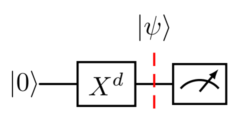
Chapter 6
Noisy Intermediate-Scale Quantum (NISQ) computation

For quantum algorithms to function properly, we need to ensure that the basic units of quantum information, the qubits, are as reliable as the bits in classical computers. These qubits must be shielded from environmental noise that can disrupt their states, while still have to be controlled by external agents. This control involves making the qubits entangle and eventually measuring their states to extract the outputs of the quantum computation. Technically, it is feasible to minimise the impact of noise without compromising the quantum information process through the development of Quantum Error Correction (QEC) and Quantum Error Mitigation (QEM) protocols, see Chap. 7.

Many quantum algorithms that come with guaranteed performance need millions of physical qubits to effectively use QEC methods. It could take decades to build fault-tolerant quantum computers capable of reaching this scale. Presently, quantum devices typically have around 100-1000 physical qubits, often referred to as noisy intermediate-scale quantum (NISQ) devices. These devices lack error correction and are imperfect, yet in the NISQ era, the aim is to maximize the quantum computational capabilities of current devices, while working on techniques for fault-tolerant quantum computation.

Here, we study how noises impact quantum circuits. The following circuit will be considered as a basis of the study:



(6.1)

where the gate X is repeated d times. The value of d is also called the depth of the circuit. The state before the measurement, when no noise is considered, is given by

$$|\psi\rangle = (i\hat{R}_x(\theta = \pi))^d |0\rangle = i^d \left[\cos(d\pi/2) |0\rangle - \sin(d\pi/2) |1\rangle\right], \tag{6.2}$$

indeed one has that the X gate can be realised as a rotation of an angle π around the x axis: $\hat{\sigma}_x = i\hat{R}_x(\pi)$. Notably, the factor i^d is just a negligible global phase. The expectation value of the polarisation is

$$\langle Z \rangle = \langle \psi | \hat{\sigma}_z | \psi \rangle = \cos^2(d\pi/2) - \sin^2(d\pi/2) = \cos(d\pi), \tag{6.3}$$

which is shown in the left panel of Fig. 6.1. Clearly, the value of $\langle Z \rangle$ jumps from $+1$ to -1 depending on the value of d. However, when we perform such a simple experiment the result is quite different due to the noises and errors acting on the system. This is represented in the right panel Fig. 6.1. Such a result account for the presence of different noises and errors. These are listed and studied below.

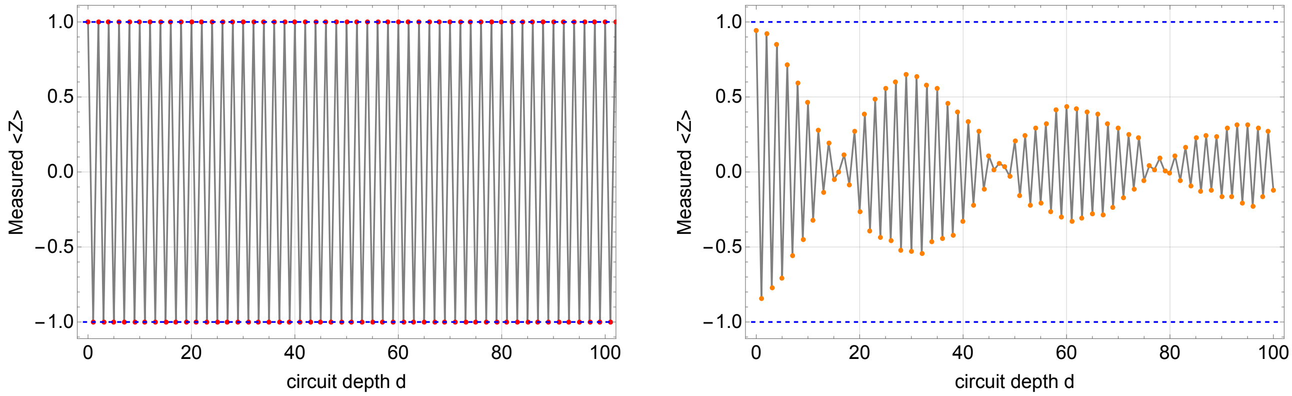
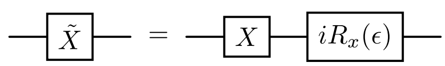


Fig. 6.1: Expectation value of the polarisation $\langle Z \rangle$ for the circuit in Eq. (6.1) with respect to the depth d of the circuit: (left panel) in the case of no noises; (right panel) when noises are accounted.

6.1 Miscalibrated gates

As we saw previously, the gate X can be performed as a rotation of an angle π around the x axis. Now, let us suppose that the gate is systematic miscalibrated. Specifically, one performs a rotation of an angle of $\pi + \epsilon$ in place of only π. Then, we have that the actual gate $\tilde{X}$ is given by

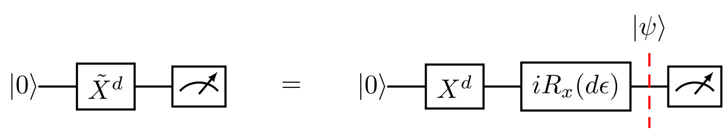


(6.4)

indeed one has that

$$R_x(\pi + \epsilon) = R_x(\pi) R_x(\epsilon). \tag{6.5}$$

Then, when running the circuit with d repetitions, one simply has



(6.6)

where

$$|\psi\rangle = i^d \left[\cos\left(d\frac{(\pi+\epsilon)}{2} \right) |0\rangle - i \sin\left(d\frac{(\pi+\epsilon)}{2} \right) |1\rangle \right]. \tag{6.7}$$

Correspondingly, one has that the expectation value for the polarisation is

$$\langle Z \rangle = \cos^2\left(d\frac{(\pi+\epsilon)}{2} \right) - \sin^2\left(d\frac{(\pi+\epsilon)}{2} \right) = \cos(d(\pi+\epsilon)), \tag{6.8}$$

which is shown in the left panel of Fig. 6.2. For small values of ϵ and d, one performs an error of

$$| \langle Z \rangle_{\text{noiseless}} - \langle Z \rangle_{\text{miscalibrated}} | \sim \tfrac{1}{2} d^2 \epsilon^2, \tag{6.9}$$

which scales quadratically with the miscalibration ϵ. As it is shown in the right panel of Fig. 6.2, the difference with respect to the noiseless result can be substantial. Indeed, for values of d such that $d\epsilon \sim (2n+1)\pi$, with $n \in \mathbb{N}$, we have that

$$\langle Z \rangle_{\text{miscalibrated}} = - \langle Z \rangle_{\text{noiseless}} . \tag{6.10}$$

This means that the error completely inverts the output signal.

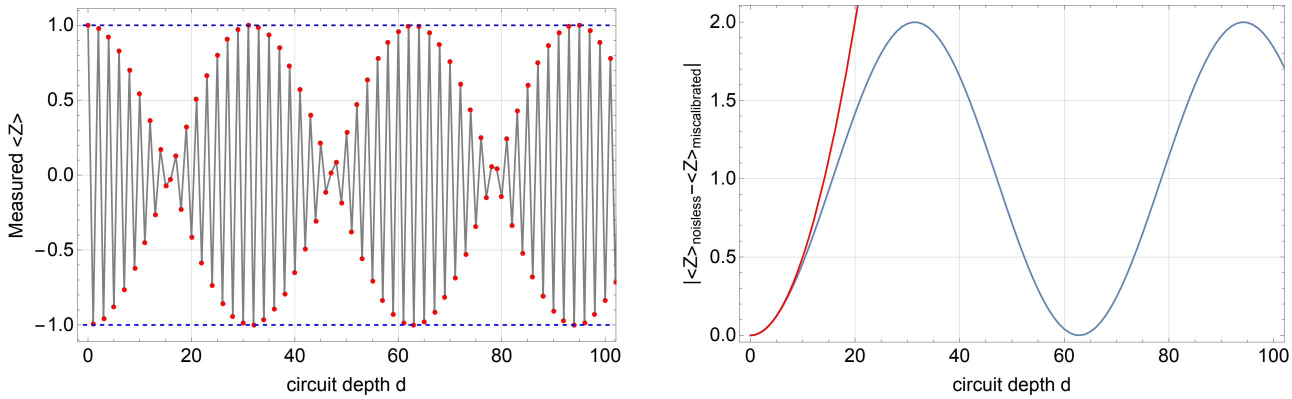
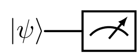


Fig. 6.2: (Left panel) Expectation value (red dots) of the polarisation $\langle Z \rangle$ for the circuit in Eq. (6.1) with respect to the depth d of the circuit when miscalibrated gates are considered. (Right panel) Difference with respect to the noiseless case (blue line) and the small ϵ and d expansion (red line). Here we considered $\epsilon = 0.1$.

6.2 Projection noise and sampling error

Consider the following trivial circuit



(6.11)

whose possible values of the polarisation are $z = +1$, meaning that the state after the collapse is $|0\rangle$, and $z = -1$, with corresponding state $|1\rangle$.

Let us define a measurement operator $\hat{M}$ that indicates if the state of the qubit is in the $|1\rangle$ state. Such an operator can be constructed as

$$\begin{aligned} \hat{M} &= |1\rangle \langle 1| , \\ &= 1\,|1\rangle \langle 1| + 0\,|0\rangle \langle 0| , \\ &= \sum_{m=0,1} m \hat{\Pi}_m , \end{aligned} \tag{6.12}$$

where m are the eigeinvalues of $\hat{M}$ and $\hat{\Pi}_m = |m\rangle \langle m|$, with $\sum_m \hat{\Pi}_m = \hat{\mathbb{1}}$. Namely, the expectation value of $\hat{M}$ indicates the probability $\mathbb{p}(m = 1)$ that the state $|\psi\rangle$ is equal to $|1\rangle$:

$$\langle \psi | \hat{M} | \psi \rangle = \langle \psi | 1 \rangle \langle 1 | \psi \rangle = | \langle 1 | \psi \rangle |^2 = \mathbb{p}(m = 1). \tag{6.13}$$

Correspondingly, one has $\mathbb{p}(m = 0) = 1 - \mathbb{p}(m = 1)$. To ease the notation, in the following we use $\mathbb{p} = \mathbb{p}(m = 1)$. The outcomes of the operator $\hat{M}$ are distributed via a binomial distribution $\mathcal{B}(\mathbb{p})$. To be more explicit, let us consider the repetition of the above circuit N times, meaning that we have N samplings of the protocol. Then, we can construct a series of outputs: $\{ m_n \}_{n=1}^N = \{ 0, 0, 1, 0, 1, 1, 0, \dots \}$, where each entry is a random variable. This corresponds in a series of states into which the system has collapsed after the measurement: $\{ \,|\,0\rangle, \,|\,0\rangle, \,|\,1\rangle, \,|\,0\rangle, \,|\,1\rangle, \,|\,1\rangle, \,|\,0\rangle, \dots \}$. Suppose we have $N = 3$, then we have a total of $2^N = 2^3$ outputs (m_1, m_2, m_3). Namely

(m_1, m_2, m_3)	$(\|m_1\rangle, \|m_2\rangle, \|m_3\rangle)$	S
$(0,0,0)$	$(\|0\rangle, \|0\rangle, \|0\rangle)$	0
$(0,0,1)$	$(\|0\rangle, \|0\rangle, \|1\rangle)$	1/3
$(0,1,0)$	$(\|0\rangle, \|1\rangle, \|0\rangle)$	1/3
$(0,1,1)$	$(\|0\rangle, \|1\rangle, \|1\rangle)$	2/3
$(1,0,0)$	$(\|1\rangle, \|0\rangle, \|0\rangle)$	1/3
$(1,0,1)$	$(\|1\rangle, \|0\rangle, \|1\rangle)$	2/3
$(1,1,0)$	$(\|1\rangle, \|1\rangle, \|0\rangle)$	2/3
$(1,1,1)$	$(\|1\rangle, \|1\rangle, \|1\rangle)$	1

(6.14)

where we also computed the corresponding sampling mean S, which is defined as

$$S = \frac{1}{N} \sum_{n=1}^{N} m_n. \tag{6.15}$$

The statement then is that the sampling mean S, which is also a random variable, is distributed as a binomial distribution $\mathcal{B}(\mathbb{p})$, with mean $\mathbb{E}[S] = \mathbb{p}$ and variance $\mathbb{V}[S] = \mathbb{p}(1-\mathbb{p})/N$. Fig. 6.3 shows some examples. Now,

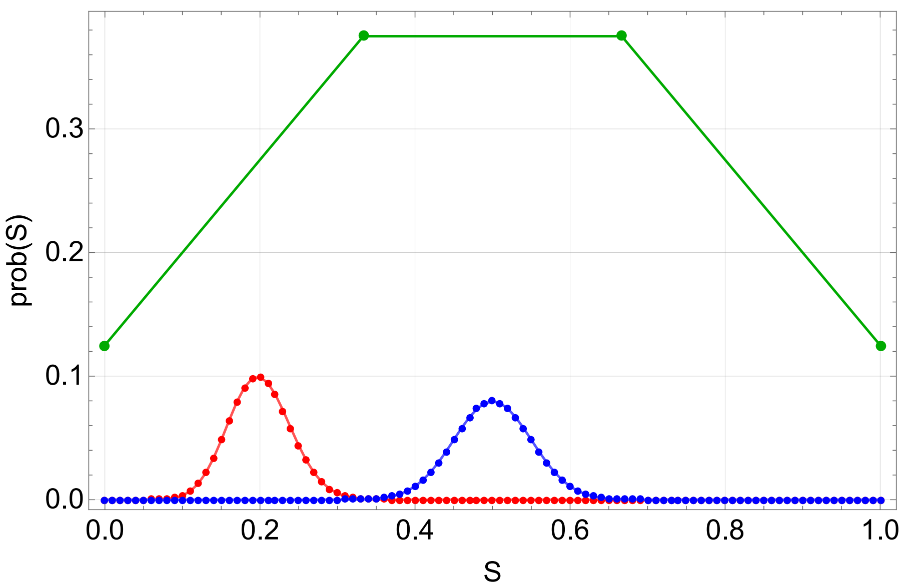


Fig. 6.3: Binomial distribution followed by the sampling mean S for $N = 100$ samplings with $\mathbb{p} = 0.2$ (red plot) and $\mathbb{p} = 0.5$ (blue plot), and for $N = 3$ with $\mathbb{p} = 0.5$ (green plot).

by assuming that the state $|\psi\rangle$ in Eq. (6.11) is that one has before the measurement in Eq. (6.6), i.e.

$$|\psi\rangle = i^d \left[\cos\left(d\tfrac{(\pi+\epsilon)}{2}\right)|0\rangle - i\sin\left(d\tfrac{(\pi+\epsilon)}{2}\right)|1\rangle\right], \tag{6.16}$$

then the corresponding distribution $\mathcal{B}(\mathbb{p}_d)$ depends on the probability of being in the state $|1\rangle$ after a depth d of the circuit. This is given by

$$\mathbb{p}_d = \sin^2\left(d\tfrac{(\pi+\epsilon)}{2}\right). \tag{6.17}$$

The corresponding expectation value of the polarisation is shown in Fig. 6.4, where we also report its difference with respect to the case shown in Fig. 6.2.

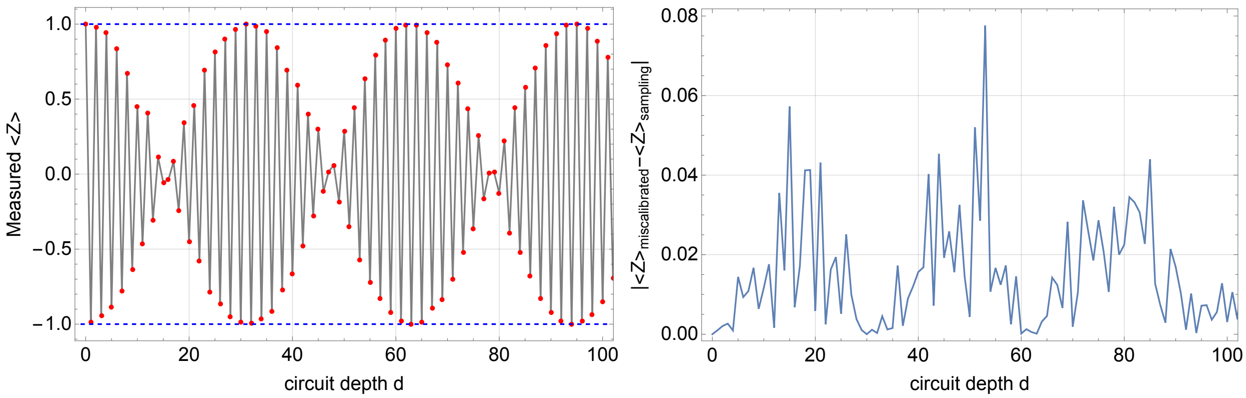
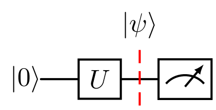


Fig. 6.4: (Left panel) Expectation value (red dots) of the polarisation $\langle Z\rangle$ for the circuit in Eq. (6.1) with respect to the depth d of the circuit when miscalibrated gates and error sampling are considered. (Right panel) Difference with respect to the miscalibrated case. Here we considered $\epsilon = 0.1$ and $N = 10$.

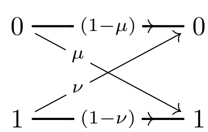
6.3 Measurement error

Another source of errors is related to the performance of the measurement apparatus. To be explicit, consider the following trivial circuit



(6.18)

Suppose that is the state $|\psi\rangle = |0\rangle$ to be fed to the measurement apparatus. Then, if the apparatus is perfectly set, it will give as the polarisation value $z = +1$ corresponding to the output $m = 0$. However, there might be errors and there could be a non-null probability μ that the measurement apparatus gives as an output $m = 1$. Similarly, if $|\psi\rangle = |1\rangle$, one could have a probability ν to have $m = 0$. The following scheme represents the supposed outcome and the actual outcome with the corresponding probabilities:

supposed $m =$  $=$ actual $\tilde{m}$. (6.19)

We can define the probability vectors for the supposed and actual outcomes, which respectively are

$$\mathbb{P}_M = \begin{pmatrix} 1 - \mathbb{p} \\ \mathbb{p} \end{pmatrix} \quad \text{and} \quad \mathbb{P}_{\tilde{M}} = \begin{pmatrix} 1 - \tilde{\mathbb{p}} \\ \tilde{\mathbb{p}} \end{pmatrix}, \tag{6.20}$$

where $\mathbb{p} = \mathbb{p}(m = 1)$ and $\tilde{\mathbb{p}} = \tilde{\mathbb{p}}(\tilde{m} = 1)$ are the probability of having the outcome $m = 1$ in the supposed and actual case respectively. Such vectors are related by the matrix A, which quantifies the performances of measurement apparatus: $\mathbb{P}_{\tilde{M}} = A\mathbb{P}_M$, where

$$A = \begin{pmatrix} P(\tilde{m} = 0|m = 0) & P(\tilde{m} = 0|m = 1) \\ P(\tilde{m} = 1|m = 0) & P(\tilde{m} = 1|m = 1) \end{pmatrix}, \tag{6.21}$$

where $P(\tilde{m}|m)$ is the conditional probability of having the actual outcome $\tilde{m}$ given the supposed outcome m. The diagram in Eq. (6.19) sets the entries of A to

$$A = \begin{pmatrix} 1-\mu & \nu \\ \mu & 1-\nu \end{pmatrix}. \tag{6.22}$$

By merging the latter and Eq. (6.20), we find

$$\tilde{\mathbb{p}} = \mathbb{p} + \mu - (\nu + \mu)\mathbb{p}, \tag{6.23}$$

which is represented in Fig.6.5. To recalibrate the measurement apparatus, i.e. to quantify experimentally the

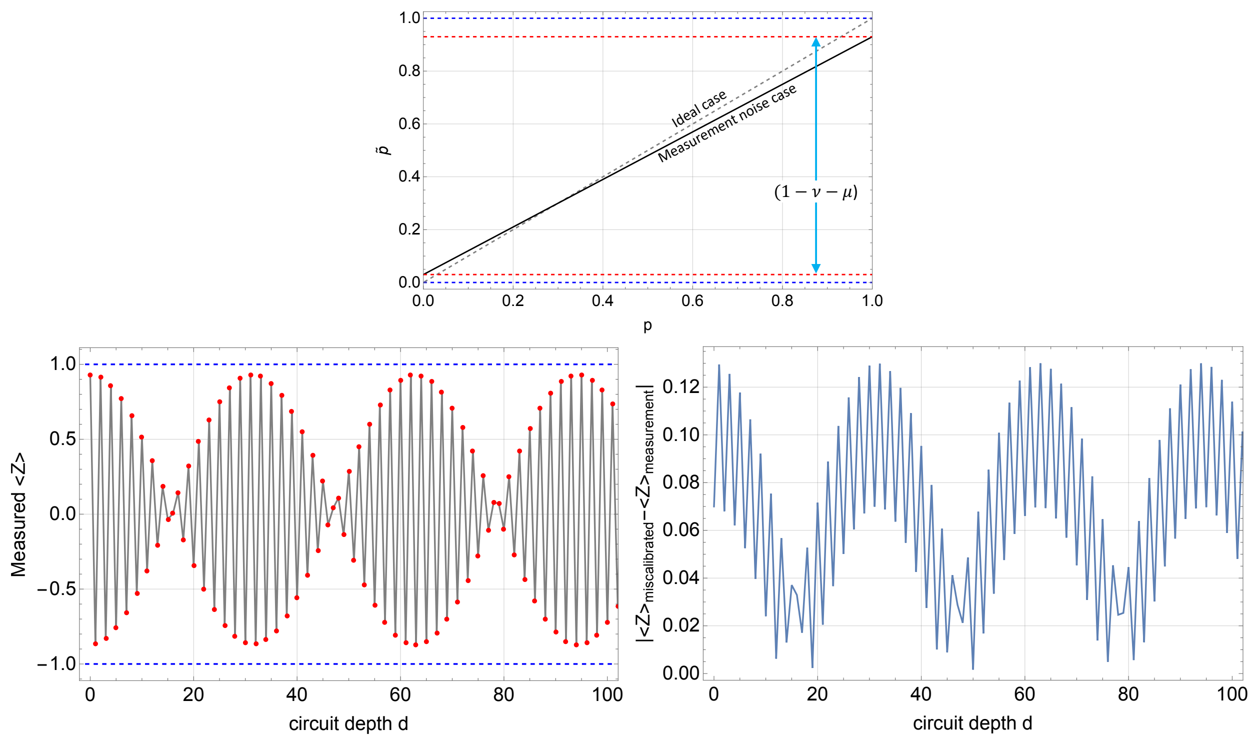
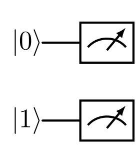


Fig. 6.5: (Top panel) Representation of Eq. (6.23). (Bottom left panel) Expectation value (red dots) of the polarisation $\langle Z \rangle$ for the circuit in Eq. (6.1) with respect to the depth d of the circuit when miscalibrated gates and measurement errors are considered. (Bottom right panel) Difference with respect to the miscalibrated case. Here we considered $\epsilon = 0.1$, $\nu = 0.07$ and $\mu = 0.03$.

values of μ and ν one would need to run the following two trivial circuits:

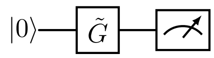


(6.24)

They should give 100% of the time the outcomes $m = 0$ and $m = 1$ respectively. Variations of such percentage will characterise the values of μ and ν. The bottom panels in Fig. 6.5 show the corresponding polarisation and difference with the miscalibrated case.

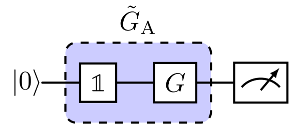
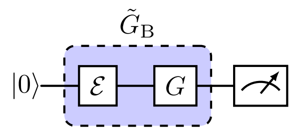
6.3.1 Environmental noise

We now dwell in the most interesting source of noise, which is the one due to the external environment. As already saw in Chapter 2 and Chapter 3, an external environment can modify the unitary evolution of the system. Thus, in a circuit-based representation of the evolution, one now has



(6.25)

where $\tilde{G}$ represents the noisy version of the gate G. There are different possible ways how one can account for the noise action. Here, we will consider the following way. Every gate G of the noiseless case is substituted by $\tilde{G}$, where an extra gate, say $\mathcal{E}$, is added with a probability $\mathbb{p}_{\mathrm{E}}$. Namely,

Case	Probability	Effective circuit
A	$1-\mathbb{p}_{\mathrm{E}}$	
B	$\mathbb{p}_{\mathrm{E}}$	

(6.26)

For the sake of simplicity, we will consider the case of $\mathcal{E}=X$ and $G=X$. With this choice, in the case A, one has $\tilde{G}=\mathbb{1}X=X$, and the gate is properly implemented. While the case B, one has $\tilde{G}=XX=\mathbb{1}$, which nullify the action of the original gate. Now, in the ideal case ($\mathcal{E}=\mathbb{1}$ always), the state before the measurement is $|\psi\rangle=|1\rangle$, so the outcome is

$$\langle m\rangle=\operatorname{Tr}\left[\hat{M}\hat{\rho}\right]=1, \tag{6.27}$$

indeed, the probability of having $m=1$ is $\mathbb{p}(m=1)=1$. In the case with the environmental noise, one has

$$\begin{aligned}\langle m\rangle&=\sum_{m=0,1}m\,\mathbb{p}(m),\\&=(1-\mathbb{p}_{\mathrm{E}})+\mathbb{p}_{\mathrm{E}}\times 0,\\&=(1-\mathbb{p}_{\mathrm{E}}).\end{aligned} \tag{6.28}$$

In particular, this result can be constructed as follows. Starting from the initial state $|0\rangle$, we construct the corresponding initial statistical operator $\hat{\rho}_0$. Then, with a probability $\mathbb{p}_{\mathrm{A}}=(1-\mathbb{p}_{\mathrm{E}})$ evolves as in the case A, and with a probability $\mathbb{p}_{\mathrm{B}}=\mathbb{p}_{\mathrm{E}}$ as in the case B. Namely, one has

$$\hat{\rho}=\mathbb{p}_{\mathrm{A}}\hat{\rho}_{\mathrm{A}}+\mathbb{p}_{\mathrm{B}}\hat{\rho}_{\mathrm{B}}, \tag{6.29}$$

where

$$\hat{\rho}_{\mathrm{B}}=\hat{\sigma}_x\hat{\rho}_0\hat{\sigma}_x,\quad\text{and}\quad\hat{\rho}_{\mathrm{B}}=\hat{\mathbb{1}}\hat{\rho}_0\hat{\mathbb{1}}. \tag{6.30}$$

This gives

$$\hat{\rho}=(1-\mathbb{p}_{\mathrm{E}})\,|1\rangle\langle 1|+\mathbb{p}_{\mathrm{E}}\,|0\rangle\langle 0|\,, \tag{6.31}$$

which, in the computational basis, is represented as

$$\rho=\begin{pmatrix}\mathbb{p}_{\mathrm{E}}&0\\0&(1-\mathbb{p}_{\mathrm{E}})\end{pmatrix}. \tag{6.32}$$

The corresponding expectation value of the measurement operator $\hat{M}$ is

$$\begin{aligned}\langle m\rangle &= \mathrm{Tr}\left[\hat{M}\hat{\rho}\right],\\ &= \mathrm{Tr}\left[\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbb{p}_{\mathrm{E}} & 0\\ 0 & (1-\mathbb{p}_{\mathrm{E}})\end{pmatrix}\right],\\ &= \mathrm{Tr}\left[\begin{pmatrix}0 & 0\\ 0 & (1-\mathbb{p}_{\mathrm{E}})\end{pmatrix}\right],\\ &= (1-\mathbb{p}_{\mathrm{E}}),\end{aligned} \tag{6.33}$$

as expected. Conversely, the polarisation is

$$\langle Z\rangle = (2\mathbb{p}_{\mathrm{E}} - 1). \tag{6.34}$$

Applying the noisy gate d times, we find

$$\langle Z\rangle = (2\mathbb{p}_{\mathrm{E}} - 1)^d, \tag{6.35}$$

which is represented in Fig. 6.6.

> ***Exercise 6.1***
> *Verify the relation in Eq. (6.35).*

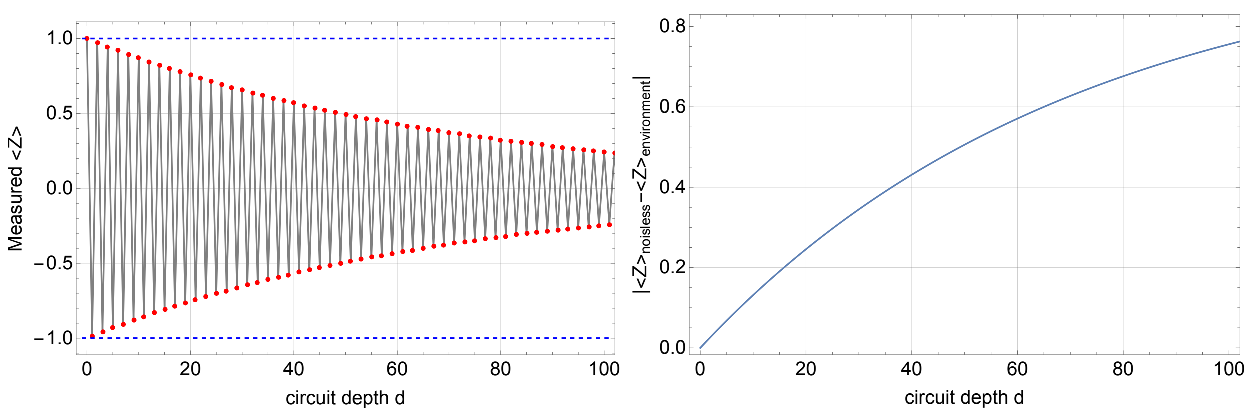
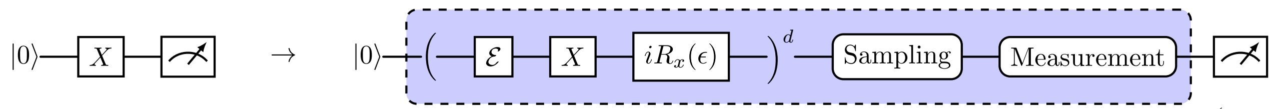


Fig. 6.6: (Left panel) Expectation value (red dots) of the polarisation $\langle Z\rangle$ for the circuit in Eq. (6.1) with respect to the depth d of the circuit when the environment noise considered. (Right panel) Difference with respect to the noiseless case. Here we considered $\mathbb{p}_{\mathrm{E}} = 0.007$.

6.3.2 Global noise action

The last step of this section is to put together the different noises and errors discussed here. Namely, when one has a circuit, to account for the error, such a circuit needs to be substituted as represented



(6.36)

The action of the environment noise and miscalibration lead the qubit in the state

$$\hat{\rho} = (1 - \mathbb{p}_{\mathrm{E}}) \left|\psi_{\mathrm{A}}\right\rangle \left\langle\psi_{\mathrm{A}}\right| + \mathbb{p}_{\mathrm{E}} \left|\psi_{\mathrm{B}}\right\rangle \left\langle\psi_{\mathrm{B}}\right|, \tag{6.37}$$

where

$$\begin{aligned} \left|\psi_{\mathrm{A}}\right\rangle &= \cos\left(d\tfrac{(\pi+\epsilon)}{2}\right)\left|0\right\rangle - i\sin\left(d\tfrac{(\pi+\epsilon)}{2}\right)\left|1\right\rangle, \\ \left|\psi_{\mathrm{B}}\right\rangle &= \cos\left(d\tfrac{\epsilon}{2}\right)\left|0\right\rangle - i\sin\left(d\tfrac{\epsilon}{2}\right)\left|1\right\rangle. \end{aligned} \tag{6.38}$$

Specifically, such a state can be rewritten as

$$\hat{\rho} = \left(\rho_{00}\left|0\right\rangle\left\langle 0\right| + \rho_{11}\left|1\right\rangle\left\langle 1\right| + \rho_{10}\left|1\right\rangle\left\langle 0\right| + \rho_{01}\left|0\right\rangle\left\langle 1\right|\right), \tag{6.39}$$

where the ρ_{00} and ρ_{11} populations are fundamental in computing the expectation value of the polarisation:

$$\begin{aligned} \langle Z\rangle &= \rho_{00} - \rho_{11}, \\ &= (1 - \mathbb{p}_{\mathrm{E}})\cos\left(d(\pi+\epsilon)\right) + \mathbb{p}_{\mathrm{E}}\cos\left(d\epsilon\right). \end{aligned} \tag{6.40}$$

On the other hand, to compute the effect of the sampling error, one needs only ρ_{11}, which is equal to the probability of having $m = 1$, i.e. of being in the state $|1\rangle$. Thus, Eq. (6.17) becomes

$$\mathbb{p}_d = (1 - \mathbb{p}_{\mathrm{E}})\frac{(1 - \cos(d(\pi+\epsilon)))}{2} + \mathbb{p}_{\mathrm{E}}\frac{(1 - \cos(d\epsilon))}{2}, \tag{6.41}$$

with respect to which one constructs the statistics of the outcomes that are distributed following a binomial distribution $\mathcal{B}(\mathbb{p}_d)$.

Finally, the measurement error can be accounted by mapping the sampling means S_d in those accounting for diagram in Eq. (6.19). This consists in

$$S_d \to S_d + \mu - (\nu + \mu)S_d. \tag{6.42}$$

The result of considering all the noises and errors is given by the right panel of Fig. 6.1.