A.9 Solution to Exercise 4.1

To express the Hadamard gate H as a rotation, we proceed as follows. We consider the general rotation

$$\hat{R}^{\mathbf{n}}(\theta) = \cos(\theta/2)\hat{\mathbb{1}} - i\sin(\theta/2)\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}, \tag{A.31}$$

of an angle θ around $\mathbf{n}$, where the Pauli matrices are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.32}$$

From such a rotation, we want to obtain

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A.33}$$

First thing, we highlight that the sum $X + Z$ gives

$$X + Z = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{A.34}$$

then it follows that

$$H = \frac{X+Z}{\sqrt{2}} = \{\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\}\cdot\hat{\boldsymbol{\sigma}}. \tag{A.35}$$

Such an expression recalls the last term in Eq. (A.31). Finally, we need to set the angle θ so that the first term in Eq. (A.31) vanishes. This is $\theta = \pi$. Then

$$H = i\hat{R}^{\mathbf{n}}(\pi), \quad \text{where} \quad \mathbf{n} = \{\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\}, \tag{A.36}$$

gives the solution.

A.10 Solution to Exercise 4.2

To prove that, given two fixed non-parallel normalised vectors $\mathbf{n}$ and $\mathbf{m}$, any unitary $\hat{U}$ can be expressed as

$$\hat{U} = e^{i\alpha}\hat{R}^{\mathbf{n}}(\beta)\hat{R}^{\mathbf{m}}(\gamma)\hat{R}^{\mathbf{n}}(\delta), \tag{A.37}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then one needs to recast $\hat{U}$ in the form

$$\hat{U} = e^{i\alpha}\hat{R}^{\mathbf{t}}(\omega), \tag{A.38}$$

with $\alpha \in \mathbb{R}$ and $\mathbf{t} \in \mathbb{R}^3$ suitably chosen.

The first step of the proof is to write $(\mathbf{m}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})$ in terms of a single Pauli matrix vector $\hat{\boldsymbol{\sigma}}$. We have

$$\begin{aligned}(\mathbf{m}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}) &= (m_1\hat{\sigma}_x + m_2\hat{\sigma}_y + m_3\hat{\sigma}_z)(n_1\hat{\sigma}_x + n_2\hat{\sigma}_y + n_3\hat{\sigma}_z), \\ &= \mathbf{m}\cdot\mathbf{n} + m_1n_2\hat{\sigma}_x\hat{\sigma}_y + m_1n_3\hat{\sigma}_x\hat{\sigma}_z + m_2n_1\hat{\sigma}_y\hat{\sigma}_x + m_2n_3\hat{\sigma}_y\hat{\sigma}_z + m_3n_1\hat{\sigma}_z\hat{\sigma}_x + m_3n_2\hat{\sigma}_z\hat{\sigma}_y.\end{aligned} \tag{A.39}$$

By applying Eq. (A.15), Eq. (A.39) becomes

$$(\mathbf{m}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}) = (\mathbf{m}\cdot\mathbf{n})\hat{\mathbb{1}} + i(\mathbf{m}\times\mathbf{n})\cdot\hat{\boldsymbol{\sigma}}. \tag{A.40}$$

The second step is to consider the composition of two rotations:

$$\begin{aligned}\hat{R}^{\mathbf{m}}(\gamma)\hat{R}^{\mathbf{n}}(\delta) &= \left(\cos(\gamma/2)\hat{\mathbb{1}} - i\sin(\gamma/2)\mathbf{m}\cdot\hat{\boldsymbol{\sigma}}\right)\left(\cos(\delta/2)\hat{\mathbb{1}} - i\sin(\delta/2)\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}\right), \\ &= \cos(\gamma/2)\cos(\delta/2)\hat{\mathbb{1}} - i\cos(\gamma/2)\sin(\delta/2)\mathbf{n}\cdot\hat{\boldsymbol{\sigma}} - i\cos(\delta/2)\sin(\gamma/2)\mathbf{m}\cdot\hat{\boldsymbol{\sigma}} \\ &\quad - \sin(\gamma/2)\sin(\delta/2)(\mathbf{m}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}).\end{aligned} \tag{A.41}$$

Substituting Eq. (A.40) in the last expression, we find that

$$\hat{R}^{\mathbf{m}}(\gamma)\hat{R}^{\mathbf{n}}(\delta) = \hat{R}^{\mathbf{h}}(\epsilon) = \cos(\epsilon/2)\hat{\mathbb{1}} - i\sin(\epsilon/2)\mathbf{h}\cdot\hat{\boldsymbol{\sigma}}, \tag{A.42}$$

where ϵ and $\mathbf{h}$ are taken such that

$$\begin{aligned}\cos(\epsilon/2) &= \cos(\gamma/2)\cos(\delta/2) - \sin(\gamma/2)\sin(\delta/2)\mathbf{m}\cdot\mathbf{n}, \\ \sin(\epsilon/2)\mathbf{h} &= \cos(\gamma/2)\sin(\delta/2)\mathbf{n} + \cos(\delta/2)\sin(\gamma/2)\mathbf{m} + \sin(\gamma/2)\sin(\delta/2)(\mathbf{m}\times\mathbf{n}).\end{aligned} \tag{A.43}$$

Then, we have

$$\hat{R}^{\mathbf{n}}(\beta)\hat{R}^{\mathbf{m}}(\gamma)\hat{R}^{\mathbf{n}}(\delta) = \hat{R}^{\mathbf{n}}(\beta)\hat{R}^{\mathbf{h}}(\epsilon) = \hat{R}^{\mathbf{t}}(\omega), \tag{A.44}$$

where we applied again the composition of two rotations, which ends the proof.

A.11 Solution to Exercise 4.3

Consider two qubits, where the first is prepared in the superposition

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \tag{A.45}$$

while the second is initialised in the ground state $|\psi_2\rangle = |0\rangle$. The total state is

$$|\psi_{12}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}. \tag{A.46}$$

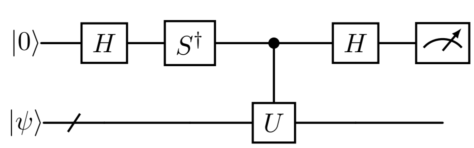
From the first expression, one clearly sees that the state is separable. By appling the CNOT gate, we find that the state becomes

$$|\psi_{12}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{A.47}$$

which is a fully entangled state.

A.12 Solution to Exercise 4.4

Consider the circuit



(A.48)

Its action is the following

$$
\begin{aligned}
|0\rangle\,|\psi\rangle &\xrightarrow{\hat{H}\otimes\hat{\mathbb{1}}} \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|\psi\rangle \\
&\xrightarrow{\hat{S}^\dagger\otimes\hat{\mathbb{1}}} \tfrac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)\,|\psi\rangle \\
&\xrightarrow{C(U)} \tfrac{1}{\sqrt{2}}(|0\rangle\,|\psi\rangle - i\,|1\rangle\,\hat{U}\,|\psi\rangle) \\
&\xrightarrow{\hat{H}\otimes\hat{\mathbb{1}}} \tfrac{1}{2}\Big[(|0\rangle + |1\rangle)\,|\psi\rangle - i(|0\rangle - |1\rangle)\hat{U}\,|\psi\rangle\Big] \\
&= \tfrac{1}{2}\Big[|0\rangle\,(\hat{\mathbb{1}} - i\hat{U})\,|\psi\rangle + |1\rangle\,(\hat{\mathbb{1}} + i\hat{U})\,|\psi\rangle\Big]\,.
\end{aligned}
\tag{A.49}
$$

Finally, one measures qubit 0, and the probability of finding the qubit in $|0\rangle$ is

$$
P(|0\rangle) = \tfrac{1}{4}\,\langle\psi|\,\Big(\hat{\mathbb{1}} + i\hat{U}^\dagger\Big)\,\Big(\hat{\mathbb{1}} - i\hat{U}\Big)\,|\psi\rangle = \tfrac{1}{2}\,\Big(1 + \Im\,\langle\psi|\hat{U}|\psi\rangle\Big)\,, \tag{A.50}
$$

which ends the exercise.