Appendix A
Solutions of the exercises

A.1 Solution to Exercise 1.1

This is straightforward to verify Eq. (1.13) using Eq. (1.11). Such a relation does not apply to a statistical mixture. Indeed, by considering its spectral decomposition, we have

$$\begin{aligned}\hat{\rho}^2 &= \left(\sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right|\right)\left(\sum_n p_n \left|\psi_n\right\rangle \left\langle\psi_n\right|\right), \\ &= \sum_{nk} p_n p_k \left|\psi_k\right\rangle \left\langle\psi_k|\psi_n\right\rangle \left\langle\psi_n\right| = \sum_{nk} p_n p_k \delta_{k,n} \left|\psi_k\right\rangle \left\langle\psi_n\right|, \\ &= \sum_k p_k^2 \left|\psi_k\right\rangle \left\langle\psi_k\right| \neq \hat{\rho}.\end{aligned} \tag{A.1}$$

A.2 Solution to Exercise 1.2

Let us consider the spectral decomposition of $\hat{\rho}^2$

$$\hat{\rho}^2 = \sum_k p_k^2 \left|\psi_k\right\rangle \left\langle\psi_k\right|, \tag{A.2}$$

which was derived in Eq. (A.1). Then, it follows that its trace is

$$\mathrm{Tr}\left[\hat{\rho}^2\right] = \sum_k p_k^2 \leq \big(\sum_k p_k\big)^2 = 1, \tag{A.3}$$

where we exploited the triangular inequality. For a pure state, the inequality becomes an equality, since there is only one coefficient equal to 1.

A.3 Solution to Exercise 1.3

Starting from

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } a, b, c, d \in \mathbb{C}, \tag{A.4}$$

we impose *i)* $\hat{\rho}^\dagger = \hat{\rho}$. This gives that

$$a^* = a, \quad d^* = d, \quad c^* = b, \tag{A.5}$$

thus giving

$$\rho = \begin{pmatrix} \alpha & \beta + i\gamma \\ \beta - i\gamma & \delta \end{pmatrix}, \tag{A.6}$$

where $\alpha = a \in \mathbb{R}$, β, $\gamma \in \mathbb{R}$ such that $b = \beta + i\gamma$ and $\delta = d$.

We impose *ii)* $\mathrm{Tr}\,[\hat{\rho}] = 1$, which gives

$$\alpha + \delta = 1, \tag{A.7}$$

thus imposing

$$\delta = 1 - \alpha. \tag{A.8}$$

Finally, *iii)* $\hat{\rho} \geq 0$ is equivalent (this holds only for two dimensional systems) to

$$\mathrm{Tr}\,[\hat{\rho}] > 0, \quad \det[\hat{\rho}] > 0. \tag{A.9}$$

The first is already covered, while the second gives

$$\det[\hat{\rho}] = \alpha(1-\alpha) - \beta^2 - \gamma^2. \tag{A.10}$$

By defining

$$r_x = 2\beta, \quad r_y = 2\gamma, \quad r_z = 2\alpha - 1, \tag{A.11}$$

we obtain

$$\det[\hat{\rho}] = \tfrac{1}{4}(1 - r_x^2 - r_y^2 - r_z^2). \tag{A.12}$$

The latter expression givers $\det[\hat{\rho}] > 0$ only if the vector $\mathbf{r} = (r_x, r_y, r_z)$ has $||\mathbf{r}||^2 \leq 1$. This covers the exercise.

A.4 Solution to Exercise 1.4

Consider the expression

$$\begin{aligned} \hat{\boldsymbol{\sigma}}\hat{\rho} =& \frac{\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}}\mathbf{r}\cdot\hat{\boldsymbol{\sigma}}}{2}, \\ =& \frac{\hat{\boldsymbol{\sigma}}}{2} + \frac{1}{2}(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)(r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z), \\ =& \frac{\hat{\boldsymbol{\sigma}}}{2} + \frac{1}{2}\left(r_x\hat{\mathbb{1}} + r_y\hat{\sigma}_x\hat{\sigma}_y + r_z\hat{\sigma}_x\hat{\sigma}_z, r_x\hat{\sigma}_y\hat{\sigma}_x + r_y\hat{\mathbb{1}} + r_z\hat{\sigma}_y\hat{\sigma}_z, r_x\hat{\sigma}_z\hat{\sigma}_x + r_y\hat{\sigma}_z\hat{\sigma}_y + r_z\hat{\mathbb{1}}\right) \end{aligned} \tag{A.13}$$

Here, the Pauli matrices obey to the following rules:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k, \quad \text{and} \quad \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{\mathbb{1}}, \tag{A.14}$$

whose sum gives

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\hat{\mathbb{1}} + i\epsilon_{ijk}\hat{\sigma}_k. \tag{A.15}$$

By applying this expression to Eq. (A.13), we get

$$\hat{\boldsymbol{\sigma}}\hat{\rho} = \frac{\hat{\boldsymbol{\sigma}}}{2} + \frac{1}{2}\left(r_x\hat{\mathbb{1}} + ir_y\hat{\sigma}_z - ir_z\hat{\sigma}_y, -ir_x\hat{\sigma}_z + r_y\hat{\mathbb{1}} + ir_z\hat{\sigma}_x, ir_x\hat{\sigma}_y - ir_y\hat{\sigma}_x + r_z\hat{\mathbb{1}}\right). \tag{A.16}$$

However, the Pauli matrices are traceless, while $\mathrm{Tr}\left[\hat{\mathbb{1}}\right] = 2$. Thus, one has

$$\mathrm{Tr}\,[\hat{\boldsymbol{\sigma}}\hat{\rho}] = (r_x, r_y, r_z) = \mathbf{r}, \tag{A.17}$$