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Corso di Termofluidodinamica Computazionale

Homework No. 2
AA 2023/2024

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Proposed problem

Description

Following *Homework 1 - AA. 2023-24*, consider a cylindrical (pin) fin, as shown in figure 1, which is made with a uniform, isotropic material with a thermal conductivity value of $k = 40$ W/(m K). The fin has a length L and a diameter d.

The fin is cooled only by convection with a convective heat transfer coefficient h =400 W/(m^2 K), and the temperature of the surrounding fluid is $T_\infty = 25$ °C. The temperature of the base of the fin is maintained at a temperature $T_b = 200$ °C, while also the tip of the fin contributes, with the same heat transfer coefficient, to the overall heat flux.

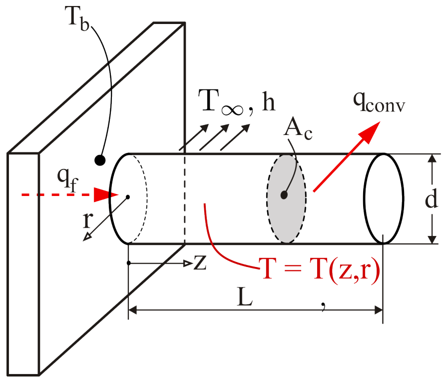

Figure 1: Axisymmetrical cylindrical (pin) fin.

In this case, disregard the usual assumption of 1D temperature distribution (see [1, 2]), i.e.

$$T \approx T(z)$$

and consider a full *2D, axisymmetric* temperature distribution

$$T = T(z, r)$$

The governing equation

The general heat (conduction) equation for an isotropic material in cylindrical coordinates (z, r, ϕ) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_g = \rho\, c_p \frac{\partial T}{\partial \tau} \tag{1}$$

which, under the assumption of steady, 2D axisymmetric temperature field with no heat generation, reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0 \tag{2}$$

The problem

Using MATLAB, or any other language of choice, develop a 2D axisymmetric steady numerical model for the fin and, using an *adequate* number of finite volumes, compute the heat flux q_{FV2D} [W] and plot a contour map of the temperature field for the same cases considered in *Homework 1 - AA. 2023-24*

1. $L = 30$ mm and $d = 3$ mm.
2. $L = 30$ mm and $d = 20$ mm.

Compare the result with that obtained with the 1D model of *Homework 1*. What is the $\%$ error using the 1D assumption?

The detailed description of the 2D axisymmetric model and the derivation of the discretized equations are given in the Appendix.

Optional problem

Using the MATLAB *PDE Toolbox*, develop a 2D axisymmetric steady numerical model for the fin and, using an *adequate* number of finite elements, compute the heat flux q_{FE2D} [W] and plot a contour map of the temperature field. Compare the result with that obtained with the 1D model of *Homework 1* and the 2D axisymmetric FV model.
Do the results compare well with those from the 2D axisymmetric FV model?

References

[1] G. Comini, G. Cortella, *Fondamenti di trasmissione del calore*, 4a Ed., S.G.E. Editore, (2013).

[2] F. P. Incropera, D. P. Dewitt, T. L. Bergman, A. S. Lavine, *Fundamentals of Heat and Mass Transfer*, 6th Ed., Wiley, (2007).

Appendix

For the present axisymmetric heat conduction problem, the computational domain is illustrated in figure 2.

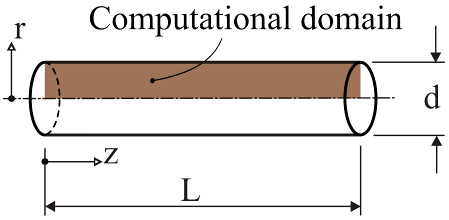

Figure 2: Computational domain.

A possible way to derive the discrete equation is to integrate, for the generic Finite Volume depicted in figure 3, equation (2). One should note that, while for Cartesian 2D problems the generic cell has a volume $V = \Delta x\, \Delta y\, 1$, in the axisymmetric case the volume (see figure 4) is $V = \Delta z\, \Delta r\, r_P\, 1$

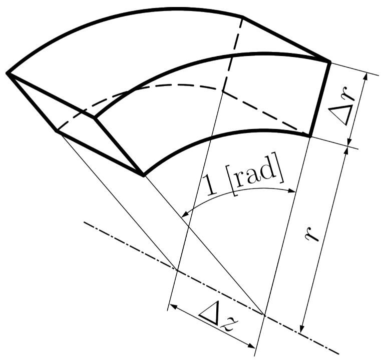

Figure 3: Axisymmetric Finite Volume.

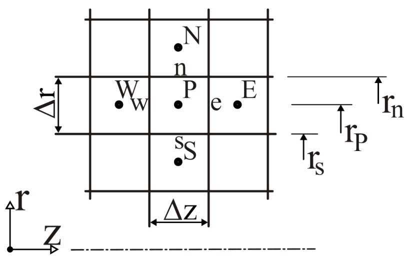

Figure 4: 2D axisymmetric grid.

$$\int_s^n \int_w^e \frac{1}{r}\frac{\partial}{\partial r}\left(k\, r\, \frac{\partial T}{\partial r}\right) 1\, r\, \mathrm{d}r\ \mathrm{d}z + \int_s^n \int_w^e \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) 1\, r\, \mathrm{d}r\ \mathrm{d}z = 0 \tag{3}$$

Reordering the integrals so that exact differentials are integrated first, and using the mean value theorem it results

$$\int_w^e \left(\left[kr \frac{\partial T}{\partial r} \right]_n - \left[kr \frac{\partial T}{\partial r} \right]_s \right) \mathrm{d}z + \int_s^n \left(\left[k \frac{\partial T}{\partial z} \right]_e - \left[k \frac{\partial T}{\partial z} \right]_w \right) r \ \mathrm{d}r = 0 \tag{4}$$

Integrating once more

$$\left(\left[kr \frac{\partial T}{\partial r} \right]_n - \left[kr \frac{\partial T}{\partial r} \right]_s \right) \Delta z + \left(\left[k \frac{\partial T}{\partial z} \right]_e - \left[k \frac{\partial T}{\partial z} \right]_w \right) \frac{r_n^2 - r_s^2}{2} = 0 \tag{5}$$

The quantity $(r_n^2 - r_s^2)/2$ may be written as $(r_n - r_s)(r_n + r_s)/2$ which, for a uniform mesh, reduces to $\Delta r \, r_P$.
Invoking, as usual for the diffusion term, the *CDS* scheme, equation (5) becames

$$r_n k_n \left(\frac{T_N - T_P}{\Delta r} \right) \Delta z - r_s k_s \left(\frac{T_P - T_S}{\Delta r} \right) \Delta z + r_P k_e \left(\frac{T_E - T_P}{\Delta z} \right) \Delta r - r_P k_w \left(\frac{T_P - T_W}{\Delta z} \right) \Delta r = 0 \tag{6}$$

Reordering and assuming, for simplicity, constant value of the thermal conductivity, we have

$$A_P \, T_P + A_E \, T_E + A_W \, T_W + A_N \, T_N + A_S \, T_S = 0 \tag{7}$$

where

$$\begin{aligned} &A_E = -r_P \frac{k}{\Delta z} \Delta r \quad A_W = -r_P \frac{k}{\Delta z} \Delta r \\ &A_N = -r_n \frac{k}{\Delta r} \Delta z \quad A_S = -r_s \frac{k}{\Delta r} \Delta z \\ &A_P = -\left(A_E + A_W + A_N + A_S \right) \end{aligned}$$

and

$$r_n = r_P + \Delta r/2 \quad r_s = r_P - \Delta r/2$$

For the cells at the axis, we have $r_s = 0$, and therefore the corresponding term in the equation vanishes and no boundary conditions are necessary for the axis.