

Riassunto :

Q : SPAZIO DELLE CONFIGURAZIONI (m -dimensionale)

$p \in Q \rightsquigarrow$ determina la configurazione del sistema,
cioè (per un sist. meccanico discreto)
la posizione di tutti gli N pt:
 $\vec{r}_i(q, t) \quad i = 1, \dots, N$

\uparrow
coord.
 (q_1, \dots, q_m)

TQ : SPAZIO DEGLI STATI

$(p, u) \in TQ \rightsquigarrow$ determina lo stato del sistema,
cioè (per un sist. meccanico discreto)
la posizione e la velocità
di tutti gli N pt:
 $\vec{r}_i(q, t), \vec{v}_i(q, \dot{q}, t) = \sum_{k=1}^m \frac{\partial \vec{r}_i(q, t)}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i(q, t)}{\partial t}$
 $i = 1, \dots, N$

\uparrow
coord.
 $(q_1, \dots, q_m, \dot{q}_1, \dots, \dot{q}_m)$

MOTO : funzioni $q_1(t), \dots, q_m(t) \rightsquigarrow$ permettono di descrivere
il moto di tutti i pt:
 $\vec{r}_i(t) = \vec{r}_i(q(t), t) \quad i = 1, \dots, N$

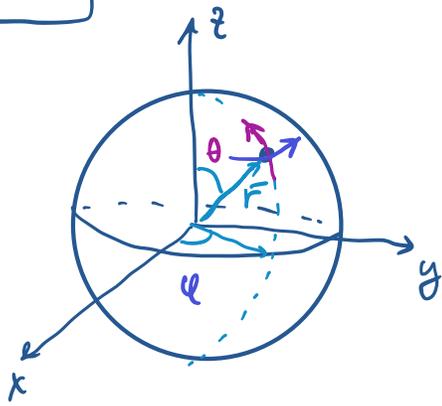
EQ. del MOTO : dinamica del sistema data dalle

Eq. di Lagrange $L(q, \dot{q}, t)$ è detta "Lagrangiana"

$$\frac{d}{dt} \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_h} - \frac{\partial L(q(t), \dot{q}(t), t)}{\partial q_h} = 0 \quad h = 1, \dots, m$$

\rightsquigarrow m eq. diff. (del secondo ordine) in m incognite

ES. PENDOLO SFERICO (pto materiale vincolato a una sfera di raggio R , soggetto alle forze di gravità)



$$x = R \sin\theta \cos\varphi$$

$$y = R \sin\theta \sin\varphi$$

$$z = R \cos\theta$$

$$\leftarrow \bar{r}(\varphi, \theta)$$

$$\frac{\partial \bar{r}}{\partial \varphi} \quad \frac{\partial \bar{r}}{\partial \theta}$$

En. cinetica

$$\dot{x} = -R \sin\varphi \sin\theta \dot{\varphi} + R \cos\varphi \cos\theta \dot{\theta}$$

$$\dot{y} = R \cos\varphi \sin\theta \dot{\varphi} + R \sin\varphi \cos\theta \dot{\theta}$$

$$\dot{z} = -R \sin\theta \dot{\theta}$$

$$\leftarrow \bar{v}(\dot{\varphi}, \dot{\theta})$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) =$$

$$= \frac{1}{2} m \left[\underbrace{(R^2 \sin^2\varphi \sin^2\theta \dot{\varphi}^2)}_{\text{orange}} + \underbrace{R^2 \cos^2\varphi \cos^2\theta \dot{\theta}^2}_{\text{orange}} - 2 \cancel{R^2 \cos\varphi \sin\varphi \cos\theta \sin\theta \dot{\varphi} \dot{\theta}}^{\cos\theta} \right. \\ \left. + \underbrace{(R^2 \cos^2\varphi \sin^2\theta \dot{\varphi}^2)}_{\text{orange}} + \underbrace{R^2 \sin^2\varphi \cos^2\theta \dot{\theta}^2}_{\text{orange}} + 2 \cancel{R^2 \cos\varphi \sin\varphi \cos\theta \sin\theta \dot{\varphi} \dot{\theta}} \right. \\ \left. + R^2 \sin^2\theta \dot{\theta}^2 \right] =$$

$$= \frac{1}{2} m \left(R^2 \sin^2\theta \dot{\varphi}^2 + \underbrace{R^2 \cos^2\theta \dot{\theta}^2}_{\text{orange}} + \underbrace{R^2 \sin^2\theta \dot{\theta}^2}_{\text{orange}} \right)$$

$$= \frac{1}{2} m \left(R^2 \sin^2\theta \dot{\varphi}^2 + R^2 \dot{\theta}^2 \right) \leftarrow a = \begin{pmatrix} mR^2 \sin^2\theta & 0 \\ 0 & mR^2 \end{pmatrix}$$

$$= \frac{1}{2} \sum_{i,k} a_{ik} \dot{q}_i \dot{q}_k$$

$$V = mgz = mgR \cos\theta$$

$n=2$ gradi di libertà
(ci aspettiamo 2 eq. di Lagrange)

$$L = \frac{mR^2}{2} (\sin^2\theta \dot{\varphi}^2 + \dot{\theta}^2) - mgR \cos\theta$$

$$\leftarrow L(\varphi, \theta, \dot{\varphi}, \dot{\theta})$$

Eq. di Lagrange

$$\frac{\partial L}{\partial \dot{\varphi}} = mR^2 \sin^2 \theta \dot{\varphi} \quad \rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = 2mR^2 \sin \theta \overset{\theta(t)}{\cos \theta} \dot{\varphi} + mR^2 \sin^2 \theta \ddot{\varphi}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta} \quad \rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mR^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \theta} = mR^2 \sin \theta \cos \theta \dot{\varphi}^2 + mgR \sin \theta$$

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 2mR^2 \sin \theta \cos \theta \dot{\varphi} + mR^2 \sin^2 \theta \ddot{\varphi}$$

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mR^2 \ddot{\theta} - mR^2 \sin \theta \cos \theta \dot{\varphi}^2 - mgR \sin \theta$$

$$\hookrightarrow \ddot{\theta} = \sin \theta \cos \theta \dot{\varphi}^2 + \frac{g}{R} \sin \theta$$

INVARIANZA PER CAMBIAMENTO DI COORDINATE

Iniziamo con un esempio.

ES. Punto vincolato sul piano non soggetto a forze

1 - coord cartesiane $\vec{r}(q_1, q_2) = \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}$

2 - coord polari: $\vec{r}(\tilde{q}_1, \tilde{q}_2) = \begin{pmatrix} \tilde{q}_1 \cos \tilde{q}_2 \\ \tilde{q}_1 \sin \tilde{q}_2 \end{pmatrix}$

legate da transf. di coord

Trasformazione di coordinate:

$$\begin{cases} q_1 = \tilde{q}_1 \cos \tilde{q}_2 \\ q_2 = \tilde{q}_1 \sin \tilde{q}_2 \end{cases} (*) \rightarrow \begin{cases} \dot{q}_1 = \dot{\tilde{q}}_1 \cos \tilde{q}_2 - \tilde{q}_1 \dot{\tilde{q}}_2 \sin \tilde{q}_2 \\ \dot{q}_2 = \dot{\tilde{q}}_1 \sin \tilde{q}_2 + \tilde{q}_1 \dot{\tilde{q}}_2 \cos \tilde{q}_2 \end{cases}$$

$$L(q_1, q_2) = \frac{m}{2} (\dot{q}_1^2 + \dot{q}_2^2) \quad \tilde{L}(\tilde{q}_1, \tilde{q}_2) = \frac{m}{2} (\dot{\tilde{q}}_1^2 + \tilde{q}_1^2 \dot{\tilde{q}}_2^2)$$

$L \circ (*)$

$L(q(\tilde{q}), \dot{q}(\tilde{q}, \dot{\tilde{q}}))$

Eq. Lagrange

$$\begin{cases} \ddot{q}_1 = 0 \\ \ddot{q}_2 = 0 \end{cases}$$

soluz. con dato iniziale

$$q_1^0 = q_2^0 = 1$$

$$\dot{q}_1^0 = \dot{q}_2^0 = v/\sqrt{2}$$

$$q_1(t) = 1 + \frac{v}{\sqrt{2}} t$$

$$q_2(t) = 1 + \frac{v}{\sqrt{2}} t$$

Eq. Lagrange

$$\begin{cases} \ddot{\tilde{q}}_1 = \tilde{q}_1 \dot{\tilde{q}}_2^2 \\ 2\tilde{q}_1 \dot{\tilde{q}}_1 \dot{\tilde{q}}_2 + \tilde{q}_1^2 \ddot{\tilde{q}}_2 = 0 \end{cases}$$

soluz. con dato iniziale

$$\tilde{q}_1^0 = \sqrt{2}$$

$$\tilde{q}_2^0 = \pi/4$$

$$\dot{\tilde{q}}_1^0 = v$$

$$\dot{\tilde{q}}_2^0 = 0$$

$$\tilde{q}_1(t) = vt$$

$$\tilde{q}_2(t) = \pi/4$$

$$\rightarrow q(t) = q(\tilde{q}(t))$$

Trasf. di coord.

$$q_n = q_n(\tilde{q}_1, \dots, \tilde{q}_m, t) \quad \text{t.c.} \quad \det \left(\frac{\partial q_n}{\partial \tilde{q}_k} \right) \neq 0 \quad (*)$$

$$\dot{q}_n = \dot{q}_n(\tilde{q}, \dot{\tilde{q}}, t) = \sum_{k=1}^m \frac{\partial q_n}{\partial \tilde{q}_k} \dot{\tilde{q}}_k + \frac{\partial q_n}{\partial t}$$

Prop. Dato un sist. lagrangiano con lagrangiana $L(\bar{q}, \dot{\bar{q}}, t)$, si consideri il camb. di coord. (regolare e invertibile) (*) e sia $\tilde{L}(\tilde{q}, \dot{\tilde{q}}, t)$ la lagrangiana ottenuta da L in SOSTITUZIONE di (*), cioè

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}, t) = L(\bar{q}(\tilde{q}, t), \dot{\bar{q}}(\tilde{q}, \dot{\tilde{q}}, t), t) \quad (o)$$

Allora $\tilde{q}(t)$ è soluzione delle eq. di Lagrange con lagrangiana \tilde{L} SE E SOLO SE $\bar{q}(t) = \bar{q}(\tilde{q}(t), t)$ è soluzione delle eq. di Lagrange con lagrangiana L

Dim.

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_n} &\stackrel{(o)}{=} \sum_{l=1}^m \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \dot{\tilde{q}}_n} \stackrel{(*)}{=} \sum_{l=1}^m \frac{\partial L}{\partial \dot{q}_l} \sum_{k=1}^m \frac{\partial q_l}{\partial \tilde{q}_k} \frac{\partial \dot{q}_k}{\partial \dot{\tilde{q}}_n} \\ &= \sum_{l=1}^m \frac{\partial L}{\partial \dot{q}_l} \frac{\partial q_l}{\partial \tilde{q}_n} \end{aligned}$$

$\frac{\partial \dot{q}_k}{\partial \dot{\tilde{q}}_n} = \delta_{kn}$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_n} &= \sum_{l=1}^m \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} \right) \frac{\partial q_l}{\partial \tilde{q}_n} + \sum_{l=1}^m \frac{\partial L}{\partial \dot{q}_l} \frac{d}{dt} \frac{\partial q_l}{\partial \tilde{q}_n}(\tilde{q}(t), t) \\ &= \sum_{m=1}^m \frac{\partial^2 q_l}{\partial \tilde{q}_n \partial \tilde{q}_m} \dot{\tilde{q}}_m + \frac{\partial^2 q_l}{\partial \tilde{q}_n \partial t} = f(\tilde{q}, \dot{\tilde{q}}, t) \text{ valutata in } \tilde{q}(t) \\ &= \frac{\partial}{\partial \tilde{q}_n} \left(\sum_{m=1}^m \frac{\partial q_l}{\partial \tilde{q}_m} \dot{\tilde{q}}_m + \frac{\partial q_l}{\partial t} \right) = \frac{\partial}{\partial \tilde{q}_n} \dot{q}_l \\ &= \sum_{l=1}^m \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} \right) \frac{\partial q_l}{\partial \tilde{q}_n} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial q_l}{\partial \tilde{q}_n} \right] \end{aligned}$$

$$\frac{\partial \tilde{L}}{\partial \tilde{q}_h} = \sum_{l=1}^m \left[\frac{\partial L}{\partial q_l} \frac{\partial q_l}{\partial \tilde{q}_h} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \tilde{q}_h} \right] \quad \sum_l J_{le} v_e$$

$$\Rightarrow \underbrace{\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \tilde{q}_h} - \frac{\partial \tilde{L}}{\partial \tilde{q}_h}}_{\text{ep. di Lagr. di } \tilde{L} \text{ (*)}} = \sum_{l=1}^m \underbrace{\begin{pmatrix} \frac{\partial q_l}{\partial \tilde{q}_h} \\ \frac{\partial \dot{q}_l}{\partial \tilde{q}_h} \end{pmatrix}}_{\equiv J_{le} \text{ invertibile}} \underbrace{\left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} \right]}_{\text{ep. di Lagr. di } L \text{ (*)}} \equiv v_l \quad h=1, \dots, m$$

Se (*) sono sodd. $\forall h \Rightarrow$ (**) sono sodd. $\forall h$ (manifest)

$$\text{Se (**) sono sodd. } \forall h \Rightarrow \sum_{l=1}^m J_{le} v_l = 0 \Leftrightarrow J \cdot \bar{v} = 0$$

\Rightarrow (*) sono sodd. $\forall h$
 J invertibile
 (moltiplicando a destr. e sinistra per J^{-1})

Diverse Lagrangiane possono portare alle stesse ep. del moto.

Prop. Per ogni scelta della funzione $F(\bar{q}, t)$ e della cost. $c \neq 0$, la Lagrangiana $L(\bar{q}, \dot{\bar{q}}, t)$ e la Lagrangiana

$$L'(\bar{q}, \dot{\bar{q}}, t) \equiv c L(\bar{q}, \dot{\bar{q}}, t) + \mathcal{Q}(\bar{q}, \dot{\bar{q}}, t)$$

$$\text{dove } \mathcal{Q}(\bar{q}, \dot{\bar{q}}, t) = \sum_{k=1}^n \frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial t}$$

"derivata totale"
 $\bar{\mathcal{Q}}(\bar{q}(t), \dot{\bar{q}}(t), t) = \frac{d}{dt} F(\bar{q}(t), t)$

condurranno alle stesse ep. di Lagrange.

"Lagrangiane che differiscono per una DERIVATA TOTALE (in t) sono (classicam.) EQUIVALENTI"

Dim.

$$\frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_h} = c \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_h} + \underbrace{\frac{d}{dt} \frac{\partial \varphi}{\partial \dot{q}_h}}_{= \frac{d}{dt} \frac{\partial F}{\partial \dot{q}_h}} = \sum_{l=1}^m \frac{\partial^2 F}{\partial \dot{q}_h \partial \dot{q}_l} \dot{q}_l + \cancel{\frac{\partial^2 F}{\partial \dot{q}_h \partial t}}$$

$$\frac{\partial L'}{\partial \dot{q}_h} = c \frac{\partial L}{\partial \dot{q}_h} + \underbrace{\frac{\partial \varphi}{\partial \dot{q}_h}}_{= \sum_{k=1}^m \frac{\partial^2 F}{\partial \dot{q}_h \partial \dot{q}_k} \dot{q}_k + \cancel{\frac{\partial^2 F}{\partial \dot{q}_h \partial t}}}$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_h} - \frac{\partial L'}{\partial \dot{q}_h} = \underset{c \neq 0}{\uparrow} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial \dot{q}_h} \right) \quad //$$