alpha\sigma} + \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\mu\nu}\,h_{,\alpha\sigma} \\
& -\frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\,h_{\alpha\beta,\gamma\sigma} + \frac{1}{4}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\,\eta\alpha\beta\,h_{,\gamma\sigma} \\
& +\frac{1}{2}\eta^{\alpha\sigma}\,h_{\mu\alpha,\nu\sigma} - \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\mu\alpha}\,h_{,\nu\sigma} \\
& +\frac{1}{2}\eta^{\alpha\sigma}\,h_{\nu\alpha,\mu\sigma} - \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\nu\alpha}\,h_{,\mu\sigma}
\end{aligned}$$

$$\begin{aligned}
\;=\; & -\frac{1}{2}\eta^{\alpha\sigma}\,h_{\mu\nu,\alpha\sigma} - \frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\,h_{\alpha\beta,\gamma\sigma} + \frac{1}{2}\eta^{\alpha\sigma}\,h_{\mu\alpha,\nu\sigma} \\
& +\frac{1}{2}\eta^{\alpha\sigma}\,h_{\nu\alpha,\mu\sigma} + \frac{1}{2}\eta^{\alpha\sigma}\,\eta_{\mu\nu}\,h_{,\alpha\sigma} - \frac{1}{2}h_{,\mu\nu}
\end{aligned}$$

Comparing with our expression for $G_{\mu\nu}$ in Appendix A2.3, we see that – after changing some repeated indices and using the fact that the Minkowski metric is symmetric – the above expression is identical. This establishes that equation (110) is indeed the correct expression for the $G_{\mu\nu}$ in terms of $\overline{h}_{\mu\nu}$.

Appendix 4: The Lorentz Gauge Condition

First we show that, if $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$, the final three terms on the right hand side of equation (110) vanish. Consider the bracketed terms in turn

$$\begin{aligned}
\overline{h}_{\alpha\beta}{}^{,\alpha\beta} &= \left(\eta_{\alpha\gamma}\,\eta_{\beta\sigma}\,\overline{h}^{\gamma\sigma}\right)^{,\alpha\beta} \\
&= \left(\eta_{\alpha\gamma}\,\eta_{\beta\sigma}\,\eta^{\alpha\tau}\,\eta^{\beta\epsilon}\,\overline{h}^{\gamma\sigma}{}_{,\tau\epsilon}\right) \\
&= \delta^{\tau}_{\gamma}\,\delta^{\epsilon}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\tau\epsilon} \\
&= \overline{h}^{\gamma\sigma}{}_{,\gamma\sigma} \quad = \quad \left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\gamma} \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
\overline{h}_{\mu\alpha,\nu}{}^{,\alpha} &= \eta_{\mu\gamma}\,\eta_{\alpha\sigma}\left(\overline{h}^{\gamma\sigma}{}_{,\nu}\right)^{,\alpha} \\
&= \eta_{\mu\gamma}\,\eta_{\alpha\sigma}\,\eta^{\alpha\tau}\,\overline{h}^{\gamma\sigma}{}_{,\nu\tau} \\
&= \eta_{\mu\gamma}\,\delta^{\tau}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\nu\tau} \\
&= \eta_{\mu\gamma}\,\overline{h}^{\gamma\sigma}{}_{,\nu\sigma} \quad = \quad \eta_{\mu\gamma}\left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\nu} \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
\overline{h}_{\nu\alpha,\mu}{}^{,\alpha} &= \eta_{\nu\gamma}\,\eta_{\alpha\sigma}\left(\overline{h}^{\gamma\sigma}{}_{,\mu}\right)^{,\alpha} \\
&= \eta_{\nu\gamma}\,\eta_{\alpha\gamma}\,\eta^{\alpha\tau}\,\overline{h}^{\gamma\sigma}{}_{,\mu\tau} \\
&\quad \eta_{\nu\gamma}\,\delta^{\tau}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\mu\tau} \\
&\quad \eta_{\nu\gamma}\,\overline{h}^{\gamma\sigma}{}_{,\mu\sigma} \quad = \quad \eta_{\nu\gamma}\left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\mu} \quad = \quad 0
\end{aligned}$$

So we see that the final three terms do indeed equal zero provided $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$.

Finally we establish the equation which must be solved in order that the Lorentz gauge condition $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$ is satisfied.

Suppose we begin with arbitrary metric perturbation components $h^{(\mathrm{old})}_{\mu\nu} \neq 0$. We define

$$h^{(\mathrm{LG})}_{\mu\nu} = h^{(\mathrm{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

where the components ξ^μ are to be determined. We can also define

$$\begin{aligned}
\overline{h}^{(\mathrm{LG})}_{\mu\nu} &= h^{(\mathrm{LG})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\beta}\,h^{(\mathrm{LG})}_{\alpha\beta} \\
&= h^{(\mathrm{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{1}{2}\eta_{\mu\nu}\left[\eta^{\alpha\beta}\left(h^{(\mathrm{old})}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}\right)\right] \\
&= h^{(\mathrm{old})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(\mathrm{old})} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\eta^{\alpha\beta}\,\xi_{\alpha,\beta} \\
&= \overline{h}^{(\mathrm{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\xi^{\beta}{}_{,\beta}
\end{aligned}$$

Now

$$\begin{aligned}
\overline{h}^{(\mathrm{LG})\,\mu\nu} &= \eta^{\mu\alpha}\,\eta^{\nu\beta}\left[\overline{h}^{(\mathrm{old})}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\,\xi^{\sigma}{}_{,\sigma}\right] \\
&= \overline{h}^{(\mathrm{old})\,\mu\nu} - \eta^{\mu\alpha}\,\eta^{\nu\beta}\,\xi_{\alpha,\beta} - \eta^{\mu\alpha}\,\eta^{\nu\beta}\,\xi_{\beta,\alpha} + \eta^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma}
\end{aligned}$$

And

$$\overline{h}^{(\mathrm{LG})\,\mu\nu}{}_{,\nu} = \overline{h}^{(\mathrm{old})\,\mu\nu}{}_{,\nu} - \eta^{\nu\beta}\,\xi^{\mu}{}_{,\beta\nu} - \eta^{\mu\alpha}\,\xi^{s}igma_{,\sigma\alpha} + \eta^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma\nu}$$

i.e.

$$\overline{h}^{(\mathrm{LG})\,\mu\nu}{}_{,\nu} = \overline{h}^{(\mathrm{old})\,\mu\nu}{}_{,\nu} - \eta^{\nu\beta}\,\xi^{\mu}{}_{,\nu\beta}$$

So we can ensure that $\overline{h}^{(\mathrm{LG})\,\mu\nu}{}_{,\nu} = 0$ provided we can find gauge components ξ^μ satisfying

$$\overline{h}^{(\mathrm{old})\,\mu\nu}{}_{,\nu} = \eta^{\nu\beta}\,\xi^{\mu}{}_{,\nu\beta} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\xi^\mu$$

We can always solve this equation for well-behaved metrics using standard methods for finding particular solutions of second order partial differential equations.