Chapter 7
Quantum Error Correction and Mitigation

7.1 Quantum Error Correction

Several techniques have been developed to correct errors due to the presence of noise. This is not only a considered issue in the quantum information context, but also in the classical one. In both cases, the key ingredient is the redundancy.

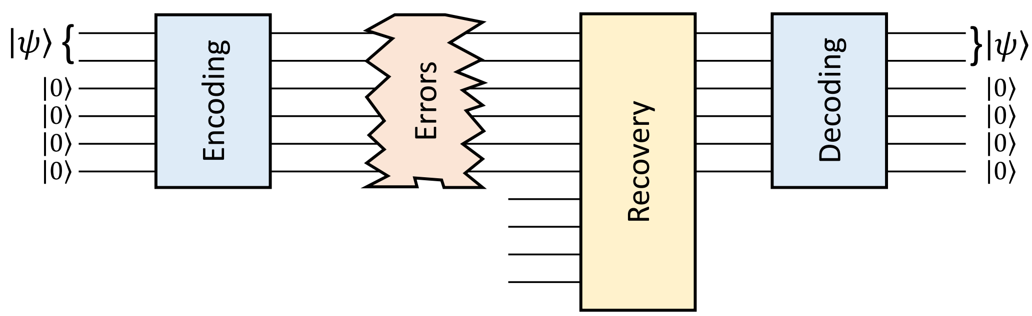


Fig. 7.1: Schematic representation of the Quantum Error Correcting approach.

7.1.1 *Classical error correction*

Consider the following example, that will make clear the usefulness of redundancies. We have Alice that wants to send a single bit information to Bob. The communication channel is not perfect: a bit-flip error noise can act on the bit with the following probabilities:

$$\text{Alice sends the bit} = \begin{tikzcd} 0 \arrow[r, "(1-\epsilon)"] \arrow[dr, "\epsilon"] & 0 \\ 1 \arrow[r, "(1-\epsilon)"'] \arrow[ur, "\epsilon"'] & 1 \end{tikzcd} = \text{is the bit the Bob receives.} \tag{7.1}$$

With this scheme, the protocol has a probability of failing that is

$$P_{\text{fail}} = \epsilon, \tag{7.2}$$

which is equal to the probability ϵ of the bit being flipped. The idea of error correction is to suitably modify the protocol so one can recover the wanted information, with a failing probability being

$$P_{\text{fail}} < \epsilon. \tag{7.3}$$

Specifically, what Alice does is to send the bit string 000 in place of only the single bit 0. This operation is called encoding, and in this specific case, one encodes the classical information in the following way

$$\begin{aligned} 0 &\to 000, \\ 1 &\to 111. \end{aligned} \tag{7.4}$$

Then, Bob receives three bits and needs to decode the information. This is performed via a majority voting. For example, let us assume the second bit is flipped while the other remain untouched: Bob receives 010, and the majority voting gives

$$010 \to 0. \tag{7.5}$$

This is a decoding of the classical information. Considering all the possible three bits strings (A, B, C) that Bob can receive if Alice sends 000, with the corresponding probabilities $p(A, B, C)$, we have

$$\text{Alice encodes 0 in 000} \to \text{Bob receives} \quad \begin{array}{c|c|c} (A,B,C) & p(A,B,C) & \text{decoded bit} \\ \hline (0,0,0) & (1-\epsilon)^3 & 0 \\ (0,0,1) & \epsilon(1-\epsilon)^2 & 0 \\ (0,1,0) & \epsilon(1-\epsilon)^2 & 0 \\ (1,0,0) & \epsilon(1-\epsilon)^2 & 0 \\ \hline (0,1,1) & \epsilon^2(1-\epsilon) & 1 \\ (1,0,1) & \epsilon^2(1-\epsilon) & 1 \\ (1,1,0) & \epsilon^2(1-\epsilon) & 1 \\ (1,1,1) & \epsilon^3 & 1 \end{array} \quad \begin{array}{c} \text{At most 1 error} \\ \\ \text{Majority voting works} \\ \\ \text{2 or more errors} \\ \\ \text{Majority voting fails} \end{array} \tag{7.6}$$

The probability that the protocol fails is given by the sum of the probabilities of the failing cases:

$$P_{\text{fail}} = 3 \times \epsilon^2(1-\epsilon) + e^3 = 3\epsilon^2 - 2\epsilon^3, \tag{7.7}$$

which is reported in Fig. 7.2.

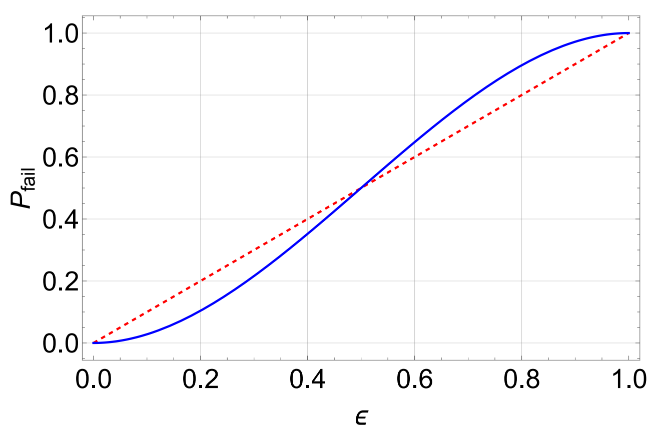


Fig. 7.2: Probability of failing P_{fail} for the single bit channel (dashed red line) and the three-bit classical correcting code (blue line).

Specifically, for $0 \leq \epsilon < 1/2$, we have that Eq. (7.3) is satisfied. This means that by using this protocol one has less probability to fail (p_{fail}) rather than using a single bit (ϵ). Therefore, the redundancy is a good approach to reduce errors, as long as Eq. (7.3) is satisfied.

7.1.2 Quantum information context

The direct application of redundancy in the quantum information context encounters some important, although not insurmountable, difficulties.

- The no-cloning theorem (see Sec. 4.1) does not allow to create copies of an unknown quantum state. This means that Alice cannot generate $|\psi\rangle\,|\psi\rangle\,|\psi\rangle$ to protect an unknown state $|\psi\rangle$.
- The second difficulty comes in how the classical error correcting code operates: one measures the state of the bit and applies a correcting operation accordingly. In the quantum case, the measurement operation would destroy the coherence of the state and thus the information encoded in the state. To be more quantitative, take the generic state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. After the measurement, the state collapses in $|0\rangle$ or in $|1\rangle$ with the respective probabilities. However, one cannot reconstruct the coherence of the state after its collapse.
- In classical information, the only possible error is a bit-flip: $0 \to 1$ and $1 \to 0$. Conversely, in quantum information the class of possible noises is far wider (see Sec. 2.3). For example, the phase-flip maps $\alpha\,|0\rangle + \beta\,|1\rangle \to \alpha\,|0\rangle - \beta\,|1\rangle$, and this does not have a classical counterpart. Moreover, in the quantum context, one can also have infinitesimal errors that can accumulate as the depth of the algorithm increases. An example can be $\alpha\,|0\rangle + \beta\,|1\rangle \to \alpha\,|0\rangle + \hat{R}^x(\epsilon)\beta\,|1\rangle$, where $\hat{R}^x(\epsilon)$ is a rotation of an infinitesimal angle ϵ around the x axis.

7.1.3 The 3-qubit bit-flip code

The first Quantum Error Correction (QEC) code we see is that correcting bit-flip errors. This is the counterpart of that seen in the classical information context.

Suppose Alice wants to send the generic state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ to Bob via a bit-flip noisy channel. We assume that the noise acts independently on each of the qubits that Alice sends. This is an important assumption for the QEC codes we will see here. We assume that the noise leaves the qubit untouched with a probability $(1-\epsilon)$, while it applies $\hat{\sigma}_x$ with a probability ϵ. Indeed, $\hat{\sigma}_x\,|0\rangle = |1\rangle$ and $\hat{\sigma}_x\,|1\rangle = |0\rangle$. This is essentially the quantum version of what shown in Eq. (7.1).

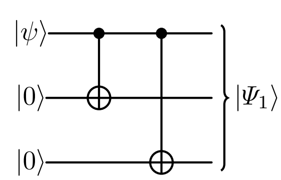
To protect the information from bit-flip errors, Alice employs the following encoding:

$$\begin{aligned} |0\rangle &\to |0_{\text{L}}\rangle = |0\rangle\,|0\rangle\,|0\rangle\,, \\ |1\rangle &\to |1_{\text{L}}\rangle = |1\rangle\,|1\rangle\,|1\rangle\,, \end{aligned} \tag{7.8}$$

where $|0\rangle$ and $|1\rangle$ are physical qubits, while $|0_{\text{L}}\rangle$ and $|1_{\text{L}}\rangle$ are logical qubits. Then, the generic state $|\psi\rangle$ is encoded in

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \to \alpha\,|0_{\text{L}}\rangle + \beta\,|1_{\text{L}}\rangle = \alpha\,|000\rangle + \beta\,|111\rangle\,, \tag{7.9}$$

with the notation $|q_1 q_2 q_3\rangle = |q_1\rangle\,|q_2\rangle\,|q_3\rangle$. This encoding can be implemented via the following circuit



(7.10)

Indeed, we have

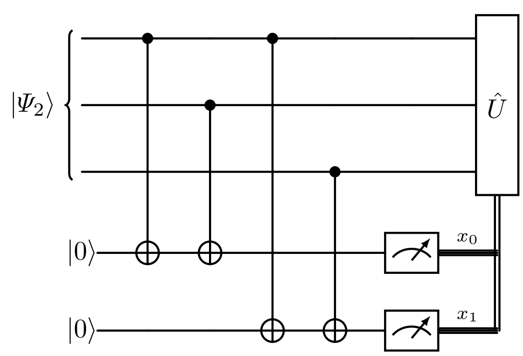
$$|\psi 00\rangle = \alpha\,|000\rangle + \beta\,|100\rangle \xrightarrow{CNOT\otimes\hat{\mathbb{1}}} \alpha\,|000\rangle + \beta\,|110\rangle \xrightarrow{\hat{\mathbb{1}}\otimes CNOT} \alpha\,|000\rangle + \beta\,|111\rangle = |\Psi_1\rangle\,. \tag{7.11}$$

We underline that the entangled state $|\Psi_1\rangle$ is not equal to $|\psi\rangle\,|\psi\rangle\,|\psi\rangle$, so the no-cloning theorem is not violated.

Now, the state $|\Psi_1\rangle$ is sent to Bob via the noisy channel. Bob receives one of the following states $|\Psi_2\rangle$ with the respective probabilities $p(|\Psi_2\rangle)$:

$$\text{Alice encodes } |\psi\rangle \text{ in } |\Psi_1\rangle \to \text{Bob receives}\quad \begin{array}{c|c l} |\Psi_2\rangle & p(|\Psi_2\rangle) & \\ \hline \alpha\,|000\rangle + \beta\,|111\rangle & (1-\epsilon)^3 & \\ \alpha\,|001\rangle + \beta\,|110\rangle & \epsilon(1-\epsilon)^2 & \\ \alpha\,|010\rangle + \beta\,|101\rangle & \epsilon(1-\epsilon)^2 & \text{At most 1 error} \\ \alpha\,|100\rangle + \beta\,|011\rangle & \epsilon(1-\epsilon)^2 & \\ \hline \alpha\,|011\rangle + \beta\,|100\rangle & \epsilon^2(1-\epsilon) & \\ \alpha\,|101\rangle + \beta\,|010\rangle & \epsilon^2(1-\epsilon) & \\ \alpha\,|110\rangle + \beta\,|001\rangle & \epsilon^2(1-\epsilon) & \text{2 or more errors} \\ \alpha\,|111\rangle + \beta\,|000\rangle & \epsilon^3 & \end{array} \tag{7.12}$$

Let us suppose that Bob receives the state $|\Psi_2\rangle = \alpha\,|100\rangle + \beta\,|011\rangle$. To correct the bit-flip, Bob would be tempted to perform a simultaneous measurement of the spin of the three qubits, i.e. $\hat{\sigma}_z^{(1)}\hat{\sigma}_z^{(2)}\hat{\sigma}_z^{(3)}$. Such an operation would give as an outcome 100 with probability $|\alpha|^2$ and 011 with probability $|\beta|^2$. Then, by applying a majority voting, Bob would understand that the first qubit is flipped. However, the coherence of the state is lost. Indeed, the measurement of the spin of the qubits collapses the state. To solve the problem, one needs to perform the so-called error syndrome. In particular, Bob employs two ancillary qubits that are prepared in the $|0\rangle$ state and coupled to the qubits carrying the encoded state. The circuit implementing the correction then is



(7.13)

To be quantitative, starting from $|\Psi_2\rangle = \alpha\,|100\rangle + \beta\,|011\rangle$, we have

$$\begin{aligned} |\Psi_2\rangle\,|00\rangle = \alpha\,|10000\rangle + \beta\,|01100\rangle &\xrightarrow{CNOT_{1,4}} \alpha\,|10010\rangle + \beta\,|01100\rangle\,, \\ &\xrightarrow{CNOT_{2,4}} \alpha\,|10010\rangle + \beta\,|01110\rangle\,, \\ &\xrightarrow{CNOT_{1,5}} \alpha\,|10011\rangle + \beta\,|01110\rangle\,, \\ &\xrightarrow{CNOT_{3,4}} \alpha\,|10010\rangle + \beta\,|01111\rangle\,, \\ &= (\alpha\,|100\rangle + \beta\,|011\rangle)\,|11\rangle\,. \end{aligned} \tag{7.14}$$

Fundamentally, the last two qubits are not entangled with the first three. Thus, the measurement on the last two qubits does not impose the collapse of the first three. After such a measurement, Bob has the outcomes $x_0 = 1$ and $x_1 = 1$. In particular, $x_0 = 1$ indicates that one among the first and the second qubit has flipped.

Similarly, $x_1 = 1$ indicates that one among the first and the third qubit has flipped. Then, Bob knows, under the assumption of single qubit errors, that the first qubit has flipped and can apply $\hat{U} = \hat{\sigma}_x^{(1)}$ to flip it back.

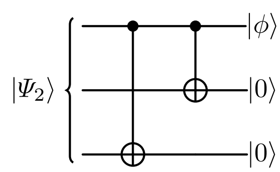
In general, Bob will apply the following unitary operations to correct the errors:

$$\begin{array}{cc|c} x_0 & x_1 & \hat{U} \\ \hline 0 & 0 & \hat{\mathbb{1}} \\ 0 & 1 & \hat{\sigma}_x^{(3)} \\ 1 & 0 & \hat{\sigma}_x^{(2)} \\ 1 & 1 & \hat{\sigma}_x^{(1)} \end{array} \tag{7.15}$$

After having applied the correction, Bob gets the state $|\Psi_3\rangle$ with the following probabilities

$$\begin{array}{c|c|c} |\Psi_2\rangle & |\Psi_3\rangle & p(|\Psi_3\rangle) \\ \hline \alpha\,|000\rangle + \beta\,|111\rangle & \alpha\,|000\rangle + \beta\,|111\rangle & (1-\epsilon)^3 \\ \alpha\,|001\rangle + \beta\,|110\rangle & \alpha\,|000\rangle + \beta\,|111\rangle & \epsilon(1-\epsilon)^2 \\ \alpha\,|010\rangle + \beta\,|101\rangle & \alpha\,|000\rangle + \beta\,|111\rangle & \epsilon(1-\epsilon)^2 \\ \alpha\,|100\rangle + \beta\,|011\rangle & \alpha\,|000\rangle + \beta\,|111\rangle & \epsilon(1-\epsilon)^2 \\ \hline \alpha\,|011\rangle + \beta\,|100\rangle & \alpha\,|111\rangle + \beta\,|000\rangle & \epsilon^2(1-\epsilon) \\ \alpha\,|101\rangle + \beta\,|010\rangle & \alpha\,|111\rangle + \beta\,|000\rangle & \epsilon^2(1-\epsilon) \\ \alpha\,|110\rangle + \beta\,|001\rangle & \alpha\,|111\rangle + \beta\,|000\rangle & \epsilon^2(1-\epsilon) \\ \alpha\,|111\rangle + \beta\,|000\rangle & \alpha\,|111\rangle + \beta\,|000\rangle & \epsilon^3 \end{array} \tag{7.16}$$

Finally, Bob applies the following decoding circuit



(7.17)

which is the inverse of the circuit in Eq. (7.10). The final state $|\phi\rangle$ is

$$\begin{aligned} |\phi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle\,, && \text{with probability} \quad p = (1-\epsilon)^3 + 3\epsilon(1-\epsilon)^2, \\ |\phi\rangle &= \alpha\,|1\rangle + \beta\,|0\rangle = \hat{\sigma}_x\,|\psi\rangle\,, && \text{with probability} \quad p = 3\epsilon^2(1-\epsilon) + \epsilon^3. \end{aligned} \tag{7.18}$$

Thus, the failing probability of this QEC code is

$$P_{\text{fail}} = 3\epsilon^2 - 2\epsilon^3, \tag{7.19}$$

which is the same as in the classical correcting algorithm seen previously and it is plotted in Fig. 7.2. Notably, Bob does not learn anything about the weights α and β thought this QEC code. The coherence of the state remains intact.

7.1.4 The 3-qubit phase-flip code

Let us consider the case of the phase-flip noise, where the following error generates with a probability ϵ:

$$\begin{aligned} |0\rangle &\to \hat{\sigma}_z\,|0\rangle = |0\rangle\,, \\ |1\rangle &\to \hat{\sigma}_z\,|1\rangle = -\,|1\rangle\,. \end{aligned} \tag{7.20}$$

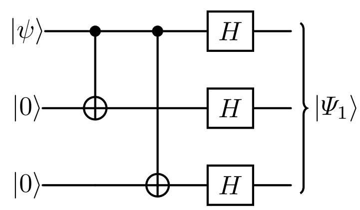
Consequently, one has

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \to \alpha\,|0\rangle - \beta\,|1\rangle\,. \tag{7.21}$$

Notably, this noise does not have a classical counterpart. Since the error is imprinted in the relative phase between $|0\rangle$ and $|1\rangle$, the bit-flip QEC code developed in Sec. 7.1.3 does not correct this type of errors. However, a phase-flip error in the computational basis $\{\,|0\rangle, |1\rangle\,\}$ corresponds to a bit-flip error in the $\{\,|+\rangle, |-\rangle\,\}$ basis. Indeed,

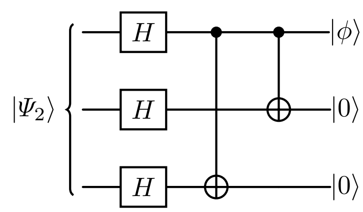
$$\begin{aligned} \hat{\sigma}_z\,|+\rangle &= |-\rangle\,, \\ \hat{\sigma}_z\,|-\rangle &= |+\rangle\,. \end{aligned} \tag{7.22}$$

Then, by simply adding a Hadamard gate, one changes basis and thus is able to employ the bit-flip QEC code to correct phase-flip errors. This is done both in the encoding and the decoding parts of the code. The encoding circuit is then



(7.23)

while the deconding becomes



(7.24)

The remaining parts of the code remain identical.