## 2.2 Quantum operations and the Kraus-Stinespring theorem

In the previous chapter we showed that the density matrix of a closed quantum system evolves according to the von Neumann-Liouville equation (1.24), which underlies a unitary evolution. From the previous examples it should be clear that the same is not true in general for the reduced density matrix of an open quantum system. Indeed, consider two systems $\mathcal{A}$ and $\mathcal{B}$, initially in two factorized and pure states: $\hat{\rho}_{\mathcal{AB}} = \hat{\rho}_{\mathcal{A}} \otimes \hat{\rho}_{\mathcal{B}}$, with $\hat{\rho}^2_{\mathcal{A}} = \hat{\rho}_{\mathcal{A}}$ and $\hat{\rho}^2_{\mathcal{B}} = \hat{\rho}_{\mathcal{B}}$. They interact with each other and after some time become entangled. While the reduced density matrix $\hat{\rho}^{(\mathcal{A})}$ of subsystem $\mathcal{A}$ initially is a pure state (since $\hat{\rho}^{(\mathcal{A})} = \hat{\rho}_{\mathcal{A}}$, see Example 2.1), at later times it becomes a statistical mixture ($(\hat{\rho}^{(\mathcal{A})})^2 \neq \hat{\rho}^{(\mathcal{A})}$, see Example 2.2). A unitary evolution cannot transform pure states into statistical mixtures, therefore the dynamics governing the evolution of the reduced density matrix must be of a different kind. The Kraus-Stinespring theorem, which is stated below, gives a first characterization of an open quantum dynamics. Before going through the theorem, with the following example we present an argument which quickly reproduces the final result.

***Example 2.3***
*Consider two interacting quantum systems $\mathcal{A}$ and $\mathcal{B}$, whose initial state is factorised. Let $\hat{\rho}_{\mathcal{A}}$ be the initial state of system $\mathcal{A}$ and $\hat{\rho}_{\mathcal{B}}$ the initial state of $\mathcal{B}$, with spectral decomposition:*

$$\hat{\rho}_{\mathcal{B}} = \sum_n p_n \left|\phi_n^{\mathcal{B}}\right\rangle \left\langle\phi_n^{\mathcal{B}}\right| .$$

*The combined state evolves with the unitary quantum dynamics $\hat{U}_t$ and after some fixed time $t$ the two system become entangled. The reduced density matrix of system $\mathcal{A}$ at time $t$ is:*

$$\begin{aligned}
\hat{\rho}^{(\mathcal{A})}(t) &= \mathrm{Tr}^{(\mathcal{B})}\left[\hat{U}_t(\hat{\rho}_{\mathcal{A}} \otimes \hat{\rho}_{\mathcal{B}})\hat{U}_t^{\dagger}\right], \\
&= \sum_{m,n} p_n \left\langle\phi_m^{\mathcal{B}}\right| \hat{U}_t \left|\phi_n^{\mathcal{B}}\right\rangle \hat{\rho}_{\mathcal{A}} \left\langle\phi_n^{\mathcal{B}}\right| \hat{U}_t^{\dagger} \left|\phi_m^{\mathcal{B}}\right\rangle, \\
&= \sum_{m,n} \hat{E}_{mn} \hat{\rho}_{\mathcal{A}} \hat{E}_{mn}^{\dagger},
\end{aligned}$$

*where we have defined the operators $\hat{E}_{mn} = \sqrt{p_n} \left\langle\phi_m^{\mathcal{B}}\right| \hat{U}_t \left|\phi_n^{\mathcal{B}}\right\rangle$, which act on $\mathbb{B}(\mathbb{H}_{\mathcal{A}})$, and are called Kraus operators. Since the dynamics $\hat{U}_t$ is unitary, it follows that:*

$$\sum_{m,n} \hat{E}_{mn} \hat{E}_{mn}^{\dagger} = \hat{\mathbb{1}}.$$

*The results above present the general form of the map of an open quantum system, as summarized by the Kraus-Stinespring theorem.*

Consider an open quantum system, whose associated Hilbert space $\mathbb{H}$ has dimensionality $N$. Let $\hat{\rho}$ describe its state at a given initial time, and let $\hat{\rho}'$ be the state after some fixed time. Let $\mathcal{T} : \mathbb{B}(\mathbb{H}) \to \mathbb{B}(\mathbb{H})$ be the map connecting these two states. The first observation is that the map $\mathcal{T}$ must be linear. The reason for this rests in the physical meaning of the density matrix as presented in Sec. 1.1, and in the motto "ignorance propagates linearly", which is valid both in a classical as well as quantum situation. If a system is in state $|\psi_1\rangle$ with probability $p_1$ and in state $|\psi_2\rangle$ with probability $p_2$, and if $|\psi_1\rangle$ evolves into $|\psi_1'\rangle$ and $|\psi_2\rangle$ into $|\psi_2'\rangle$, then the system will end up in state $|\psi_1'\rangle$ with probability $p_1$ and in state $|\psi_2'\rangle$ with probability $p_2$. In terms of statistical operators, a pure state $|\psi_k\rangle\langle\psi_k|$ is mapped into $\mathcal{T}[|\psi_k\rangle\langle\psi_k|]$. Then, for a statistical mixture we have:

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| \;\rightarrow\; \hat{\rho}' = \mathcal{T}[\hat{\rho}] = \sum_k p_k T[|\psi_k\rangle\langle\psi_k|], \tag{2.13}$$

which shows that the map $\mathcal{T}$ is linear. Let us fix an orthonormal basis $\{\,|\phi_i\rangle\,\}_{i=1}^N$ of $\mathbb{H}$. The linearity of the map $\mathcal{T}$ implies that we can express the matrix elements of $\hat{\rho}'$ as a linear combination of the matrix elements of $\hat{\rho}$:

$$\rho'_{ij} = \sum_{r,s=1}^{N} T_{ir,js}\rho_{rs}, \tag{2.14}$$

where $\rho'_{ij} = \langle\phi_i|\hat{\rho}'|\phi_j\rangle$ and $\rho_{rs} = \langle\phi_r|\hat{\rho}|\phi_s\rangle$. Since there are $N^2$ complex matrix elements $\rho'_{ij}$, there are $N^4$ complex constants $T_{ir,js}$. The output state $\hat{\rho}'$ must be Hermitian: $(\hat{\rho}')^\dagger = \hat{\rho}'$. Then, since also the input state $\hat{\rho}$ is Hermitian, and being it generic, Eq. (2.14) implies that

$$T^*_{js,ir} = T_{ir,js}. \tag{2.15}$$

We now group the pair of indices $(js)$ in a single index $m$ running from 1 to $N^2$. Similarly, we group the pair $(ir)$ in a single index $n$. The coefficients $T_{ir,js}$ can be arranged to form a $N^2 \times N^2$ square matrix[1] $\hat{\mathbf{T}}$ with matrix elements $\mathbf{T}_{mn} = T_{ir,js}$. Then, Eq. (2.15) tells that $\hat{\mathbf{T}}$ is Hermitian and therefore it has $N^2$ real eigenvalues $\lambda^\alpha$ and $N^2$ complex eigenvectors $\mathbf{e}^\alpha$, which are orthonormal: $\mathbf{e}^\alpha \cdot \mathbf{e}^\beta = \delta^{\alpha\beta}$. Resorting to the inverse map between the pair $(js)$ and $m$, the eigenvectors $\mathbf{e}^\alpha$ identify square $N \times N$ matrix $\hat{E}^\alpha$ via the relation $E^\alpha_{js} = \mathbf{e}^\alpha{}_m$. The orthonormality condition translates into:

$$\mathrm{Tr}\left[\hat{E}^\alpha \hat{E}^{\beta\dagger}\right] = \delta^{\alpha\beta}. \tag{2.16}$$

The spectral decomposition of $\hat{\mathbf{T}}$ tells that:

$$\mathbf{T}_{mn} = \sum_{\alpha=1}^{N^2} \lambda^\alpha \mathbf{e}^\alpha_m \mathbf{e}^{*\alpha}_n \quad \to \quad T_{ir,js} = \sum_{\alpha=1}^{N^2} \lambda^\alpha E^\alpha_{ir} E^{*\alpha}_{js}, \tag{2.17}$$

using which, Eq. (2.14) can be rewritten as follows:

$$\rho'_{ij} = \sum_{\alpha=1}^{N^2} \lambda^\alpha \sum_{r,s=1}^{N} E^\alpha_{ir} E^{\alpha *}_{js} \rho_{rs} \quad \to \quad \hat{\rho}' = \sum_{\alpha=1}^{N^2} \lambda^\alpha \hat{E}^\alpha \hat{\rho}\, \hat{E}^{\alpha\dagger}. \tag{2.18}$$

We expect the map $\mathcal{T}$ to preserve the trace of the density matrix: $\mathrm{Tr}\,[\hat{\rho}'] = \mathrm{Tr}\,[\hat{\rho}] = 1$. According to Eq. (2.18), this implies:

$$\mathrm{Tr}\left[\sum_{\alpha=1}^{N^2} (\lambda^\alpha \hat{E}^\alpha \hat{\rho}\, \hat{E}^{\alpha\dagger}) - \hat{\rho}\right] = \mathrm{Tr}\left[\left(\sum_{\alpha=1}^{N^2} (\lambda^\alpha \hat{E}^{\alpha\dagger}\, \hat{E}^\alpha) - \hat{\mathbb{1}}\right) \hat{\rho}\right] = 0, \tag{2.19}$$

where $\hat{\mathbb{1}}$ is the identity operator. Since the initial state $\hat{\rho}$ is generic, we conclude that:

$$\sum_{\alpha=1}^{N^2} \lambda^\alpha \hat{E}^{\alpha\dagger}\, \hat{E}^\alpha = \hat{\mathbb{1}}. \tag{2.20}$$

***Example 2.4***

*Let $\mathbb{H}$ be a two dimensional Hilbert space. Let us consider the following map:*

$$\rho_{ij} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \to \quad \rho'_{ij} = \begin{pmatrix} \rho_{22} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{pmatrix},$$

[1] Here and in the rest of the section, we denote $N^2 \times N^2$ square matrices and vectors of length $N^2$ with bold letters; non-bold letters denote $N \times N$ square matrices and vectors of length $N$.

*which exchanges the two diagonal elements. Then, with reference to Eq. (2.14), the only non-vanishing coefficients* $T_{ir,js}$ *are:* $T_{11,22} = T_{12,12} = T_{21,21} = T_{22,11} = 1$. *Accordingly, the 4-dimensional square Hermitian matrix* $\hat{\mathbf{T}}$ *has matrix elements:*

$$\mathbf{T}_{mn} = \begin{pmatrix} 0\,0\,0\,1 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 1\,0\,0\,0 \end{pmatrix},$$

*and its eigenvalues are:* $\lambda^{(1)} = \lambda^{(2)} = -\lambda^{(3)} = \lambda^{(4)} = 1$. *The corresponding eigenvectors are:*

$$\mathbf{e}_m^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_m^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ i \\ 0 \end{pmatrix}, \ \mathbf{e}_m^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{e}_m^4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

*The associated 2-dimensional matrices* $\hat{E}^\alpha$ *are:*

$$E_{js}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}, E_{js}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, E_{js}^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_{js}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix},$$

*i.e.* $\hat{E}^\alpha = \frac{1}{\sqrt{2}}\hat{\sigma}^\alpha$, *with* $\hat{\sigma}^i$ $(i = 1, 2, 3)$ *the three Pauli matrices and* $\hat{\sigma}^4 = \hat{\mathbb{1}}$. *Then, according to Eq. (2.18), the map can be written as:*

$$\hat{\rho}' = \frac{1}{2}\left[\hat{\sigma}^1\hat{\rho}\,\hat{\sigma}^1 + \hat{\sigma}^2\hat{\rho}\,\hat{\sigma}^2 - \hat{\sigma}^3\hat{\rho}\,\hat{\sigma}^3 + \hat{\rho}\right]. \tag{2.21}$$

*It is easy to check that Eq. (2.20) is satisfied.*

The last requirement to check is that $\hat{\rho}'$ in Eq. (2.14) is positive definite. Unfortunately, there is no characterization of positive maps, apart from the definition. A stronger definition which is considered is the Complete Positivity (CP). Let us consider an ancilla Hilbert space $\mathbb{H}_a$ and the map:

$$\begin{aligned} \overline{\mathcal{T}} : \mathbb{B}(\mathbb{H}_a) \otimes \mathbb{B}(\mathbb{H}) &\to \mathbb{B}(\mathbb{H}_a) \otimes \mathbb{B}(\mathbb{H}), \\ \hat{\overline{\rho}} \to \hat{\overline{\rho}}' &= (\mathrm{id} \otimes \mathcal{T})\hat{\overline{\rho}}, \end{aligned} \tag{2.22}$$

where id is the identity map $\mathrm{id} : \hat{\rho} \to \hat{\rho}$. The map $\mathcal{T}$ is completely positive if and only if the map $\overline{\mathcal{T}}$ is positive, i.e. for any $\overline{v} \in \mathbb{H}_a \otimes \mathbb{H}$ one has:

$$\langle \overline{v}|\hat{\overline{\rho}}'|\overline{v}\rangle = \langle \overline{v}|(id \otimes \mathcal{T})\hat{\overline{\rho}}|\overline{v}\rangle = \sum_{\alpha=1}^{N^2} \lambda^\alpha \langle \overline{v}|(\hat{\mathbb{1}} \otimes \hat{E}^\alpha)\hat{\overline{\rho}}(\hat{\mathbb{1}} \otimes \hat{E}^\alpha)^\dagger|\overline{v}\rangle \geq 0 \tag{2.23}$$

Now, we decompose both $|\overline{v}\rangle$ and $\hat{\overline{\rho}}$ with respect to the orthonormal bases $\{\, |\phi_n^a\rangle \,\}_{n=1}^{N_a}$ and $\{\, |\phi_m\rangle \,\}_{m=1}^{N}$ of $\mathbb{H}_a$ and $\mathbb{H}$ respectively:

$$|\overline{v}\rangle = \sum_{n=1}^{N_a}\sum_{m=1}^{N} v_{nm}\,|\phi_n^a\rangle\,|\phi_m\rangle, \quad \text{and} \quad \hat{\overline{\rho}} = \sum_{l,l'=1}^{N_a}\sum_{k,k'=1}^{N} C_{lk}C_{k'l'}^*\,|\phi_l^a\rangle\,|\phi_k\rangle\,\langle\phi_{l'}^a|\,\langle\phi_{k'}|. \tag{2.24}$$

Then, in these terms we obtain

$$
\begin{aligned}
\langle \overline{v} | \hat{\overline{\rho}}' | \overline{v} \rangle &= \sum_{\alpha=1}^{N^2} \lambda^\alpha \sum_{\substack{m,m'\\ l,l'}=1}^{N_a} \sum_{\substack{n,n'\\ k,k'}=1}^{N} v^*_{m'n'} v_{nm} C_{lk} C^*_{k'l'} E^\alpha_{m'k} E^{\alpha *}_{mk'} \delta_{n'l} \delta_{nl'}, \\
&= \sum_{\alpha=1}^{N^2} \lambda^\alpha \, \mathrm{Tr}\left[ \hat{C} \hat{D}^\dagger \hat{E}^\alpha \right] \mathrm{Tr}\left[ \hat{E}^{\alpha\dagger} \hat{D} \hat{C}^\dagger \right],
\end{aligned}
\tag{2.25}
$$

where we define the operators $\hat{C}$ and $\hat{D}$ as having for matrix elements the coefficients $C_{kl}$ and $v_{mn}$ respectively. We choose $\hat{C}$ and $\hat{D}$ such that $\hat{E}^\beta = \hat{D}\hat{C}^\dagger/\sqrt{N}$. This is possible since they descend from $\hat{\overline{\rho}}$ and $|\overline{v}\rangle$ via Eq. (2.24), which are both generic; they only need satisfy the constraint:

$$
\langle \overline{v} | \overline{v} \rangle = 1 = \mathrm{Tr}\left[ \hat{D} \hat{D}^\dagger \right] \quad \text{and} \quad \mathrm{Tr}\left[ \hat{\overline{\rho}} \right] = 1 = \mathrm{Tr}\left[ \hat{C} \hat{C}^\dagger \right]. \tag{2.26}
$$

Two possible options are: $\hat{C}^\dagger = \hat{E}^\beta, \hat{D} = \hat{\mathbb{1}}/\sqrt{N}$ or $\hat{C}^\dagger = \hat{\mathbb{1}}/\sqrt{N}, \hat{D} = \hat{E}^\beta$, and in both cases the constraints in Eq. (2.26) are satisfied. Coming back to Eq. (2.25), we have:

$$
\langle \overline{v} | \hat{\overline{\rho}}' | \overline{v} \rangle = N \sum_{\alpha=1}^{N^2} \lambda^\alpha \left( \delta^{\alpha\beta} \right)^2 = N \lambda^\beta.
$$

Therefore, the condition of having a CP map implies that the eigenvalues $\lambda^\alpha$ must be non-negative. Returning to Eq. (2.18) and (2.20), we can redefine $\sqrt{\lambda^\alpha}\hat{E}^\alpha \rightarrow \hat{E}^\alpha$ and summarize the results in the following theorem

**Theorem 2.2 (Kraus-Stinespring).** *Any linear completely positive map*

$$
\mathcal{T} : \mathbb{B}(\mathbb{H}) \rightarrow \mathbb{B}(\mathbb{H}) \tag{2.27}
$$

*admits the following decomposition:*

$$
\mathcal{T}\hat{\rho} = \sum_{k=1}^{N^2} \hat{E}_k \hat{\rho} \hat{E}_k^\dagger, \tag{2.28}
$$

*where*

$$
\sum_k \hat{E}_k^\dagger \hat{E}_k = \hat{\mathbb{1}}. \tag{2.29}
$$

Equation (2.28) gives a nice physical interpretation to the effect of the environment on the system, once it is re-written in the following way:

$$
\mathcal{T}[\hat{\rho}] = \sum_{k=1}^{N^2} p_k \frac{\hat{E}_k \hat{\rho} \hat{E}_k^\dagger}{\mathrm{Tr}[\hat{E}_k \hat{\rho} \hat{E}_k^\dagger]}, \qquad p_k = \mathrm{Tr}[\hat{E}_k \hat{\rho} \hat{E}_k^\dagger]. \tag{2.30}
$$

The coefficients $p_k$ can be interpreted as probabilities since they are non-negative and sum to 1.

## 2.3 Quantum operations on qubits

We present some of the most common types of quantum maps acting of a single qubit, which are of relevance for quantum information processing.

**Bit flip.** As the name suggests, under this map the state of the qubit is flipped ($|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$) with a given probability $p$, otherwise it is left unchanged. In mathematical terms:

$$\mathcal{T}_{\rm bf}[\hat\rho] = (1-p)\hat\rho + p\,\hat\sigma_x\hat\rho\hat\sigma_x. \tag{2.31}$$

According to Eq. (2.28), the Kraus operators are: $\hat E_1 = \sqrt{1-p}\ \hat{\mathbb{1}}$ and $\hat E_2 = \sqrt{p}\ \hat\sigma_x$. An easy calculation shows that the Bloch vector $\mathbf{r}$ changes as follows:

$$\begin{aligned} r_x &\to r_x, \\ r_y &\to (1-2p)r_y, \\ r_z &\to (1-2p)r_z. \end{aligned} \tag{2.32}$$

The effect is summarized in Fig. 2.1.

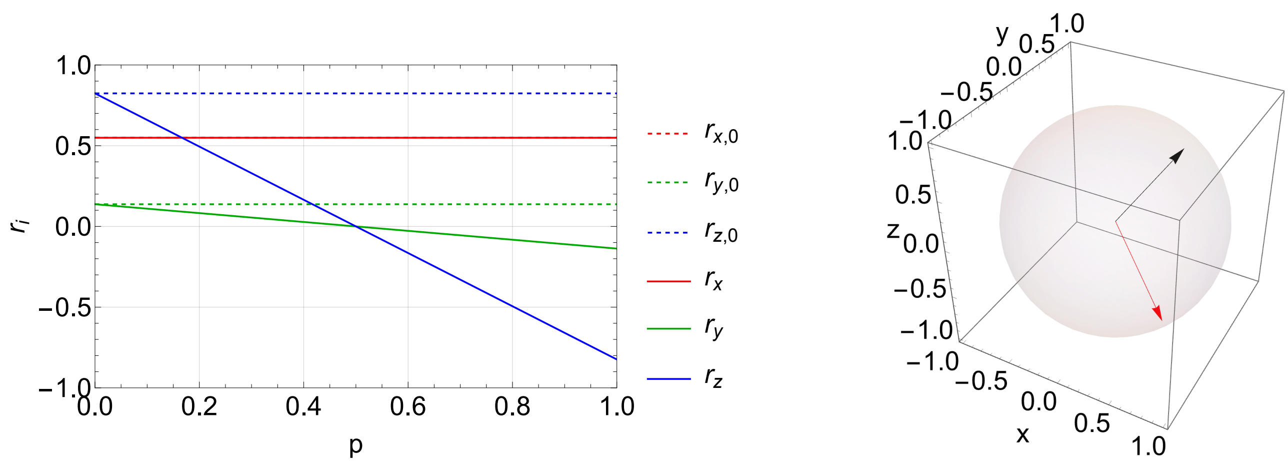


Fig. 2.1: Graphical representation of the bit flip channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $p$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $p = 1$. The initial state corresponds to $\mathbf{r}_0 = (4,1,6)/\sqrt{53}$.

**Phase flip.** In this case the phase of the qubit is flipped (namely, $|0\rangle \to |0\rangle$ and $|1\rangle \to -|1\rangle$) with probability $p$, otherwise it is left unchanged:

$$\mathcal{T}_{\rm pf}[\hat\rho] = (1-p)\hat\rho + p\,\hat\sigma_z\hat\rho\hat\sigma_z. \tag{2.33}$$

The corresponding Kraus operators are: $\hat E_1 = \sqrt{1-p}\ \hat{\mathbb{1}}$ and $\hat E_2 = \sqrt{p}\ \hat\sigma_z$. The Bloch vector $\mathbf{r}$ changes as follows:

$$\begin{aligned} r_x &\to (1-2p)r_x, \\ r_y &\to (1-2p)r_y, \\ r_z &\to r_z. \end{aligned} \tag{2.34}$$

The effect is summarized in Fig. 2.2.

**Bit-Phase flip.** This is the combination of the previous two maps, and it is described as follows:

$$\mathcal{T}_{\rm bp}[\hat\rho] = (1-p)\hat\rho + p\,\hat\sigma_y\hat\rho\hat\sigma_y. \tag{2.35}$$

The Kraus operators are: $\hat E_1 = \sqrt{1-p}\ \hat{\mathbb{1}}$ and $\hat E_2 = \sqrt{p}\ \hat\sigma_y$. The Bloch vector $\mathbf{r}$ changes as follows:

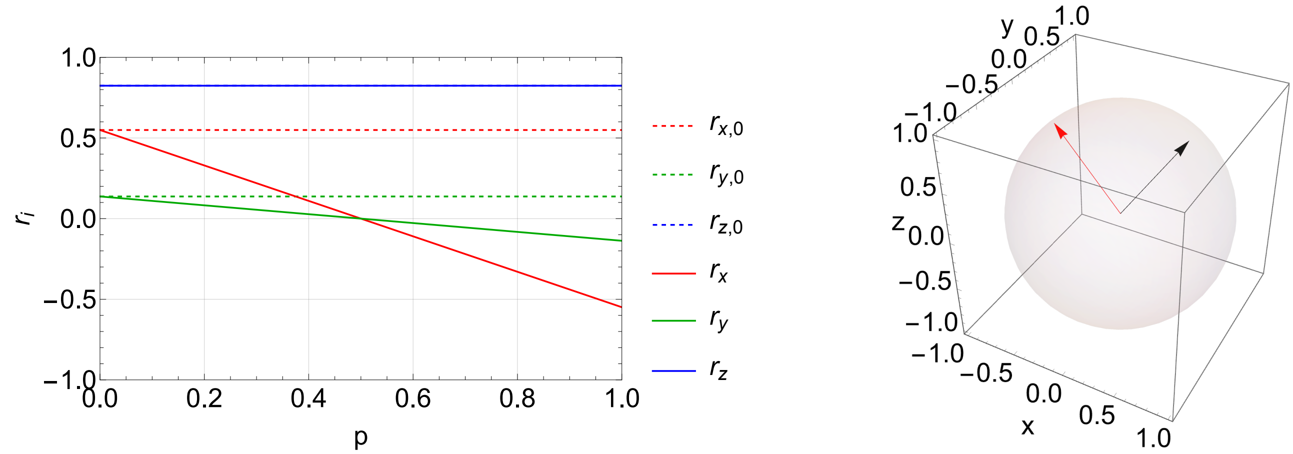


Fig. 2.2: Graphical representation of the phase flip channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $p$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $p = 1$. The initial state corresponds to $\mathbf{r}_0 = (4,1,6)/\sqrt{53}$.

$$
\begin{aligned}
r_x &\rightarrow (1-2p)r_x, \\
r_y &\rightarrow r_y, \\
r_z &\rightarrow (1-2p)r_z.
\end{aligned}
\tag{2.36}
$$

The effect is summarized in Fig. 2.3.

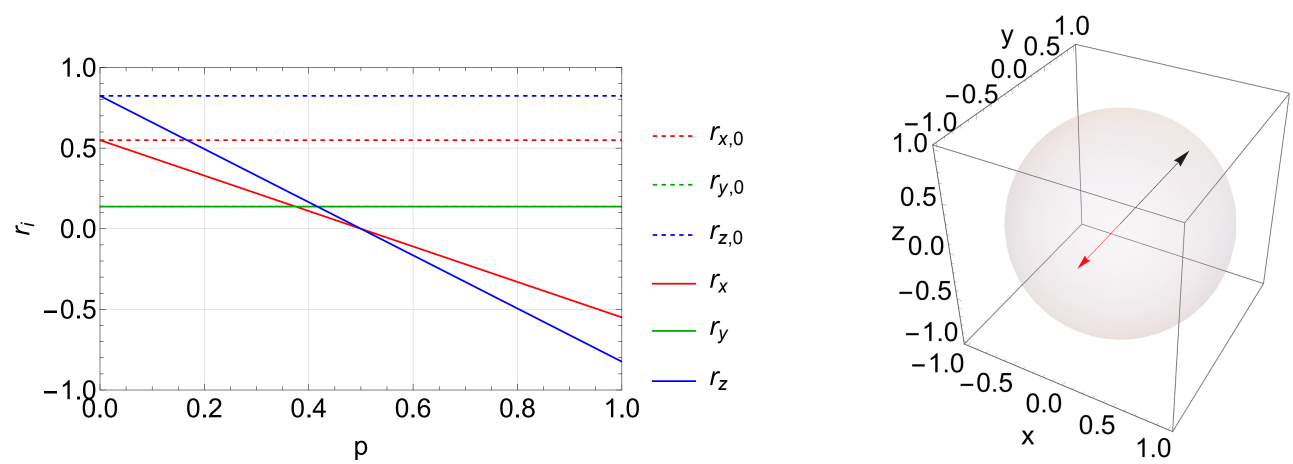


Fig. 2.3: Graphical representation of the bit-phase flip channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $p$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $p = 1$. The initial state corresponds to $\mathbf{r}_0 = (4,1,6)/\sqrt{53}$.

**Depolarizing channel.** In this map, the state $\hat{\rho}$ of the system is depolarized with probability $p$, i.e. it is replaced by the completely mixed state $\hat{\mathbb{1}}/2$. With probability $(1-p)$ the state is left unchanged. Then, we have

$$\mathcal{T}_{\rm dc}[\hat{\rho}] = (1-p)\hat{\rho} + p\,\frac{\hat{\mathbb{1}}}{2}. \tag{2.37}$$

Using the relation

$$\frac{\hat{\mathbb{1}}}{2} = \frac{\hat{\rho} + \hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z}{4}, \tag{2.38}$$

we can rewrite Eq. (2.37) as follows:

$$\mathcal{T}_{\rm dc}\hat{\rho} = \left(1 - \frac{3}{4}p\right)\hat{\rho} + \frac{p}{4}\left(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z\right), \tag{2.39}$$

showing that the Kraus operators are: $\hat{E}_1 = \sqrt{1-3p/4}\;\hat{\mathbb{1}}$, $\hat{E}_2 = \sqrt{p/4}\;\hat{\sigma}_x$, $\hat{E}_3 = \sqrt{p/4}\;\hat{\sigma}_y$ and $\hat{E}_4 = \sqrt{p/4}\;\hat{\sigma}_z$. Often it is convenient to parametrize the depolarizing channel as follows:

$$\mathcal{T}_{\rm dc}\hat{\rho} = (1-q)\,\hat{\rho} + \frac{q}{3}\left(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z\right), \tag{2.40}$$

where $q = 3p/4$. Thus, Eq. (2.40) provides an alternative way of interpreting the depolarizing channel: with probability $(1-q)$ the state is left unchanged, and the operations $\hat{\sigma}_i$ $(i = z, y, z)$ are applied each with probability $q/3$. The effect is summarized in Fig. 2.4.

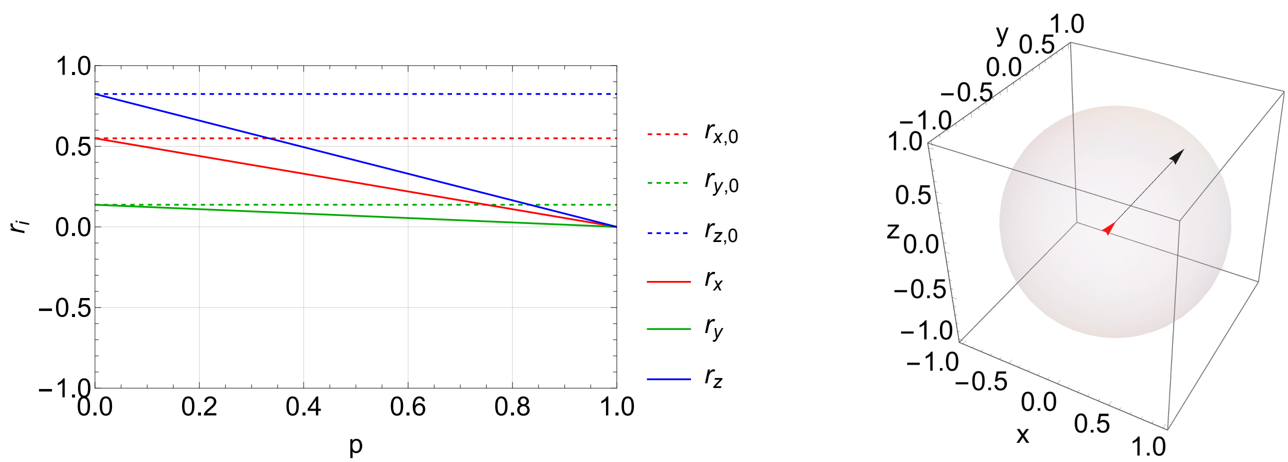


Fig. 2.4: Graphical representation of the depolarising channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $p$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $p = 1$. The initial state corresponds to $\mathbf{r}_0 = (4,1,6)/\sqrt{53}$.

***Exercise 2.1***
*Verify the relation in Eq. (2.38).*

**Amplitude damping channel.** This channel describes the situation where the state $|1\rangle$ decays to $|0\rangle$ with some probability $\gamma$ due to dissipative effects, while $|0\rangle$ remains unchanged. This is formally achieved by the operation:

$$\mathcal{T}_{\rm ad}[\hat{\rho}] = \hat{E}_1\hat{\rho}\hat{E}_1^\dagger + \hat{E}_2\hat{\rho}\hat{E}_2^\dagger, \tag{2.41}$$

with the Kraus operators $\hat{E}_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|$ and $\hat{E}_2 = \sqrt{\gamma}\,|0\rangle\langle 1|$, whose matrix representation on the computational basis is:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \tag{2.42}$$

An easy calculation shows that these operators can be written in terms of the Pauli matrices as follows:

$$\hat{E}_1 = \frac{1}{2}\left[(1+\sqrt{1-\gamma})\hat{\mathbb{1}} + (1-\sqrt{1-\gamma})\hat{\sigma}_z\right], \quad \text{and} \quad \hat{E}_2 = \frac{1}{2}\sqrt{\gamma}(\hat{\sigma}_x + i\hat{\sigma}_y). \tag{2.43}$$

The effect on the Bloch vector is:

$$\begin{aligned} r_x &\to \sqrt{1-\gamma}\, r_x, \\ r_y &\to \sqrt{1-\gamma}\, r_y, \\ r_z &\to \gamma + (1-\gamma)\, r_z, \end{aligned} \tag{2.44}$$

The effect is summarized in Fig. 2.5. The repeated application of this channel causes a progressive decay of

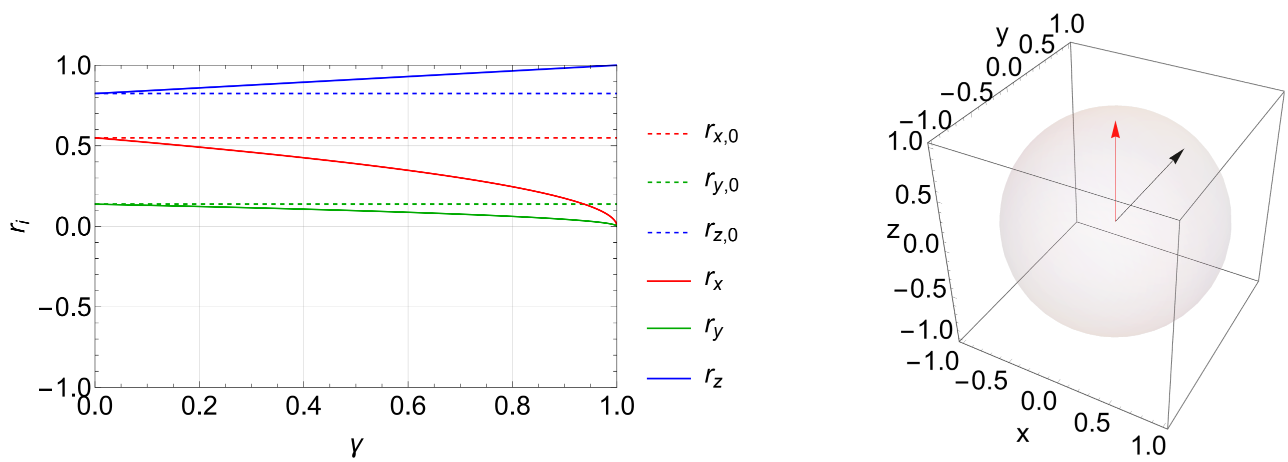


Fig. 2.5: Graphical representation of the amplitude damping channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $\gamma$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $\gamma = 1$. The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.

the state $|1\rangle$, and eventually only the state $|0\rangle$ survives. In the language of statistical mechanics, this effect is produced by an environment at 0 temperature, to which the qubit thermalizes.

**Generalised amplitude damping channel.** This channel describes the effect of an environment at finite temperature, which causes the state $|1\rangle$ to decay to $|0\rangle$, but also the state $|0\rangle$ to be excited to $|1\rangle$, with a different probability depending on the temperature. Therefore, to the Kraus operators $\hat{E}_1$ and $\hat{E}_2$ describing the decay process of the amplitude damping channel, we add the operators $\hat{E}_3 = |1\rangle\langle 1| + \sqrt{1-\gamma}\,|0\rangle\langle 0|$ and $\hat{E}_4 = \sqrt{\gamma}\,|1\rangle\langle 0|$ describing the inverse process. The two are blended together

$$\mathcal{T}_{\mathrm{ga}}[\hat{\rho}] = p(\hat{E}_1\hat{\rho}\hat{E}_1^\dagger + \hat{E}_2\hat{\rho}\hat{E}_2^\dagger) + (1-p)(\hat{E}_3\hat{\rho}\hat{E}_3^\dagger + \hat{E}_4\hat{\rho}\hat{E}_4^\dagger), \tag{2.45}$$

with a different probability to break the symmetry between them. The effect is summarized in Fig. 2.6.

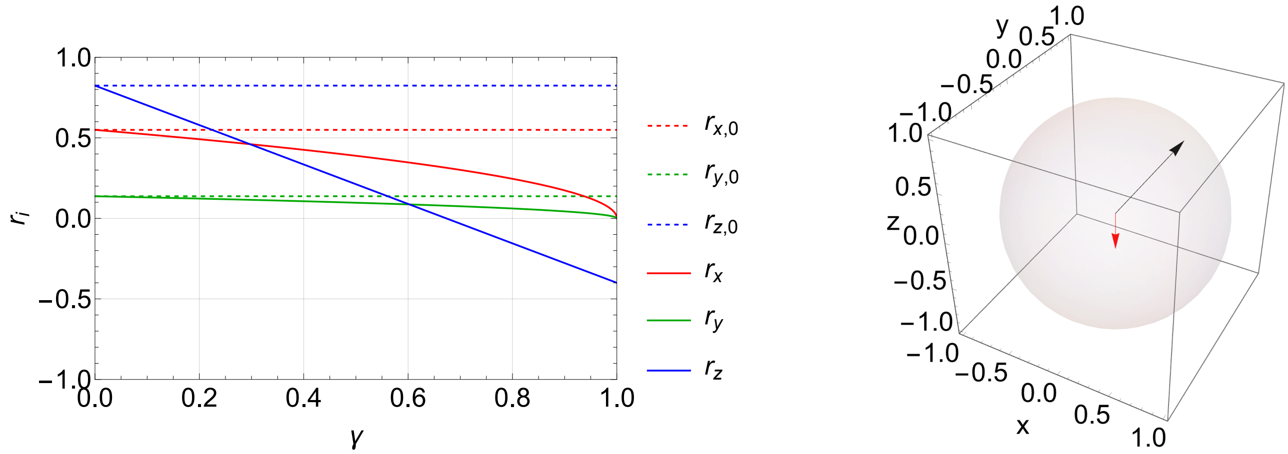


Fig. 2.6: Graphical representation of the generalised amplitude damping channel. (Left) Components of the Bloch radius (continuous lines) with respect to the value of $\gamma$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the state before (black arrow) and after (red arrow) the application of the map for $\gamma = 1$ and $p = 0.3$. The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.