Geophysical Fluid Dynamics

Lecture V: Conservation Laws

1. **Leibniz Theorem for time derivative of volume integrals**
2. Conservation of Mass - Continuity Equation
   - **Conservation Equation for a tracer**
   - **Advection-Diffusion**
   - **Diffusion**
3. Conservation of Momentum
   - **Cauchy**
   - **Navier-Stokes Equations**
   - **Euler Equation**
   - **The case of a rotating frame (towards the GFD Eq.)**
4. Conservation of Energy
   - **Kinetic, Mechanical, Potential and Total Energy**
   - **First and Second law of thermodynamics**
   - **Bernoulli's Equation - Principle**

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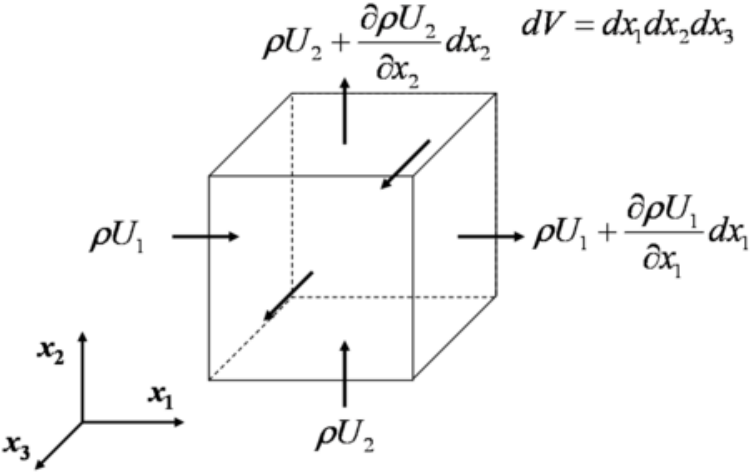
Conservation of Mass



Conservation of Mass

Conservation of Mass

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \tag{1}$$

or

$$\boxed{1/\rho \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0}. \tag{2}$$

For a steady flow, it reduces to

$$\boxed{\nabla \cdot (\rho \mathbf{u}) = 0} \tag{3}$$

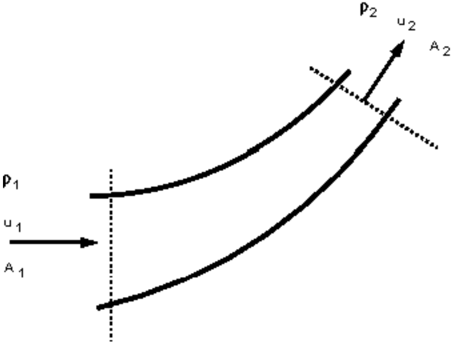
For an incompressible flow, it reduces to

$$\boxed{\nabla \cdot \mathbf{u} = 0} \tag{4}$$

Continuity

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{5}$$

so that, in this case, $u_2 > u_1$.



Conservation law for a substance/tracer

The molecular coefficient

$$\boxed{K = U^c \times l}, \tag{6}$$

where U^c is the characteristic velocity and l is the free molecular distance. Units are $m^2 s^{-1}$.
If we have advection and diffusion the balance is

$$\boxed{\frac{\partial C}{\partial t} = (\text{Advection in - Advection out}) + (\text{Diffusion in - Diffusion out})} \tag{7}$$

Conservation law for a substance/tracer

$$\boxed{\frac{DC}{Dt} = -C\nabla\mathbf{u} - K\nabla^2 C} \tag{8}$$

For incompressible / steady flows, since
$\frac{\partial\rho}{\partial t} = 0 \rightarrow \nabla\cdot\mathbf{u} = 0$
so we have the *Advection-Diffusion Equation*

$$\boxed{\frac{DC}{Dt} = -K\nabla^2 C} \tag{9}$$

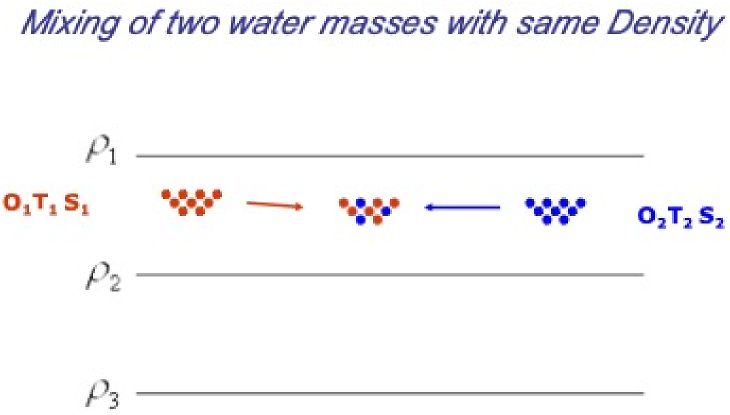
If we assume no advection ($\mathbf{u} = 0$), then we have the *Diffusion Equation*

$$\boxed{\frac{\partial C}{\partial t} = -K\nabla^2 C} \tag{10}$$

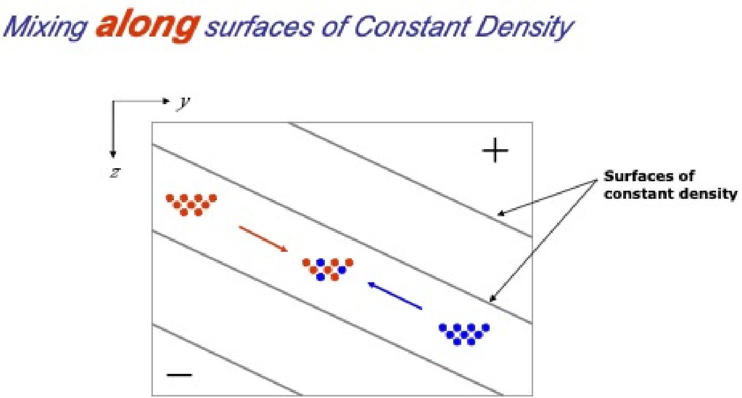
Eddy diffusivity

Diffusion and mixing in the Ocean is not referred to in depth-space bu along isopycnals (adiabatic surfaces) and across isopycnals (diapycnal mixing)

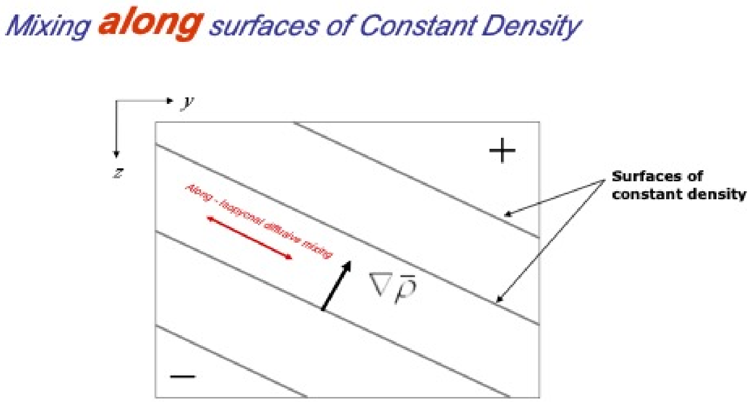
Eddy diffusivity



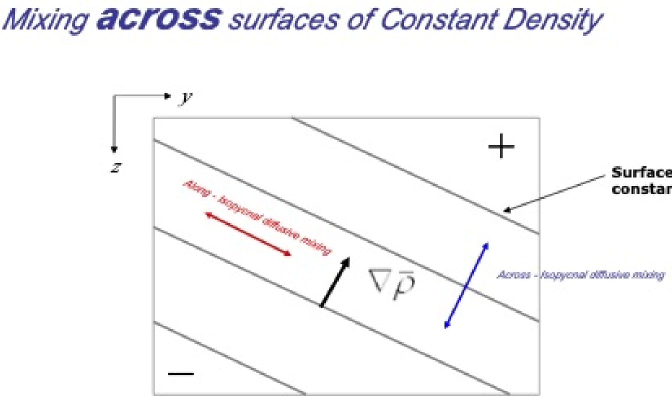
Eddy diffusivity



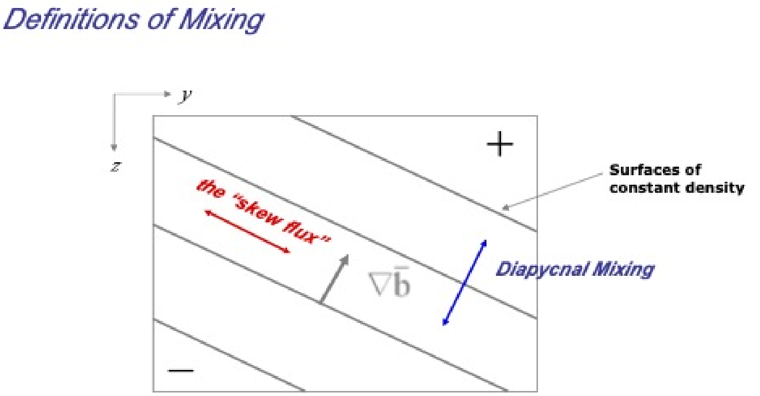
Eddy diffusivity



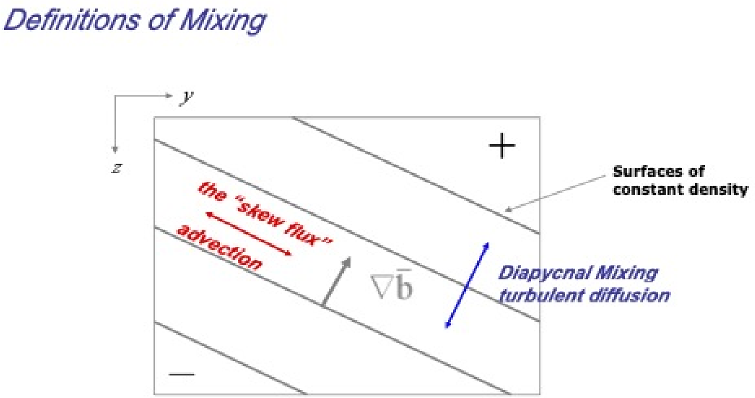
Eddy diffusivity



Eddy diffusivity



Eddy diffusivity



Eddy diffusivity

Starting from the advection-diffusion eq., performing a Reynolds decomposition, we find a relationship between the eddy flux $\overline{u'c'}$ and the mean tracer gradient $\nabla\overline{c}$

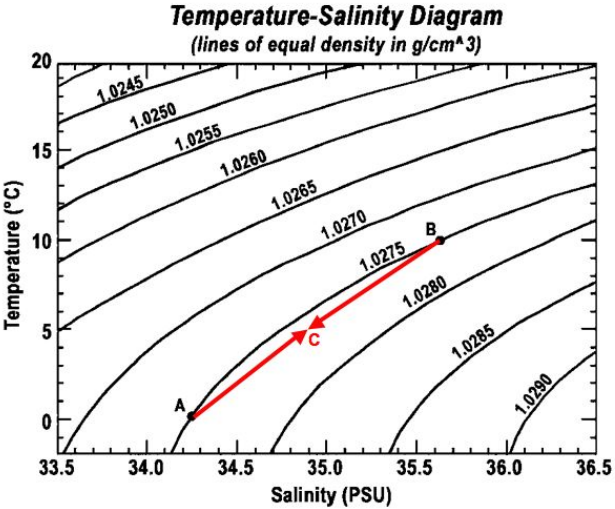
$$\overline{u'c'} = -D_{ij}\partial_{x_j}\overline{c} \tag{11}$$

The diffusivity tensor can be decomposed into its symmetric and antisymmetric components.
The symmetric represents the diffusive component.
The antisymmetric is the skew component.
The tracer flux associated to the antisymmetric component of the diffusivity is the skew flux.

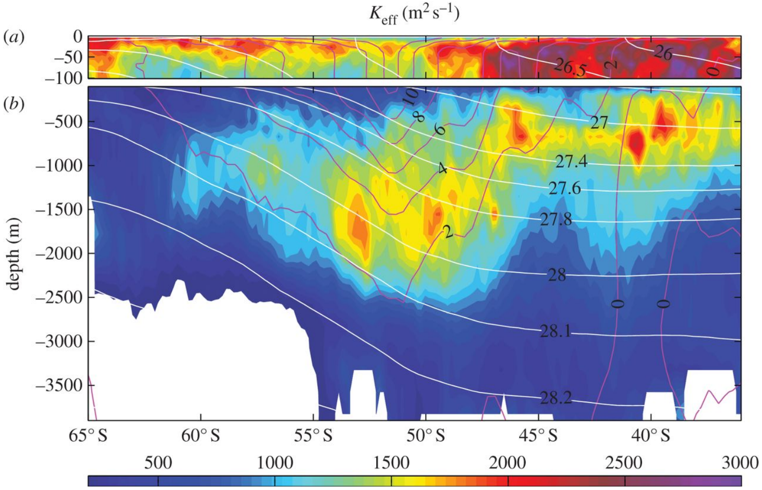
$$\overline{u'c'}^a = -D_{ij}^a\partial_{x_j}\overline{c} = \varepsilon_{ijk}\Psi_k\partial_{x_j}\overline{c} = \Psi \times \nabla\overline{c} \tag{12}$$

The skew flux is an advective flux where $\mathbf{u} = -\nabla \times \boldsymbol{\Psi}$

Cabelling

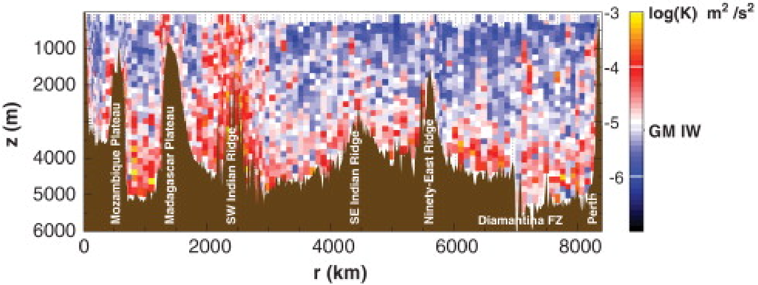

Temperature-Salinity Diagram
(lines of equal density in g/cm^3)
Temperature (°C)
Salinity (PSU)
20
15
10
5
0
33.5
34.0
34.5
35.0
35.5
36.0
36.5
1.0245
1.0250
1.0255
1.0260
1.0265
1.0270
1.0275
1.0280
1.0285
1.0290
A
B
C

Eddy diffusivity



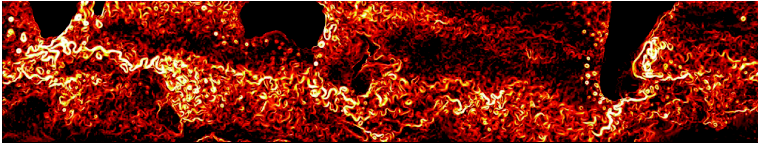
notice how K is intense at the surface and mostly along isopycnals

Eddy diffusivity



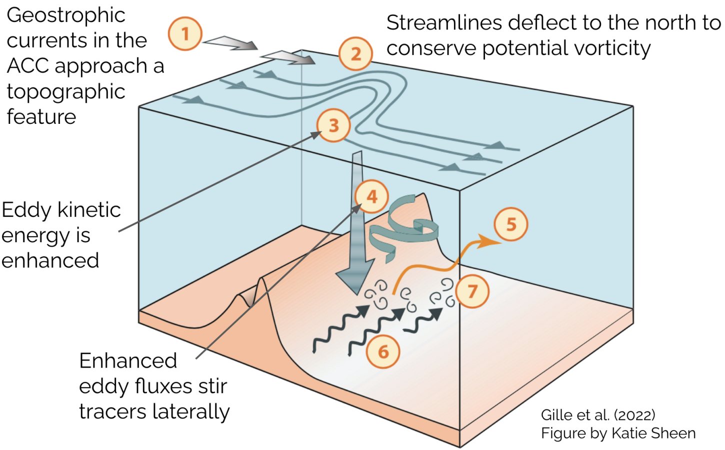
internal tides breaking enhancing turbulence and diffusivity have been observed. If the interaction with topography is strong, then the enhanced diffusivity can reach high into the water column, even reaching the pycnocline.

Eddy diffusivity



Mesoscale eddies and standing meanders transfer momentum vertically and dissipate it with topography.

Eddy diffusivity



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