## 4.5 Quantum Phase estimation

The framework of quantum phase estimation (QPE) is the following. Consider a unitary operation $\hat{U}$ where the state $|\psi\rangle$ is one of its eigenstates. In particular, one has

$$\hat{U}|\psi\rangle = e^{2\pi i\varphi}|\psi\rangle . \quad (4.39)$$

Then, the task is to determine the phase $\varphi$ with a certain given precision.

### *4.5.1 Single-qubit quantum phase estimation*

The Hadamard test described in Sec. 4.1.1 can be used to implement a single qubit phase estimation. Indeed, from Eq. (4.39) one gets that

$$\langle\psi|\hat{U}|\psi\rangle = e^{2\pi i\varphi}. \quad (4.40)$$

Then, by merging with Eq. (4.21) one has

$$P(|0\rangle) = \tfrac{1}{2}(1+\cos(2\pi\varphi)), \quad (4.41)$$

which implies

$$\varphi = \pm\frac{\arccos\left(1-2P(|0\rangle)\right)}{2\pi} + 2\pi k, \quad (4.42)$$

where $k \in \mathbb{N}$. Notice that such a circuit cannot distinguish the sign of $\varphi$. Conversely, using both Eq. (4.21) and Eq. (4.24), one has

$$\varphi = \arctan\left(\frac{1-2P(|0\rangle)}{1-2\tilde{P}(|0\rangle)}\right). \quad (4.43)$$

Now, for the sake of simplicity, let us restrict to the case of $\varphi \in [0,1[$. Suppose we would like to estimate the value of $\varphi$ with a single run of the circuit in Eq. (4.17). Then, if the outcome is $+1$ (i.e., the state collapses on $|0\rangle$), we have $P(|0\rangle) = 1$. Conversely, with the outcome being $-1$ we have $P(|0\rangle) = 0$. Then, by employing Eq. (4.42) we obtain

| outcome | $P(\|0\rangle)$ | $\bar{\varphi}$ | $\varphi_v$ |
|---|---|---|---|
| $+1$ | 1 | 0 | $[0, 1/2[$ |
| $-1$ | 0 | $1/2$ | $[1/2, 1[$ |

(4.44)

where $\bar{\varphi}$ gives the best estimation for the real value of the phase $\varphi_v$. Since there are no other possible outcomes with a single run, the phase is estimated with an error $\epsilon = 1/2$, namely $\varphi_v \in [\bar{\varphi}, \bar{\varphi}+\epsilon[$. This is a really low accuracy for a deterministic algorithm. To improve this accuracy, one should run the algorithm several times (namely, a number of times that scales as $\mathcal{O}(1/\epsilon^2)$, where $\epsilon$ is the target error bound), or consider alternative methods, as the N-qubit quantum phase estimation described below.

### *4.5.2 Kitaev's method for single-qubit quantum phase estimation*

In the fixed point representation, a natural number $k$ can be represented with a real number $\varphi \in [0,1[$ by employing $d$ bits, i.e.

$$\varphi = (.\varphi_{d-1}\ldots\varphi_0), \quad (4.45)$$

where $\varphi_k \in \{0,1\}$, as far as $k \leq 2^d - 1$.

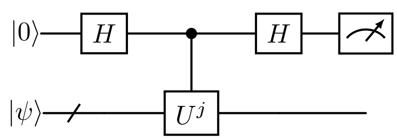
***Example 4.3***
*To make an explicit example of the fixed point representation, the value of $k = 41$ corresponds to the $d = 6$ bit's string $[101001]$ and can be represented with $\varphi = 0.640625$ being equivalent to $(.101001)$. Indeed, by employing the following expression with the string $\varphi = (.\varphi_{d-1} \dots \varphi_0) = (.101001)$ one has*

$$\sum_{i=0}^{d-1} \varphi_i 2^{i-d} = \varphi_5 2^{-1} + \varphi_4 2^{-2} + \varphi_3 2^{-3} + \varphi_2 2^{-4} + \varphi_1 2^{-5} + \varphi_0 2^{-6} = 2^{-1} + 2^{-3} + 2^{-6} = 0.640625. \quad (4.46)$$

*Such a value, when multiplied by $2^6$ gives exactly 41.*

In the simplest scenario of $d = 1$, one has $\varphi = (.\varphi_0)$ with $\varphi_0 \in \{0, 1\}$. Thus, when performing once the real Hadamard test, one has $P(|0\rangle) = 1$ if $\varphi_0 = 0$ (i.e., $\bar{\varphi} = 0$), and $P(|0\rangle) = 0$ if $\varphi_0 = 1$ (i.e., $\bar{\varphi} = 1/2$).

Next, we consider the case of $d$ bits, where $\varphi = (.0 \dots 0\varphi_0)$. Here, the first $d$ bits are 0 and the last one is $\varphi_0$. To determined the value of $\varphi_0$ one needs to reach a precision of $\epsilon < 2^{-d}$. This would require $\mathcal{O}(1/\epsilon^2) = \mathcal{O}(2^{2d})$ repeated applications of the single-qubit quantum phase estimation, or number of queries to $\hat{U}$. The observation from Kitaev's method is that if we can have access to $\hat{U}^j$ for a suitable power $j$, then the number of queries to $\hat{U}$ can be reduced. If one substitutes $\hat{U}^j$ to $\hat{U}$, with the corresponding circuit being




(4.47)

then the probability changes in

$$P(|0\rangle) = \tfrac{1}{2}(1 + \cos(2\pi j \varphi)). \quad (4.48)$$

Importantly, every time one multiplies a number by a factor 2, the bits in the fixed point representation are shifted to the left. To make an example,

$$2 \times (.00\varphi_0) = (.0\varphi_0). \quad (4.49)$$

Then, one has that $2^{d-1}\varphi = 2^{d-1}(.0 \dots 0\varphi_0) = (.\varphi_0)$. Thus, applying the circuit in Eq. (4.47) with $j = d - 1$ to estimate $(.0 \dots 0\varphi_0)$ is equivalent to apply the circuit in Eq. (4.17) to estimate $(.\varphi_0)$.

This idea can be extended to general phases with $d$ bits, i.e. $\varphi = (.\varphi_{d-1} \dots \varphi_0)$. Indeed, one has

$$\hat{U} e^{2\pi i \varphi} |\psi\rangle = \hat{U} e^{2\pi i (.\varphi_{d-1} \dots \varphi_0)} |\psi\rangle = e^{2\pi i (\varphi_{d-1} . \varphi_{d-2} \dots \varphi_0)} |\psi\rangle = e^{2\pi i \varphi_{d-1}} e^{2\pi i (.\varphi_{d-2} \dots \varphi_0)} |\psi\rangle , \quad (4.50)$$

but $e^{2\pi i \varphi_{d-1}} = 1$ independently from the value of $\varphi_{d-1}$. Thus

$$\hat{U} e^{2\pi i \varphi} |\psi\rangle = e^{2\pi i (.\varphi_{d-2} \dots \varphi_0)} |\psi\rangle , \quad (4.51)$$

i.e. the application of $\hat{U}$ shifts the bits and allows the evaluation of the first bit after the decimal point.

### *4.5.3 n-qubit quantum phase estimation*

Notably, both the previous algorithms necessitate an important classical post-processing. Employing $n$ ancillary qubits allow the reduction of such post-processing. This is based on the application of the Inverse Quantum Fourier Transform $\hat{F}^\dagger$.

***Recall 4.1 (Quantum Fourier transform)***
*The discrete Fourier transform of a N-component vector with complex components* $\{ f(0), \dots, f(N-1) \}$ *is a new complex vector* $\{ \tilde{f}(0), \dots, \tilde{f}(N-1) \}$*, defined as*

$$F(f(j), k) = \tilde{f}(k) = \tfrac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i jk/N} f(j). \tag{4.52}$$

*The Quantum Fourier transform (QFT) acts similarly: it acts as the unitary operator* $\hat{F}$ *on a quantum register of n qubits, where* $N = 2^n$*, in the computational basis as*

$$\hat{F} |j\rangle = \tfrac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} |k\rangle , \tag{4.53}$$

*where* $|j\rangle = |j_{n-1} \dots j_0\rangle$ *and* $|k\rangle = |k_{n-1} \dots k_0\rangle$*. Namely, the application of the quantum Fourier transform* $\hat{F}$ *to the state* $|j\rangle = |j_{n-1} \dots j_0\rangle$ *gives*

$$\hat{F} |j\rangle = \tfrac{1}{\sqrt{2^n}} \left( |0\rangle + e^{2\pi i (0.j_0)} |1\rangle \right) \left( |0\rangle + e^{2\pi i (0.j_1 j_0)} |1\rangle \right) \dots \left( |0\rangle + e^{2\pi i (0.j_{n-1} \dots j_0)} |1\rangle \right) . \tag{4.54}$$

*In the case of a superposition* $|\psi\rangle = \sum_j f(j) |j\rangle$*, one has*

$$|\tilde{\psi}\rangle = \hat{F} |\psi\rangle = \sum_{k=0}^{2^n-1} \tilde{f}(k) |k\rangle , \tag{4.55}$$

*where the coefficients* $\tilde{f}(k)$ *are the discrete Fourier transform of the coeficients* $f(j)$*.*
*The inverse quantum Fourier transform* $\hat{F}^\dagger$ *acts as*

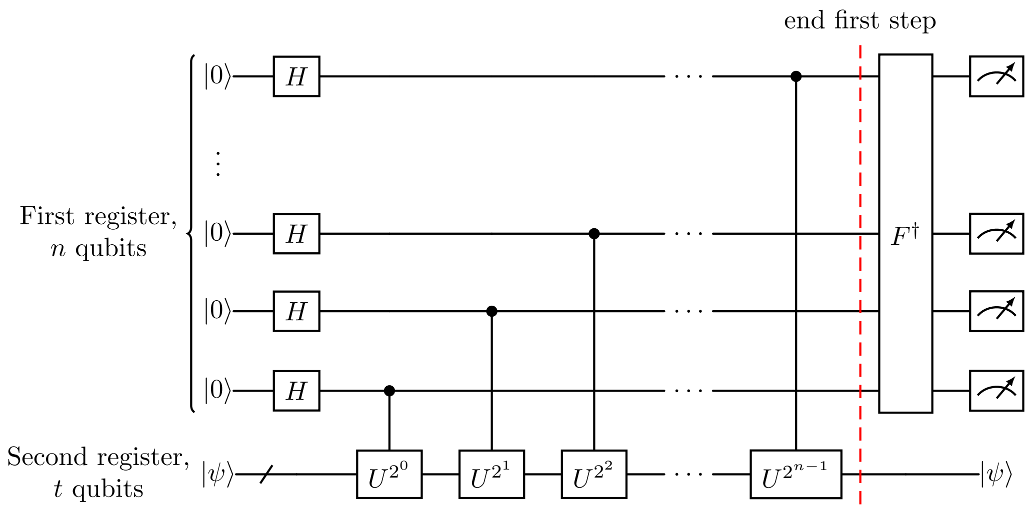
$$\hat{F}^\dagger |j\rangle = \tfrac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{-2\pi i jk/2^n} |k\rangle , \tag{4.56}$$

*in a completely similar way as Eq. (4.53) but with negative phases.*

***Example 4.4***
*The application of the quantum Fourier transform* $\hat{F}$ *to the state* $|j\rangle = |10\rangle = |j_1 = 1, j_0 = 0\rangle$ *gives*

$$\begin{aligned} \hat{F} |j\rangle &= \tfrac{1}{2} \left( |0\rangle + e^{2\pi i (0.j_0)} |1\rangle \right) \left( |0\rangle + e^{2\pi i (0.j_1 j_0)} |1\rangle \right) , \\ &= \tfrac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \tfrac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \end{aligned} \tag{4.57}$$

The algorithm implementing the (standard) quantum phase estimation uses a first register of $n$ ancillary qubits and a second register of which we want to compute the phase. The first register is initially prepared in the $|0\rangle$ state for all the qubits. The circuit implementing the algorithm is the following




(4.58)

In particular, the state of the first register after the end of the first part of the algorithm (see red dashed line) reads

$$\frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{2\pi i(2^{n-1}\varphi)}|1\rangle\right)\dots\left(|0\rangle + e^{2\pi i(2^{0}\varphi)}|1\rangle\right). \tag{4.59}$$

Now, by considering the binary representation of $\varphi = (\varphi_{n-1}\dots\varphi_0)$, the latter expression becomes

$$\frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{2\pi i(0.\varphi_0)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.\varphi_1\varphi_0)}|1\rangle\right)\dots\left(|0\rangle + e^{2\pi i(0.\varphi_{n-1}\dots\varphi_0)}|1\rangle\right), \tag{4.60}$$

which is exactly equal to $\hat{F}|j\rangle$ in Eq. (4.54) for $|j\rangle = |\varphi\rangle$. Thus, applying the inverse Fourier transform $\hat{F}^\dagger$ one gets $|\varphi\rangle$, which is then measured.