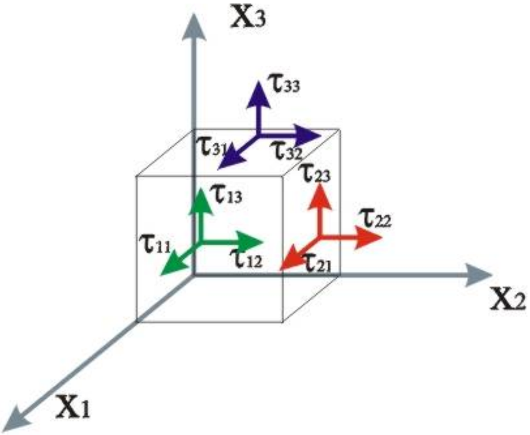
Geophysical Fluid Dynamics

Lecture III

1. Scalar, Vector, Tensors, Divergence, Curl
2. Force on a surface
3. Guass' Theorem (a.k.a. Divergence Theorem)
4. Stokes Theorem
5. lagrangian and eulerian framework
6. material derivative

Tensors

1. Vectors and Scalars are also Tensors
2. The *order* of a tensor is the dimensionality of the array needed to represent it.



Force on a surface

A two-dimensional case; what is the force on AC? If **F** is the force on the face AC:

$$F_1 = \tau_{11} dx_2 + \tau_{21} dx_1 \tag{1}$$

If $\mathbf{f} = \mathbf{F}/\mathrm{ds}$, then

$$f_1 = \tau_{11} \frac{dx_2}{ds} + \tau_{21} \frac{dx_1}{ds} \tag{2}$$

or $f_1 = \tau_{11} cos\theta_1 + \tau_{21} cos\theta_2 = \tau_{11} n_1 + \tau_{21} n_2$
Using the summation convention

$$f_1 = \tau_{j1} n_j \tag{3}$$

Generalizing to three dimensions

$$f_i = \tau_{ji} n_j \tag{4}$$

leading to nine separate stress components involving normal and shear stresses. The inner product of the stress tensor τ and the unit outward vector **n** is the force per unit area on a surface.

(Gauss') Divergence theorem

The theorem relates a volume integral to a surface integral.

Consider an infinitesimal surface element dA whose outward unit normal is **n**. The vector $n\,dA$ has magnitude dA and direction **n**. Gauss' theorem states that the volume integral of the divergence of **Q** is equal to the surface integral of the outflow of **Q**.

$$\int_v \frac{\partial Q}{\partial x_i} dV = \int_A dA_i Q \tag{5}$$

For a vector:

$$\boxed{\int_v \frac{\partial Q_i}{\partial x_i} dV = \int_A dA_i Q} \tag{6}$$

and in vector notation

$$\int_v \nabla \cdot \mathbf{Q} dV = \int_A d\mathbf{A} \cdot \mathbf{Q} \tag{7}$$

Stokes' theorem

The theorem relates a surface over an open surface to a line integral.

Consider an open surface A, with bounding curve C. Let $d\mathbf{r}$ be an element of the bounding curve whose direction is the tangent.

$$\boxed{\int_A (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_C \mathbf{F} \cdot d\mathbf{r}} \tag{8}$$

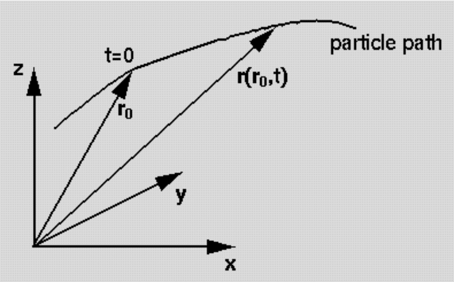
the surface integral of the curl of a vector field $\mathbf{F}$ is equal to the line integral along the bounding curve. The line integral of a vector around a closed curve C is the *circulation of the field about* C.

Kinematics

- **Kinematics:** description of the fluid and its movement (trajectories, streamlines, vorticity, deformations, ...). Only space and time.
- **Dynamics:** reasons for the movement (its forces) (conservation of mass, momentum, energy, vorticity,...)

Lagrangian and Eulerian Framework

- In the LAGRANGIAN description of motion, one essentially follows the history of an individual particle. A flow variable $F(r_0, t)$ and its velocity is given by $u_i = d(r_i)/dt$
- In the EULERIAN description one focuses on what happens at a spatial point r, so the flow variable is $F(r, t)$.
- In the Eulerian case, d/dt gives the local rate of change of F at each point r and is not the total rate of change seen by a fluid particle ...



Lagrangian Derivative

We seek to calculate the rate of change of F at each point following a particle of fixed identity.

$$\boxed{\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}} \tag{9}$$

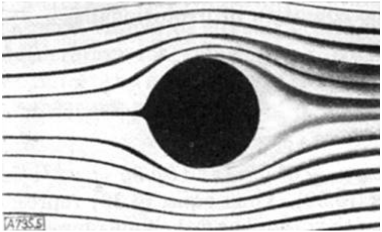
The Lagrangian Derivative $\frac{DF}{Dt}$ is made of (1) the local rate of change at a given point (zero for steady flows...) and (2) the advective derivative.

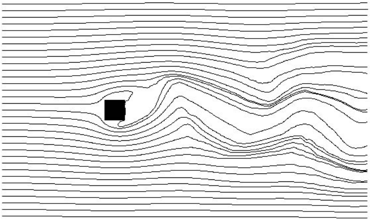
$\frac{\partial F}{\partial t}$ is the local rate of change of F at a given point.

$u_i \frac{\partial F}{\partial x_i}$ is the advective derivative, it is the change in F as a result of advection of the particle from one location to another where F is different.

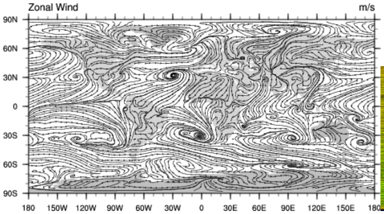
Streamlines

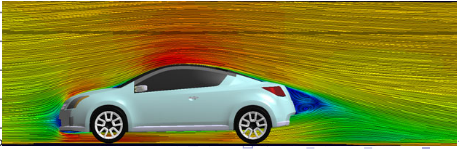
- At $t = t_0$, streamlines are curves that are tangent to direction of flow.
- For unsteady flows, streamlines change with time.





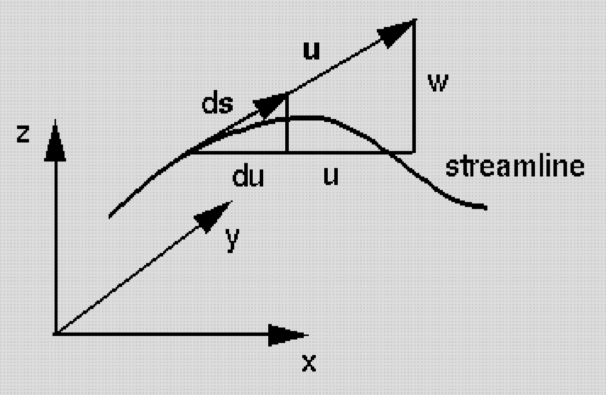
Example of a streamline plot





Streamlines

Let $ds = (dx, dy, dz)$ be an element of arc length along a streamline, and let $u = (u, v, w)$ be the local velocity vector along that streamline, then $dx/u = dy/v = dz/w$.

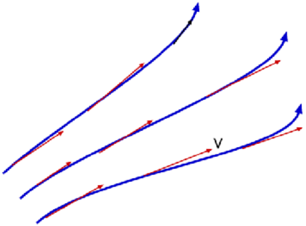


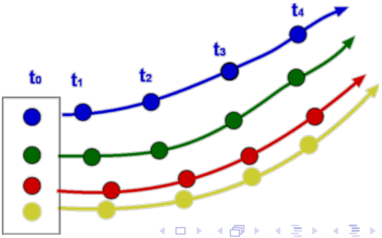
Streamlines

- Close to a solid boundary, streamlines are parallel to that boundary.
- The direction of the streamline is the direction of the fluid velocity.
- Fluid can not cross a streamline.
- Streamlines can not cross each other.
- Any particle starting on one streamline will stay on that same streamline.
- In unsteady flow, streamlines can change position with time.

Streamlines

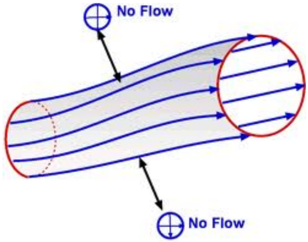
- **Streamlines** are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction a fluid element will travel in at any point in time.
- **Pathlines** are the trajectories that individual fluid particles follow. These can be thought of as a "recording" of the path a fluid element in the flow takes over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.
- For a steady flow, the two are the same.





Streamtubes

- All streamlines passing through any closed curve C at some time form a tube called STREAMTUBE.



- The walls of the streamtubes are streamlines
- Fluid can not flow across a streamline, so fluid can not cross a streamtube surface
- A streamtube can change position with time if the flow is unsteady