3.5 Lindblad evolution in Quantum Information theory

We report some of the most common Lindblad dynamics used in quantum information theory.

Complete depolarising channel. Consider the dynamics given by

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \sum_{i=x,y,z} \left[\hat{\sigma}_i \hat{\rho} \hat{\sigma}_i - \frac{1}{2} \left\{ \hat{\sigma}_i \hat{\sigma}_i, \hat{\rho} \right\} \right], \tag{3.62}$$

where $\hat{\sigma}_i$ is the i-th Pauli matrix. We notice that $\hat{\sigma}_i \hat{\sigma}_i = \hat{\mathbb{1}}$, and thus the dynamics reduces to

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \left(\sum_{i=x,y,z} \hat{\sigma}_i \hat{\rho} \hat{\sigma}_i \right) - 3\hat{\rho}. \tag{3.63}$$

To compute the action of this map, we consider the Bloch representation of the state $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{r} \cdot \hat{\boldsymbol{\sigma}})$. Then, one has

$$\begin{aligned} \sum_i \hat{\sigma}_i \hat{\rho} \hat{\sigma}_i &= \frac{1}{2} \sum_i \hat{\sigma}_i \hat{\mathbb{1}} \hat{\sigma}_i + \frac{1}{2} \sum_{ij} r_j \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_i, \\ &= \frac{3}{2} \hat{\mathbb{1}} + \frac{1}{2} \sum_{ij} r_j \hat{\sigma}_i \left(\delta_{ij} - i \sum_k \epsilon_{ijk} \hat{\sigma}_k \right), \\ &= \frac{1}{2} \left(3\hat{\mathbb{1}} + \sum_j r_j \hat{\sigma}_j + \sum_{ijkl} r_j \epsilon_{ijk} \epsilon_{ikl} \hat{\sigma}_l \right). \end{aligned} \tag{3.64}$$

However, $\sum_{ik} \epsilon_{ijk} \epsilon_{ikl} = -2\delta_{jl}$, which gives

$$\sum_i \hat{\sigma}_i \hat{\rho} \hat{\sigma}_i = \frac{1}{2} \left(3\hat{\mathbb{1}} + \mathbf{r} \cdot \hat{\boldsymbol{\sigma}} \right). \tag{3.65}$$

By exploiting that $\mathbf{r} \cdot \hat{\boldsymbol{\sigma}} = 2\hat{\rho} - \hat{\mathbb{1}}$, one obtains

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -4\hat{\rho} + 2\hat{\mathbb{1}}. \tag{3.66}$$

Such an equation can be easily integrated, and it gives

$$\hat{\rho}(t) = e^{-4t} \hat{\rho}(0) + \frac{\left(1 - e^{-4t}\right)}{2} \hat{\mathbb{1}}. \tag{3.67}$$

Straigtforwardly, one can see that the action of the dynamics is to completely depolarise the state, i.e. any state is sent to the origin of the Bloch sphere. The effect is summarized in Fig. 3.1.

Exercise 3.1
Verify that the map $\mathcal{T}_t$, that defines the dynamics $\mathcal{T}_t[\hat{\rho}(0)] = \hat{\rho}(t)$ in Eq. (3.67), satisfies the conditions to be a strongly continuous semigroup of operators. Namely, verify that
1) $\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}$ $\forall t, s \geq 0$,
2) $\mathcal{T}_0 = \mathrm{id}$,
3) $\lim_{h\to 0} \mathcal{T}_{t+h}[\hat{\rho}(0)] = \mathcal{T}_t[\hat{\rho}(0)]$, $\forall \hat{\rho}(0)$ *and* $\forall t, h \geq 0$.

Amplitude damping channel. Consider the following master equation, which describes the amplitude damping channel:

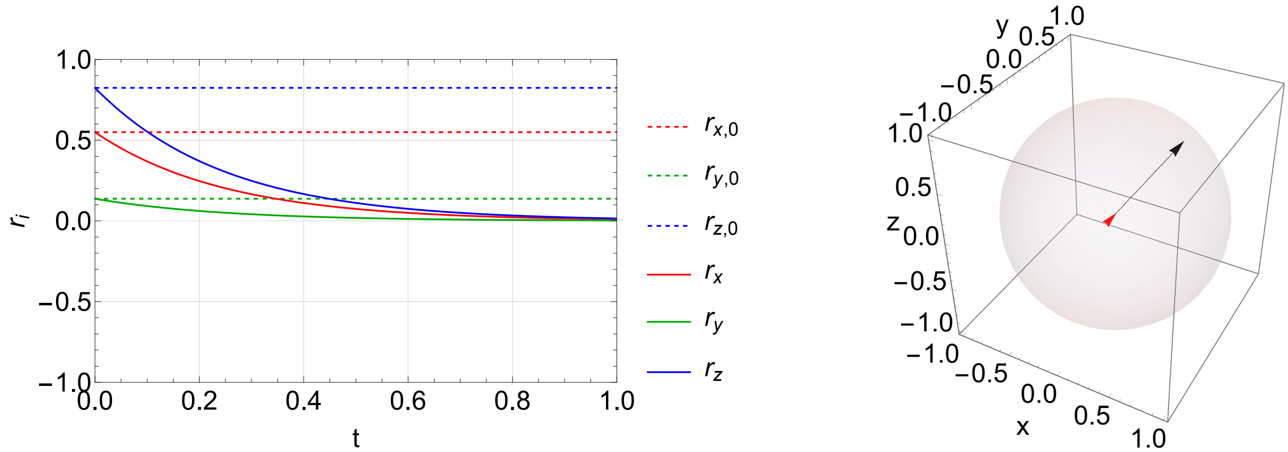

Fig. 3.1: Graphical representation of the complete depolarising channel. (Left) Dynamics of the components of the Bloch radius (continuous lines). The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the initial state (black arrow) and after a time $t = 1$ (red arrow). The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \gamma\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\left\{\hat{\sigma}_+\hat{\sigma}_-, \hat{\rho}\right\}\right), \tag{3.68}$$

where $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$ and $\hat{H} = -\frac{1}{2}\hbar\omega\hat{\sigma}_z$ is the free Hamiltonian of the system. By exploiting the following relations expressed in the computational basis $\{\,|\,0\rangle,\,|\,1\rangle\,\}$

$$\begin{aligned} \sigma_-\sigma_+ = \sigma_-\sigma_z\sigma_+ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_-\sigma_x\sigma_+ = \sigma_-\sigma_y\sigma_+ = 0, \\ \sigma_+\sigma_- = -\sigma_+\sigma_z\sigma_- &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_+\sigma_x\sigma_- = \sigma_+\sigma_y\sigma_- = 0, \end{aligned} \tag{3.69}$$

and the Bloch representation, the master equation in the computational basis reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r_z & r_x + ir_y \\ r_x + ir_y & -r_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\gamma(1+r_z) & -\frac{1}{2}(\gamma - 2i\omega)(r_x - ir_y) \\ -\frac{1}{2}(\gamma + 2i\omega)(r_x + ir_y) & \gamma(1+r_z) \end{pmatrix}. \tag{3.70}$$

This determines a set of first-order differential equations for $\mathbf{r}$:

$$\begin{aligned} \frac{\mathrm{d}r_x}{\mathrm{d}t} &= -\frac{\gamma}{2}r_x + \omega r_y, \\ \frac{\mathrm{d}r_y}{\mathrm{d}t} &= -\frac{\gamma}{2}r_y - \omega r_x, \\ \frac{\mathrm{d}r_z}{\mathrm{d}t} &= -\gamma(1 + r_z), \end{aligned} \tag{3.71}$$

whose solutions are

$$\begin{aligned} r_x(t) &= e^{-\frac{\gamma}{2}t}(\cos(\omega t)r_x(0) + \sin(\omega t)r_y(0)), \\ r_y(t) &= e^{-\frac{\gamma}{2}t}(\cos(\omega t)r_y(0) - \sin(\omega t)r_x(0)), \\ r_z(t) &= e^{-\gamma t}(1 + r_z(0)) - 1. \end{aligned} \tag{3.72}$$

Thus, one can clearly see that the asymptotic state is $|1\rangle$ corresponding to the south pole of the Bloch sphere. The effect is summarized in Fig. 3.2.

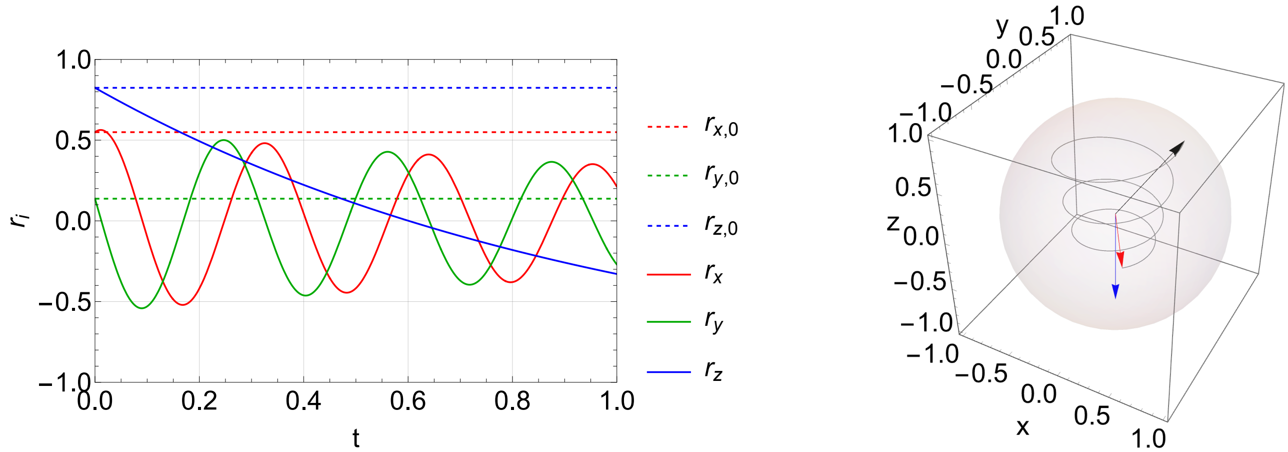

Fig. 3.2: Graphical representation of the amplitude damping channel. (Left) Dynamics of the components of the Bloch radius (continuous lines) for $\gamma = 1$ and $\omega = 20$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the dynamics (gray line) with the initial state (black arrow), that after a time $t = 1$ (red arrow). The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.

Thermalisation of a qubit. Consider the following master equation, which describes the thermalisation of a qubit to temperature T:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \gamma(n+1)\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\left\{\hat{\sigma}_+\hat{\sigma}_-, \hat{\rho}\right\}\right) + \gamma n\left(\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \frac{1}{2}\left\{\hat{\sigma}_-\hat{\sigma}_+, \hat{\rho}\right\}\right), \tag{3.73}$$

where $n = n(T)$ is the mean number of excitations at the temperature T. By exploiting Eq. (3.69) and the Bloch representation, we obtains the following set of first-order differential equations for the components of the Bloch vector

$$\begin{aligned} \frac{\mathrm{d}r_x}{\mathrm{d}t} &= -\frac{\gamma}{2}(2n+1)r_x, \\ \frac{\mathrm{d}r_y}{\mathrm{d}t} &= -\frac{\gamma}{2}(2n+1)r_y, \\ \frac{\mathrm{d}r_z}{\mathrm{d}t} &= -\gamma(1+(2n+1)r_z), \end{aligned} \tag{3.74}$$

which can be easily solved and gives

$$\begin{aligned} r_x(t) &= e^{-\frac{\gamma}{2}(2n+1)t}r_x(0), \\ r_y(t) &= e^{-\frac{\gamma}{2}(2n+1)t}r_y(0), \\ r_z(t) &= e^{-\gamma(2n+1)t}(r_z(0) - r_{z,\infty}) + r_{z,\infty}, \end{aligned} \tag{3.75}$$

where we defined $r_{z,\infty} = -1/(2n+1)$ being the asymptotic value of $r_z = \langle\hat{\sigma}_z\rangle$. The effect is summarized in Fig. 3.3.

Decoherence due to continuous non-selective measurement. A continuous non-selective measurement induce a decoherence, details on the derivation are not discussed here. For example, a measurement of the z

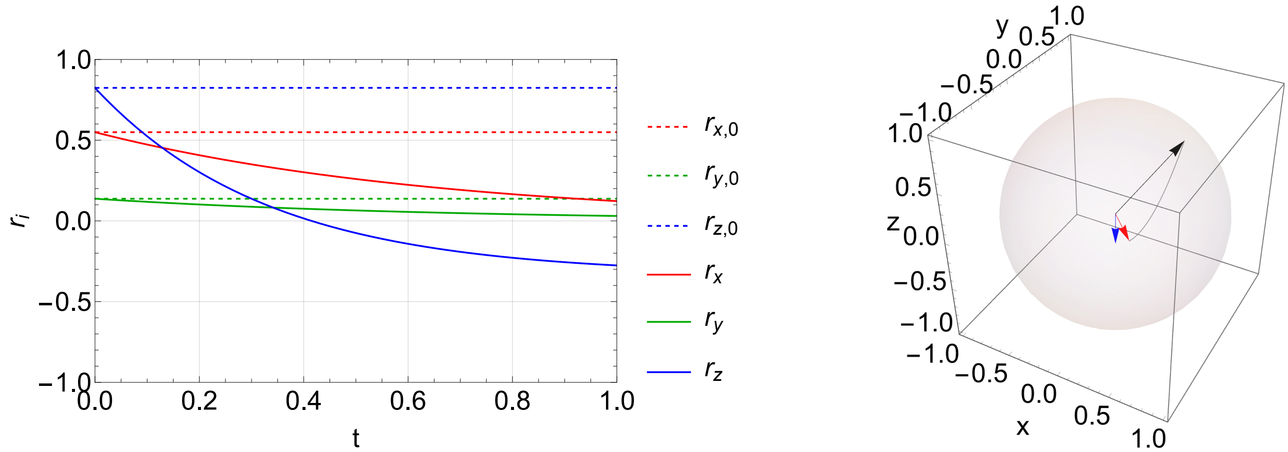

Fig. 3.3: Graphical representation of the thermalisation channel. (Left) Dynamics of the components of the Bloch radius (continuous lines) for $\gamma = n = 1$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the dynamics (gray line) with the initial state (black arrow), that after a time $t = 1$ (red arrow) and the asymptotic state (blue arrow). The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.

component of the qubit leads to the following master equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] - \frac{\gamma}{2}\left[\hat{\sigma}_z, \left[\hat{\sigma}_z, \hat{\rho}\right]\right], \tag{3.76}$$

where $\hat{H} = -\frac{1}{2}\hbar\omega\hat{\sigma}_x$ is the free Hamiltonian of the system. By exploiting the Bloch representation, one easily finds the following set of differential equations

$$\begin{aligned} \frac{\mathrm{d}r_x}{\mathrm{d}t} &= -2\gamma r_x, \\ \frac{\mathrm{d}r_y}{\mathrm{d}t} &= -2\gamma r_y + \omega r_z, \\ \frac{\mathrm{d}r_z}{\mathrm{d}t} &= -\omega r_y, \end{aligned} \tag{3.77}$$

whose solution is

$$\begin{aligned} r_x(t) &= e^{-2\gamma t} r_x(0), \\ r_y(t) &= \frac{e^{-\gamma t}}{\Omega}\left[\left(\Omega\cos(\Omega t) - \gamma\sin(\Omega t)\right) r_y(0) + \omega\sin(\Omega t) r_z(0)\right], \\ r_z(t) &= \frac{e^{-\gamma t}}{\Omega}\left[\left(\Omega\cos(\Omega t) + \gamma\sin(\Omega t)\right) r_z(0) - \omega\sin(\Omega t) r_y(0)\right], \end{aligned} \tag{3.78}$$

where $\Omega^2 = \omega^2 - \gamma^2$. Notably, the asymptotic state corresponds to the origin of the Bloch sphere. The effect is summarized in Fig. 3.4.

3.6 Unravelling formalism for noises

An alternative way to account for noises is to unravel the master equation and consider explicitly the noise acting on the wavefunction. Consider the following master equation, which — for the sake of simplicity — is taken as Markovian

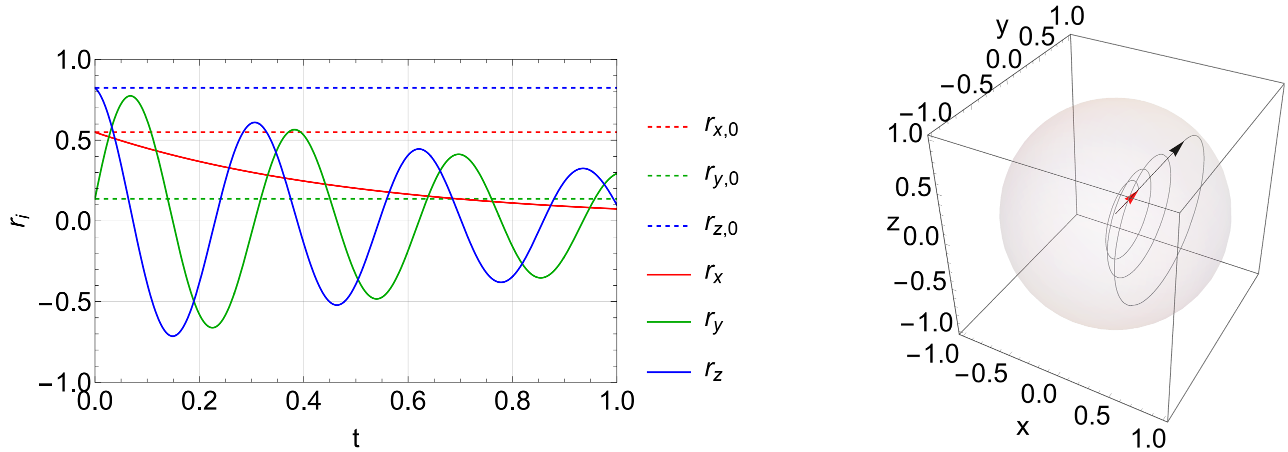

Fig. 3.4: Graphical representation of the non-selective measurement decoherence. (Left) Dynamics of the components of the Bloch radius (continuous lines) for $\gamma = 1$ and $\omega = 20$. The initial values (dashed lines) are reported for comparison. (Right) Bloch representation of the dynamics (gray line) with the initial state (black arrow), that after a time $t = 1$ (red arrow). The initial state corresponds to $\mathbf{r}_0 = (4, 1, 6)/\sqrt{53}$.

$$\frac{\mathrm{d}\hat{\rho}_t}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}_t\right] + \epsilon^2 \sum_k \left(\hat{L}_k \hat{\rho}_t \hat{L}_k^\dagger - \tfrac{1}{2}\left\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}_t\right\}\right), , \tag{3.79}$$

where $\hat{H}$ is the Hamiltonian, and ϵ quantifies the coupling of the noises, while the Lindblad operator $\hat{L}_k$ can embed also relative strengths between different noise channels.

We want to construct a stochastic unravelling of the Lindblad dynamics in Eq. (3.79). This is a dynamical stochastic equation for the wavefunction $|\psi_t\rangle$ from which one can derive exactly Eq. (3.79) for the corresponding statistical operator obtained as $\hat{\rho}_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$, where $\mathbb{E}$ indicates the average over the stochastic process. There are two advantages in using the unravelling approach in place of that based on the master equation. The first one is that, for a N level system, the master equation approach is equivalent to solve N^2 ordinary coupled differential equations of the first order, while the unravelling approach has N stochastic ordinary coupled differential equation of the first order. Clearly, there is a computational advantage in the scaling, however it is only polynomial and it has to be compared with the necessity of performing stochastic averages. The second advantage is that for every master equation there are infinite equivalent unravellings corresponding to it. Depending on the specific problem, some of these can be solved or simulated more easily than others or than the master equation.

In this family of equivalent unravellings, the linear stochastic unravelling has a special place due to its simplicity. In the so-called Ito form, it reads

$$\mathrm{d}|\psi_t\rangle = \left[-\tfrac{i}{\hbar}\hat{H}\mathrm{d}t + \sum_k \left(i\epsilon \hat{L}_k \mathrm{d}W_{k,t} - \tfrac{1}{2}\epsilon^2 \hat{L}_k^\dagger \hat{L}_k \mathrm{d}t\right)\right]|\psi_t\rangle\,, \tag{3.80}$$

where $\mathrm{d}W_{k,t}$ are differentials of standard independent Wiener processes, such that

$$\mathbb{E}[\mathrm{d}W_{k,t}] = 0, \quad \text{and} \quad \mathbb{E}[\mathrm{d}W_{k,t}\mathrm{d}W_{k',t}] = \delta_{k,k'}\mathrm{d}t. \tag{3.81}$$

The first term of Eq. (A.27) is the standard Schrödinger equantion. The second term introduces the stochasticity of the noise process, while the last term is necessary to preserve the normalisation of $|\psi_t\rangle$ in time. We now proceed in showing that the dynamics in Eq. (A.27) is equivalent to that in Eq. (3.79). We start by differentiating the statistical operator:

$$\mathrm{d}\hat{\rho}_t = \mathrm{d}\mathbb{E}[|\psi_t\rangle\langle\psi_t|] = \mathbb{E}[\mathrm{d}(|\psi_t\rangle\langle\psi_t|)], \tag{3.82}$$

where the second equality follows from the linearity of the average. Then, one has

$$\mathrm{d}\hat{\rho}_t = \mathbb{E}[|\mathrm{d}\psi_t\rangle\langle\psi_t|] + \mathbb{E}[|\psi_t\rangle\langle\mathrm{d}\psi_t|] + \mathbb{E}[|\mathrm{d}\psi_t\rangle\langle\mathrm{d}\psi_t|], \tag{3.83}$$

where the last term is needed to account all the terms of the first order in $\mathrm{d}t$, which includes that in the second order in $\mathrm{d}W$, see Eq. (3.81). Now, one substitutes, up to the first order in $\mathrm{d}t$ and second order in $\mathrm{d}W_{k,t}$, Eq. (A.27) in place of $|\mathrm{d}\psi_t\rangle$, and its conjugate in place of $\langle\mathrm{d}\psi_t|$. Then, we obtain

$$\begin{aligned}\mathrm{d}\hat{\rho}_t = \mathbb{E}&\left\{\left[-\tfrac{i}{\hbar}\hat{H}\mathrm{d}t + \sum_k\left(i\epsilon\hat{L}_k\mathrm{d}W_{k,t} - \tfrac{1}{2}\epsilon^2\hat{L}_k^\dagger\hat{L}_k\mathrm{d}t\right)\right]|\psi_t\rangle\langle\psi_t|\right\}\\ &+\mathbb{E}\left\{|\psi_t\rangle\langle\psi_t|\left[\tfrac{i}{\hbar}\hat{H}\mathrm{d}t + \sum_k\left(-i\epsilon\hat{L}_k\mathrm{d}W_{k,t} - \tfrac{1}{2}\epsilon^2\hat{L}_k^\dagger\hat{L}_k\mathrm{d}t\right)\right]\right\}\\ &-\mathbb{E}\left[\sum_k i\epsilon\hat{L}_k\mathrm{d}W_{k,t}|\psi_t\rangle\langle\psi_t|\sum_{k'} i\epsilon\hat{L}_{k'}\mathrm{d}W_{k',t}\right].\end{aligned} \tag{3.84}$$

Under the Markovian assumption, i.e. that the noises $\mathrm{d}W_{k,t}$ and $\mathrm{d}W_{k,s}$ for $t \neq s$ are independent, then in the state $|\psi_t\rangle$ there are only noises up to time $s < t$ and thus independent from $\mathrm{d}W_{k,t}$. Thus, in Eq. (3.84) one can separate the average acting on $\mathrm{d}W_{k,t}$ and that acting on the state $|\psi_t\rangle$. Namely, we find

$$\begin{aligned}\mathbb{E}[\mathrm{d}W_{k,t}|\psi_t\rangle\langle\psi_t|] &= \mathbb{E}[\mathrm{d}W_{k,t}]\,\mathbb{E}[|\psi_t\rangle\langle\psi_t|] = \mathbb{E}[\mathrm{d}W_{k,t}]\,\hat{\rho}_t = 0,\\ \mathbb{E}[\mathrm{d}W_{k,t}|\psi_t\rangle\langle\psi_t|\,\mathrm{d}W_{k',t}] &= \mathbb{E}[\mathrm{d}W_{k,t}\mathrm{d}W_{k',t}]\,\mathbb{E}[|\psi_t\rangle\langle\psi_t|] = \delta_{k,k'}\mathrm{d}t\,\hat{\rho}_t.\end{aligned} \tag{3.85}$$

By substituting these expressions in Eq. (3.84), we obtain

$$\mathrm{d}\hat{\rho}_t = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}_t\right]\mathrm{d}t + \epsilon^2\sum_k\left(\hat{L}_k\hat{\rho}_t\hat{L}_k^\dagger - \tfrac{1}{2}\left\{\hat{L}_k^\dagger\hat{L}_k,\hat{\rho}_t\right\}\right)\mathrm{d}t, \tag{3.86}$$

which can be easily recasted in the form of Eq. (3.79).

Exercise 3.2
Derive the master equation associated to Eq. (A.27) for $\epsilon = i\epsilon_0$, with $\epsilon_0 \in \mathbb{R}$.

Exercise 3.3
Derive the master equation associated to Eq. (A.27) for $\hat{L}_k = \hat{A}_k - \langle\psi_t|\hat{A}_k|\psi_t\rangle$ with $\hat{A}_k^\dagger = \hat{A}_k$.