Chapter 3
Quantum Dynamical Semigroups

We describe the dynamical evolution, being continuous in time, of density matrices in the framework of open quantum systems. The key result will be the Lindblad-Gorini-Kossakowski-Sudarshan theorem.

3.1 On the linearity of the dynamics

As we understood from the discussion in Sec. 2.2, a good quantum operation — and thus also a dynamics — should be completely positive and trace preserving (CPTP). Here, we heuristically demonstrate that the dynamics should be also linear.

Consider two sets of states $\{\,|\,\phi_i\rangle\,\}_{i=1}^n$ and $\{\,|\,\psi_j\rangle\,\}_{j=1}^m$ in the Hilbert space $\mathbb{H}$. Here, we are not assuming that these two sets have any particular proprieties, we only assume that $n > m$. Consider also two sets of parameters $\{\,p_i\,\}_{i=1}^n$ and $\{\,q_j\,\}_{j=1}^m$, where $p_i, q_j \in]0,1]$, such that they define the same statistical operator $\hat{\rho}$ in the following way

$$\hat{\rho} = \sum_{i=1}^n p_i \,|\phi_i\rangle\,\langle\phi_i| = \sum_{j=1}^m q_j \,|\psi_j\rangle\,\langle\psi_j| \,. \tag{3.1}$$

Specifically, these are two representation of the same statistical operator. Nevertheless, their physical interpretation can be significantly different as they represent different physical situations. An example of this situation is provided by the Example 1.3 and Example 1.4. Nevertheless, as they have the same statistical operator $\hat{\rho}$, they represent equivalent statistical ensembles.

Now, let us consider an ancillary Hilbert space $\mathbb{H}_{\text{A}}$, and a state $|\chi\rangle \in \mathbb{H}\otimes\mathbb{H}_{\text{A}}$ such that

$$|\chi\rangle = \sum_i \sqrt{p_i}\,|\phi_i\rangle \otimes |\alpha_i\rangle \,, \tag{3.2}$$

where $|\alpha_i\rangle$ are elements of a basis of $\mathbb{H}_{\text{A}}$.

Now, since the states $|\phi_i\rangle$ and $|\psi_j\rangle$ provide the same $\hat{\rho}$, it means that the span the same subspace of $\mathbb{H}$ (possibly the entire $\mathbb{H}$). Thus, one can express the former as a linear superposition of the latter. Namely,

$$|\phi\rangle_i = \sum_{j=1}^m b_{ij}\,|\psi_j\rangle \,, \tag{3.3}$$

where $b_{ij} = \langle\psi_j|\phi_i\rangle$. Accordingly, the state $|\chi\rangle$ can be rewritten as

$$
\begin{aligned}
|\chi\rangle &= \sum_i \sqrt{p_i} \sum_j b_{ij} \, |\psi_j\rangle \otimes |\alpha_i\rangle \,, \\
&= \sum_j |\psi_j\rangle \otimes |\tilde{\beta}_j\rangle \,,
\end{aligned}
\tag{3.4}
$$

where we defined

$$
|\tilde{\beta}_j\rangle = \sum_i \sqrt{p_i} b_{ij} \, |\alpha_i\rangle \,. \tag{3.5}
$$

This is a non-normalised state in $\mathbb{H}_{\mathrm{A}}$. Its normalisation can be computed as the following. Consider

$$
\begin{aligned}
\langle\tilde{\beta}_j|\tilde{\beta}_l\rangle &= \sum_{ik} \sqrt{p_i p_k} b^*_{ij} b_{kl} \, \langle\alpha_i|\alpha_k\rangle \,, \\
&= \sum_i p_i b^*_{ij} b_{il}, \\
&= \sum_i p_i \, \langle\psi_l|\phi_i\rangle \, \langle\phi_i|\psi_j\rangle \,, \\
&= \langle\psi_l|\hat{\rho}|\psi_j\rangle \,,
\end{aligned}
\tag{3.6}
$$

where we exploited the first expression in Eq. (3.1). Then, by exploiting Eq. (3.1), we have

$$
\begin{aligned}
\langle\tilde{\beta}_j|\tilde{\beta}_l\rangle &= \sum_k q_k \, \langle\psi_l|\psi_k\rangle \, \langle\psi_k|\psi_j\rangle \,, \\
&= q_j \delta_{lj},
\end{aligned}
\tag{3.7}
$$

where we assumed that the vectors $|\psi_j\rangle$ are orthonormal. It follows, that we can defined the normalised state

$$
|\beta_j\rangle = \frac{1}{\sqrt{q_j}} \, |\tilde{\beta}_j\rangle \,, \tag{3.8}
$$

and thus

$$
|\chi\rangle = \sum_j \sqrt{q_j} \, |\psi_j\rangle \otimes |\beta_j\rangle \,. \tag{3.9}
$$

This means that, by starting from two equivalent expressions for the statistical operator $\hat{\rho}$ of a subsystem, we can demonstrate that there are two equivalent ways to express the state of a state $|\chi\rangle$ shared (actually, entangled) between two subsystems.

Owning this information, we consider the following physical situation. Suppose we have Alice and Bob sharing the state $|\chi\rangle$, where Alice is associated to the Hilbert space $\mathbb{H}_{\mathrm{A}}$ and Bob to $\mathbb{H}$. Suppose Alice can perform on her part of the state two operations, which correspond to the operators $\hat{A}$ and $\hat{B}$, having the following spectral decompositions

$$
\begin{aligned}
\hat{A} &= \sum_i \alpha_i \, |\alpha_i\rangle \, \langle\alpha_i| \,, \\
\hat{B} &= \sum_j \beta_j \, |\beta_j\rangle \, \langle\beta_j| \,.
\end{aligned}
\tag{3.10}
$$

Conversely, Bob does nothing and can only observe his part of the state. Now, if Alice applies the operator $\hat{A}$, then her part of the state collapses in the basis $|\alpha_i\rangle$. After having performed a statistic, Bob has in his hands the state

$$
\sum_i p_i \, |\phi_i\rangle \, \langle\phi_i| = \hat{\rho}_1. \tag{3.11}
$$

Similarly, if Alice applies $\hat{B}$, the state in Bob hands becomes

$$\sum_j q_j \left|\psi_j\right\rangle \left\langle\psi_j\right| = \hat{\rho}_2. \tag{3.12}$$

However, by construction $\hat{\rho} = \hat{\rho}_1 = \hat{\rho}_2$, and thus one cannot perform faster then light signaling. Indeed, in the opposite case, to communicate at a distance, Alice could apply the operator $\hat{A}$ to communicate 0, and the operator $\hat{B}$ to communicate 1. However, since the two ensembles represented by $\hat{\rho}_1$ and $\hat{\rho}_2$ are equivalent, such a communication is not possible.

We now introduce the dynamics in the picture. Suppose that Bob, instead of measuring right away his part of the state, he waits for a time t. Then, the state ρ_i $(i = 1, 2)$ is evolved with respect to a dynamical map $\mathcal{T}_t$ with $t \geq 0$, where

$$\mathcal{T}_t[\hat{\rho}] = \hat{\rho}_t. \tag{3.13}$$

Specifically, one has

$$\begin{aligned}\hat{\rho}_1 &= \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \to \mathcal{T}_t[\hat{\rho}_1] = \hat{\rho}_1(t),\\ \hat{\rho}_2 &= \sum_j q_j \left|\psi_j\right\rangle \left\langle\psi_j\right| \to \mathcal{T}_t[\hat{\rho}_2] = \hat{\rho}_2(t).\end{aligned} \tag{3.14}$$

If the dynamics is not linear, namely $\mathcal{T}_t$ is not a linear operator, then one might have that $\hat{\rho}_1(t) \neq \hat{\rho}_2(t)$, as the action of the dynamical map can depend on the state itself. Thus, in such a case, Bob would be able to know which is the operator Alice applied to her part of the state. This would allow faster then light signalling.

Thus, one needs to require the linearity of the dynamical map $\mathcal{T}_t$ to ensure that $\hat{\rho}_1(t) = \hat{\rho}_2(t)$. In such a way, equivalent ensembles are mapped into equivalent ensembles.

3.2 Strongly Continuous Semigroup

We introduce a (strongly continuous) semigroup of operators and discuss its physical meaning.

Definition 3.1 (Strongly Continuous Semigroup of Operators) *Let $\mathbb{B}$ be a Banach space. A family $\{\mathcal{T}_t\}_{t\geq 0}$ of bounded linear operators $\mathcal{T}_t : \mathbb{B} \to \mathbb{B}$ is called a strongly continuous semigroup of operators if and only if:*

1. $\mathcal{T}_{t+s} = \mathcal{T}_t \mathcal{T}_s, \qquad \forall\, t, s \geq 0,$
2. $\mathcal{T}_0 = \mathrm{id},$
3. $\lim_{h\to 0} \mathcal{T}_{t+h} x = \mathcal{T}_t x \qquad \forall x \in \mathbb{B}, \quad \forall t, h \geq 0.$

The first two properties define the semigroup structure, while the third property makes sure that for every $x \in \mathbb{B}$, $\mathcal{T}_t x$ is (strongly) continuous in t. From the physical point of view, $\{\mathcal{T}_t\}_{t\geq 0}$ represents the (continuous) dynamical evolution of a system over time t, which does not need to be reversible, as it will be the case with open quantum systems. The key element of a semigroup of operators is its generator.

A comment here is due. The request 1. implies the Markovian nature of the semigroup evolution. In the Markovian case, the knowledge of the state at time t is sufficient to obtain the state at any time $t + \tau$ for $\tau > 0$. In the non-Markovian case, such a knowledge is not sufficient. The evolution might depend also on the past states: there is a memory effect in the dynamics. An example might be an evolution of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \int_{-\infty}^{t} \mathrm{d}s\, F(\hat{\rho}(s)). \tag{3.15}$$

Definition 3.2 (Infinitesimal generator) *The operator $\mathcal{L} : D(\mathcal{L}) \in \mathbb{B} \to \mathbb{B}$ defined such that*

$$\mathcal{L}x = \lim_{h\to 0^+} \frac{\mathcal{T}_h x - x}{h}, \qquad \forall x \in D(\mathcal{T}), \tag{3.16}$$

where the domain $D(\mathcal{L})$ is the set of all $x \in \mathbb{B}$ such that the above limit exists, is called infinitesimal generator of the semigroup $\{\mathcal{T}_t\}_{t\geq 0}$.

The importance of the infinitesimal generator if given by the following theorem.

Theorem 3.1 (Properties of the infinitesimal generator). *Let $\{\mathcal{T}_t\}_{t\geq 0}$ be a strongly continuous semigroup of operators in the Banach space $\mathbb{B}$ and let $\mathcal{L}$ be its infinitesimal generator with domain $D(\mathcal{L})$. Then:*

1. *$D(\mathcal{L})$ is a linear (and dense) subspace of $\mathbb{B}$ and $\mathcal{L}$ is a linear operator.*
2. *If $x \in D(\mathcal{L})$ then also $\mathcal{T}_t x \in D(\mathcal{L})$ for every $t \geq 0$. Moreover, $\mathcal{T}_t x$ is strongly differentiable in t and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t x = \mathcal{L}\mathcal{T}_t x = \mathcal{T}_t\mathcal{L}x, \qquad \forall t \geq 0. \tag{3.17}$$

The above theorem tells that the whole family of operators constituting the semigroup is controlled by the infinitesimal generator, through the formal relation $\mathcal{T}_t = e^{t\mathcal{L}}$, which justifies its name. As such, a strongly continuous semigroup can be seen as the generalization of the exponential function. This is made even clearer by the following theorem.

Theorem 3.2 (Solution of first order differential equations). *Let $\mathcal{L}$ with domain $D(\mathcal{L}) \in \mathbb{B}$ be the infinitesimal generator of a strongly continuous semigroup of operators $\{\mathcal{T}_t\}_{t\geq 0}$, with $\mathcal{T}_t = e^{t\mathcal{L}}$. Then, the Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}x_t = \mathcal{L}x_t, \qquad \forall t \geq 0, \tag{3.18}$$

with initial condition $x_0 \in D(\mathcal{L})$ has the unique solution

$$x_t = \mathcal{T}_t x_0, \qquad t \geq 0. \tag{3.19}$$

The theorem above also clarifies the physical meaning of the semigroup property, which describes a Markovian dynamics. Namely, the present state of the system (x_0) is sufficient to infer its future state (x_t) via Eq. (3.18). The latter is a first-order differential equation, whose solution is uniquely identified by the initial condition. An important question is when a strongly continuous semigroup admits a generator. The theorem by Hille, Yosida and Philipps presents necessary and sufficient conditions that an operator $\mathcal{L}$ with domain $D(\mathcal{L})$ in a Banach space $\mathbb{B}$ is the generator of a strongly continuous semigroup of operators.

The fact that a strongly continuous semigroup can be seen as the exponential of the generator becomes a mathematically correct statement if the semigroup is uniformly continuous (or norm continuous). A strongly continuous semigroup is called a uniformly continuous semigroup if

$$\lim_{h\to 0} \|\mathcal{T}_{t+h} - \mathcal{T}_t\| = 0, \qquad \forall\, t, t+h \geq 0. \tag{3.20}$$

Notice that uniform continuity implies strong continuity, but not viceversa. In such a case, it can be shown that the infinitesimal generator exists and is a bounded operator. Specifically, there exists a bounded linear operator $\mathcal{L}$ on $\mathbb{B}$ such that $\mathcal{T}_t = e^{t\mathcal{L}}$. Such an operator is given by:

$$\mathcal{L} = \lim_{h\to 0} \frac{\mathcal{T}_h - \mathrm{id}}{h}, \tag{3.21}$$

where the limit is taken in the uniform operator topology.

3.3 Quantum Dynamical Semigroup

A quantum dynamical semigroup specializes the notion of a strongly continuous semigroups of operators to quantum systems, according to the following definition.

Definition 3.3 (Quantum Dynamical Semigroup) *Let $\mathcal{B}_{Tr}(\mathbb{H})$ be the Banach space of trace-class operators over the Hilbert space $\mathbb{H}$, with associated trace norm $||\hat{A}||_{Tr}$. A family $\{\mathcal{T}_t\}_{t\geq 0}$ of bounded linear operators $\mathcal{T}_t : \mathcal{B}_{Tr}(\mathbb{H}) \to \mathcal{B}_{Tr}(\mathbb{H})$ is called a Quantum Dynamical Semigroup (QDS) if and only if:*

1. *$\{\mathcal{T}_t\}_{t\geq 0}$ is a strongly continuous semigroup of operators,*
2. *$\mathcal{T}_t\hat{\rho} \geq 0, \qquad \forall\hat{\rho} \in \mathcal{B}_{Tr}(\mathbb{H}), \hat{\rho} \geq 0, \forall t \geq 0,$*
3. *$\mathrm{Tr}\left[\mathcal{T}_t\hat{\rho}\right] = \mathrm{Tr}\left[\hat{\rho}\right], \qquad \forall\hat{\rho} \in \mathcal{B}_{Tr}(\mathbb{H}), \hat{\rho} \geq 0, \forall t \geq 0.$*

The meaning of the above requests is straightforward. The first conditions summarises a continuous Markovian dynamics, as discussed above, while the last two conditions make sure that a density matrix is mapped into a density matrix.

The important question is how to characterize the infinitesimal generator of a QDS, which according to Theorem 3.2 amounts to asking which kind of dynamical equation the density matrix is subject to. The answer was provided by two articles, both appeared in 1976: one by Lindblad, under the assumption that the QDS is uniformly continuous, and one by Gorini, Kossakowski and Sudarshan, under the assumption that the Hilbert space is finite dimensional, hence the common name of LGKS theorem. In both cases, a further assumption was enforced: that the QDS is completely positive, i.e. that $\mathcal{T}_t$ is a completely positive operator for any t (see Sec. 2.2). We present the theorem in the finite-dimensional case.

Theorem 3.3 (Lindblad-Gorini-Kossakowski-Sudarshan). *Let $\mathbb{H}$ be a finite-dimensional Hilbert space of dimension N. Let $\{\mathcal{T}_t\}_{t\geq 0}$ be a completely positive QDS. Its infinitesimal generator $\mathcal{L}$ has the following structure:*

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_{\alpha=1}^{N^2-1}\left[\hat{L}^\alpha\hat{\rho}\hat{L}^{\alpha\dagger} - \frac{1}{2}\left\{\hat{L}^{\alpha\dagger}\hat{L}^\alpha, \hat{\rho}\right\}\right], \tag{3.22}$$

where $\hat{H}$ is an hermitian operator and $\hat{L}^\alpha$ are called Lindblad operators.

Here we present a simple proof. From the previous discussion, we know that the generator exists. To derive its explicit form, since the map $\hat{\rho} \to \mathcal{T}_t[\hat{\rho}]$ is completely positive, we can express it in the Kraus form (see Theorem 2.2):

$$\mathcal{T}_t[\hat{\rho}] = \sum_{\alpha=1}^{N^2}\lambda_t^\alpha\,\hat{E}_t^\alpha\,\hat{\rho}\,\hat{E}_t^{\alpha\dagger}, \tag{3.23}$$

where now we indicated explicitly the time dependence. For the sake of simplicity, we recall the notation used in Sec. 2.2. Namely, we rewrite Eq. (3.23) in terms of its matrix elements on a arbitrary orthonormal basis $\{\,|\,\phi_i\rangle\,\}_i$ of $\mathbb{H}$:

$$(\rho'_t)_{ij} = \sum_{r,s=1}^{N}(\mathcal{T}_t)_{ir,js}\rho_{rs}, \tag{3.24}$$

where $\hat{\rho}'_t = \mathcal{T}_t[\hat{\rho}]$, $\rho_{rs} = \langle\phi_r|\hat{\rho}|\phi_s\rangle$, and $\mathbf{T}_{mn} = (\mathcal{T}_t)_{ir,js}$ (where we used the grouping of the indices) has the following spectral decomposition

$$(\mathcal{T}_t)_{ir,js} = \sum_{\alpha=1}^{N^2}\lambda^\alpha(E_t^\alpha)_{ir}(E_t^{*\alpha})_{js}, \tag{3.25}$$

with $(E_t^\alpha)_{ir} = \langle\phi_i|\hat{E}_t^\alpha|\phi_r\rangle$. For $t = 0$, one has the initial state, so we have $\mathcal{T}_0 = \mathrm{id}$. This means that $\mathbf{T}_{t=0}$ must leave the state unchanged. Then, its eigenvalues λ^α and eigenvectors $\hat{E}^\alpha$ are:

$$\lambda_{t=0}^\alpha = \begin{cases} N, & \alpha = N^2, \\ 0, & \text{otherwise}, \end{cases} \tag{3.26}$$

and

$$\hat{E}^{\alpha}_{t=0} = \begin{cases} \frac{1}{\sqrt{N}}\hat{\mathbb{1}}, & \alpha = N^2, \\ \hat{K}^{\alpha}, & \text{otherwise}, \end{cases} \tag{3.27}$$

where the operators $\hat{K}^{\alpha}$ are arbitrary chosen, such that $\{\hat{K}^{\alpha}, \hat{\mathbb{1}}/\sqrt{N}\}$ form an orthonormal basis of $\mathcal{B}(\mathbb{H})$ (see Example 3.1).

Example 3.1
Let us consider a two-dimensional case. Assume that $\hat{\rho}' = \hat{\rho}$. Then, Eq. (3.24) implies

$$T_{ir,js} = \delta_{ir}\delta_{js} \tag{3.28}$$

Considering the two indices $m = (i, r)$ and $n = (j, s)$, we can represent $\mathcal{T}_t$ in the basis defined in Eq. (2.4) as

$$\mathbf{T}_{m,n} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}_{m,n},$$

Its eigenvalues are:

$$\lambda = 0, \text{ with degeneration 3,}$$
$$\lambda = 2, \text{ without degeneration.}$$

Since the only non-vanishing eigenvalue $\lambda^{N^2=4} = 2$ is associated to the last eigenvector $\hat{E}^{N^2=4}$, this result is in accord with the above statement: $\lambda^{N^2}_{t=0} = N$ and $\lambda^{\alpha}_{t=0} = 0$ for $\alpha \neq N^2$.

Since the map is continuous, we can consider a linear change proportional to the infinitesimal time increment dt. Then, the eigenvalues and eigenvectors of the matrix $\mathbf{T}_{\mathrm{d}t}$ take the form:

$$\lambda^{\alpha}_{\mathrm{d}t} = \begin{cases} N(1 - c^{N^2}\mathrm{d}t), & \alpha = N^2, \\ c^{\alpha}\mathrm{d}t, & \text{otherwise}, \end{cases} \tag{3.29}$$

and

$$\hat{E}^{\alpha}_{\mathrm{d}t} = \begin{cases} \frac{1}{\sqrt{N}}\left(\hat{\mathbb{1}} + \hat{B}\mathrm{d}t\right), & \alpha = N^2, \\ \hat{K}^{\alpha} + \hat{R}^{\alpha}\mathrm{d}t, & \text{otherwise}, \end{cases} \tag{3.30}$$

Then, Eq. (3.23) at time dt becomes

$$\mathcal{T}_{\mathrm{d}t}\hat{\rho} = \hat{\rho} - c^{N^2}\hat{\rho}\mathrm{d}t + \hat{B}\,\hat{\rho}\mathrm{d}t + \hat{\rho}\,\hat{B}^{\dagger}\mathrm{d}t + \sum_{\alpha=1}^{N^2-1} c^{\alpha}\left(\hat{K}^{\alpha}\hat{\rho}\,\hat{K}^{\alpha\dagger}\right)\mathrm{d}t, \tag{3.31}$$

where we retained all terms up to first order in dt. By writing $\hat{B} = -i\hat{H}/\hbar + \hat{A}$, with both $\hat{H}$ and $\hat{K}$ Hermitian operators, Eq. (3.31) in differential form becomes:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - c^{N^2}\hat{\rho} + \hat{A}\,\hat{\rho} + \hat{\rho}\,\hat{A} + \sum_{\alpha=1}^{N^2-1} c^{\alpha}\left(\hat{K}^{\alpha}\,\hat{\rho}\hat{K}^{\alpha\dagger}\right). \tag{3.32}$$

We still have to impose that the map in Eq. (3.31) is trace preserving. Namely, by imposing $\mathrm{Tr}\left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}\right] = 0$ and the cyclic property of the trace, we obtain:

$$c^{N^2}\hat{\mathbb{1}} = 2\hat{A} + \sum_{\alpha=1}^{N^2-1} c^{\alpha}\hat{K}^{\alpha\dagger}\hat{K}^{\alpha}. \tag{3.33}$$

We can write $c^{N^2}\hat{\mathbb{1}}\hat{\rho} = c^{N^2}(\hat{\mathbb{1}}\hat{\rho} + \hat{\rho}\hat{\mathbb{1}})/2$, and the trace preserving form of Eq. (3.32) reads:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \mathbb{L}[\hat{\rho}] = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \sum_{\alpha=1}^{N^2-1} c^{\alpha}\left[\hat{K}^{\alpha}\hat{\rho}\,\hat{K}^{\alpha\dagger} - \frac{1}{2}\left\{\hat{K}^{\alpha}\hat{K}^{\alpha\dagger}, \hat{\rho}\right\}\right]. \tag{3.34}$$

Complete positivity makes sure that all the constants c^{α} are positive. Then, by defining the Lindblad operators $\hat{L}^{\alpha} = \sqrt{c^{\alpha}}\hat{K}^{\alpha}$, we arrive at the final form of the generator given in Eq. (3.22).

Remark 3.1. As consequence of the Theorem 3.3, the Lindblad operators $\hat{L}^{\alpha}$ are not arbitrary operators. They are restricted by the orthogonality condition, which can be derived from Eq. (2.16):

$$\mathrm{Tr}\left[\hat{L}^{\alpha}\hat{L}^{\beta\dagger}\right] = c^{\alpha}\delta^{\alpha\beta}.$$

It can be proved that the constrains on the number of Lindblad operators and their orthogonality can be released. Indeed, we can always redefine, through a unitary transformation, the Lindblad operators $\hat{L}'^{\alpha} = \sum_{\beta}\hat{U}^{\alpha\beta}\hat{L}^{\beta}$ in a way that the argument of the Theorem 3.3 is recovered.

Remark 3.2. As consequence of the Theorem 3.3, the dynamics given by Eq. (3.22) leads to decoherence, which is the reduction of coherences (off-diagonal terms of the density operator). This can be seen as follows. Consider Eq. (3.22) with a single Lindblad operator $\hat{L}$ and where we neglect the Hamiltonian. If represented on the basis of eigenstates $\{\,|\,l_i\rangle\,\}_i$ of $\hat{L}$, we find

$$\frac{\mathrm{d}\rho_{ij}}{\mathrm{d}t} = -\frac{1}{2}(l_i - l_j)^2\rho_{ij} \tag{3.35}$$

where $\rho_{ij} = \langle l_i|\hat{\rho}|l_j\rangle$. The solution is

$$\rho_{ij}(t) = \rho_{ij}(0)e^{-\frac{1}{2}(l_i-l_j)^2 t}, \tag{3.36}$$

which shows that the off-diagonal terms of ρ (or better, when $l_i \neq l_j$) decay exponentially with a rate proportional to the distance squared $(l_i - l_j)^2$. Conversely, the populations (diagonal terms, i.e. with $i = j$) do not decay.