4.6 Harrow-Hassidim-Lloyd algorithm

The Harrow-Hassidim-Lloyd (HHL) algorithm allows for the resolution of linear system problems on a quantum computer. To be precise, the problem to be solved is described as finding the N_b complex entries of $\mathbf{x}$ that solve the following problem

$$A\mathbf{x} = \mathbf{b}, \tag{4.61}$$

where A is an hermitian and non-singular $N_b \times N_b$ matrix and $\mathbf{b}$ is a N_b vector, both defined on $\mathbb{C}$. Classically, the solution is given by

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{4.62}$$

The question is then how one can implement this on a quantum computer.

First, let us assume that the entries of $\mathbf{b}$ are such that $||\mathbf{b}|| = 1$. Then, $\mathbf{b}$ can be stored in a n_b-qubit state $|b\rangle$, through the following mapping:

$$\mathbf{b} = \begin{pmatrix} b_0 \\ \vdots \\ b_{N_b-1} \end{pmatrix} \leftrightarrow b_0 |0\rangle + \cdots + b_{N_b-1} |N_b - 1\rangle = |b\rangle , \tag{4.63}$$

where $N_b = 2^{n_b}$. For example, this can be done via a unitary operation $\hat{U}_{\mathbf{b}}$. Now, we define $|x\rangle = \hat{A}^{-1} |b\rangle$, where $\hat{A}$ in the computational representation gives the classical matrix A. Notably, the state $|x\rangle$ needs to be normalised to be stored in a quantum register. Thus, one has

$$|x\rangle = \frac{\hat{A}^{-1} |b\rangle}{||\hat{A}^{-1} |b\rangle ||}, \tag{4.64}$$

where the normalisation problem can be tackled in a second moment.

Consider the spectral decomposition of $\hat{A}$:

$$\hat{A} |v_j\rangle = \lambda_j |v_j\rangle , \tag{4.65}$$

where λ_j and $|v_j\rangle$ are respectively the eigeinvalues and eigeinstates of $\hat{A}$. We also assume that the ordering of the eigeinvalues is such that

$$0 < \lambda_0 \leq \cdots \leq \lambda_{N_b-1} < 1. \tag{4.66}$$

In general this will not be the case, but one can remap the problem in order to fall within this case. We also assume that all the N_b eigeinvalues have an exact d-bit representation.

By applying what in Sec. 4.5, we can query $\hat{A}$ via an unitary operation $\hat{U} = e^{2\pi i \hat{A}}$ using QPE. For example, suppose $|b\rangle = |v_j\rangle$, then we have

$$\hat{U}_{\text{QPE}} |0\rangle^{\otimes d} |v_j\rangle = |\lambda_j\rangle |v_j\rangle . \tag{4.67}$$

In particular, the (not-normalised) solution of the linear system problem would be

$$\hat{A}^{-1} |b\rangle = \hat{A}^{-1} |v_j\rangle = \tfrac{1}{\lambda_j} |v_j\rangle . \tag{4.68}$$

More generally, one can decompose the state $|b\rangle$ on the basis of $\hat{A}$, i.e.

$$|b\rangle = \sum_{j=0}^{2^{n_b}-1} \beta_j |v_j\rangle , \tag{4.69}$$

where β_j are a linear combination of b_j. Then the QPE procedure gives

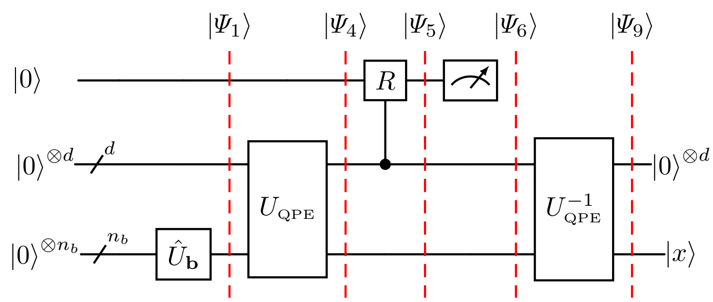
$$\hat{U}_{\text{QPE}} |0\rangle^{\otimes d} |b\rangle = \sum_j \beta_j |\lambda_j\rangle |v_j\rangle , \tag{4.70}$$

and the solution of the problem is given by

$$\hat{A}^{-1} |b\rangle = \sum_{j=0}^{2^{n_b}-1} \frac{\beta_j}{\lambda_j} |v_j\rangle . \tag{4.71}$$

The aim of the HHL algorithm is to generate the normalised version of the state in Eq. (4.71) from the general state $|b\rangle$ as shown in Eq. (4.69).

The algorithm works with three registers. The first one is an ancillary register made of a single qubit, the second is also an ancillary register but made of d qubits, the third register is made of n_b qubits and will encode the solution of the problem. The HHL circuit is the following



(4.72)

The algorithm works as the following. Initially, all the qubits are prepared in $|0\rangle$:

$$|\Psi_0\rangle = |0\rangle |0\rangle^{\otimes d} |0\rangle^{\otimes n_b} , \tag{4.73}$$

then the information about $\mathbf{b}$ is encoded in the last register:

$$|\Psi_1\rangle = \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}^{\otimes d} \otimes \hat{U}_{\mathbf{b}} |\Psi_0\rangle = |0\rangle |0\rangle^{\otimes d} |b\rangle . \tag{4.74}$$

We apply the QPE procedure, which is here broke down in the corresponding three steps. The first is the application of the Hadamard gate:

$$|\Psi_2\rangle = \hat{\mathbb{1}} \otimes \hat{H}^{\otimes d} \otimes \hat{\mathbb{1}} |\Psi_1\rangle = |0\rangle \tfrac{1}{2^{d/2}} (|0\rangle + |1\rangle)^{\otimes d} |b\rangle . \tag{4.75}$$

This is followed by the controlled unitary $\hat{U}^j$:

$$|\Psi_3\rangle = \hat{\mathbb{1}} \otimes C(U^j) |\Psi_2\rangle = |0\rangle \tfrac{1}{2^{d/2}} \sum_{k=0}^{2^d-1} e^{2\pi i k \varphi} |k\rangle |b\rangle , \tag{4.76}$$

where $\hat{U} |b\rangle = e^{2\pi i \varphi} |b\rangle$ with $\varphi \in [0, 1[$. Finally, we apply the inverse Fourier transform to the second register

$$\begin{aligned} |\Psi_4\rangle &= \hat{\mathbb{1}} \otimes \hat{F}^\dagger \otimes \hat{\mathbb{1}}^{\otimes n_b} |\Psi_3\rangle , \\ &= |0\rangle \tfrac{1}{2^{d/2}} \sum_{k=0}^{2^d-1} e^{2\pi i k \varphi} \hat{F}^\dagger |k\rangle |b\rangle , \\ &= |0\rangle \tfrac{1}{2^d} \sum_{k=0}^{2^d-1} e^{2\pi i k \varphi} \sum_{y=0}^{2^d-1} e^{-2\pi i y k / 2^d} |y\rangle |b\rangle . \end{aligned} \tag{4.77}$$

However, one has that

$$\sum_{k=0}^{2^d-1} e^{2\pi i k(\varphi - y/2^d)} = \begin{cases} \sum_{k=0}^{2^d-1} e^0 = 2^d, & \text{if } \varphi = y/2^d, \\ 0, & \text{if } \varphi \neq y/2^d, \end{cases} \tag{4.78}$$

meaning that the k sum selects the value of $y = \varphi 2^d$. Thus,

$$|\Psi_4\rangle = |0\rangle |\varphi 2^d\rangle |b\rangle . \tag{4.79}$$

In general, $|b\rangle$ is in a superposition of $|v_j\rangle$, then

$$\hat{U} |v_j\rangle = e^{2\pi i \hat{A}} |v_j\rangle = e^{2\pi i \lambda_j} |v_j\rangle . \tag{4.80}$$

Then, the entire QPE gate maps

$$|\Psi_1\rangle = |0\rangle |0\rangle^{\otimes d} \sum_{j=0}^{2^{n_b}-1} \beta_j |v_j\rangle \xrightarrow{U_{\mathrm{QPE}}} |\Psi_4\rangle = |0\rangle \sum_{j=0}^{2^{n_b}-1} \beta_j |\lambda_j 2^d\rangle |v_j\rangle . \tag{4.81}$$

We apply a controlled rotation on the first register, such that

$$|\Psi_5\rangle = C(R) \otimes \hat{\mathbb{1}}^{\otimes n_b} |\Psi_4\rangle = \sum_{j=0}^{2^{n_b}-1} \beta_j \left(\sqrt{1 - \tfrac{C^2}{\lambda_j^2}} |0\rangle + \tfrac{C}{\lambda_j} |1\rangle \right) |\lambda_j 2^d\rangle |v_j\rangle , \tag{4.82}$$

where $C \in \mathbb{R}$ is an arbitrary constant. At this point we perform the measurement of the first register. If the outcome is $+1$ and the state collapses in $|0\rangle$ then we discard the run; if the outcome is -1 with the state collapsed in $|1\rangle$ then we retain the run. To increase the probabilities of having the outcome -1, we make C as large as possible. After the collapse of the first register in $|1\rangle$, the state of the second and third register is

$$|\Psi_6\rangle = \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2 \right)^{1/2}} \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} |\lambda_j 2^d\rangle |v_j\rangle , \tag{4.83}$$

where we exploited that $C \in \mathbb{R}$. Now, we apply the inverse QPE, which has also three steps. The first is the application of the QFT:

$$\begin{aligned} |\Psi_7\rangle &= \hat{F} \otimes \hat{\mathbb{1}}^{\otimes n_b} |\Psi_6\rangle , \\ &= \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2 \right)^{1/2}} \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} \hat{F} |\lambda_j 2^d\rangle |v_j\rangle , \\ &= \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2 \right)^{1/2}} \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} \tfrac{1}{2^{d/2}} \sum_{y=0}^{2^d-1} e^{2\pi i y(\lambda_j 2^d)/2^d} |y\rangle |v_j\rangle . \end{aligned} \tag{4.84}$$

Then, we apply the controlled unitary $C(U^{-j})$, which gives

$$\begin{aligned} |\Psi_8\rangle &= C(U^{-j}) |\Psi_7\rangle , \\ &= \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2 \right)^{1/2}} \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} \tfrac{1}{2^{d/2}} \sum_{y=0}^{2^d-1} e^{2\pi i y \lambda_j} |y\rangle e^{-2\pi i \lambda_j y} |v_j\rangle , \end{aligned} \tag{4.85}$$

where the two phases cancel and thus

$$|\Psi_8\rangle = \tfrac{1}{2^{d/2}} \sum_{y=0}^{2^d-1} |y\rangle \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2\right)^{1/2}} |v_j\rangle \,. \tag{4.86}$$

Finally, the application of Hadamard's gates on the second register gives

$$|\Psi_9\rangle = \hat{H}^{\otimes d} \otimes \hat{\mathbb{1}}^{\otimes n_b} |\Psi_8\rangle = |0\rangle^{\otimes d} \sum_{j=0}^{2^{n_b}-1} \tfrac{\beta_j}{\lambda_j} \frac{1}{\left(\sum_{j=0}^{2^{n_b}-1} |\beta_j/\lambda_j|^2\right)^{1/2}} |v_j\rangle \,, \tag{4.87}$$

where the third register is exactly in the form in Eq. (4.71) after the proper normalisation. Thus,

$$|\Psi_9\rangle = |0\rangle^{\otimes d} |x\rangle \,, \tag{4.88}$$

embeds the solution of the linear system $A\mathbf{x} = \mathbf{b}$.