7.1.5 The 9-qubit Shor code

We saw how the 3-qubit bit-flip and phase flip QEC codes can correct respectively bit-flip and phase flip errors. Here, we show that concatenating these two codes, one can protects for generic single qubit errors. Indeed, consider the situation of a single qubit initially prepared in the state $|\psi\rangle$. Suppose it is coupled to the surrounding enviroment, whose state is initially $|e\rangle$, and that the latter entangles with the system. Such a transformation is described as

$$|\psi\rangle |e\rangle \to c_0 \hat{\mathbb{1}} |\psi\rangle |e_0\rangle + c_1 \hat{\sigma}_x |\psi\rangle |e_1\rangle + c_2 \hat{\sigma}_y |\psi\rangle |e_2\rangle + c_3 \hat{\sigma}_z |\psi\rangle |e_3\rangle \,, \tag{7.25}$$

where c_i are suitable constants, and $|e_i\rangle$ are states of the environment. Then, the state of the system is transformed via the application of the four Pauli operators. Here, $\hat{\sigma}_0 = \hat{\mathbb{1}}$ does not imply any change in the state, so no error needs to be corrected. The errors due to $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are respectively corrected via bit-flip and phase-flip QEC codes. It remains that due to $\hat{\sigma}_y$. However, one can notice that, since the Pauli matrices form a Lie algebra, one can express $\hat{\sigma}_y$ in terms of $\hat{\sigma}_x$ and $\hat{\sigma}_z$. Namely, $\hat{\sigma}_y = i\hat{\sigma}_x\hat{\sigma}_z$. Then, one needs only to correct two consecutive errors (phase-flip and then bit-flip) to correct a bit-phase flip. The following QEC code is sufficient to perform such a correction.

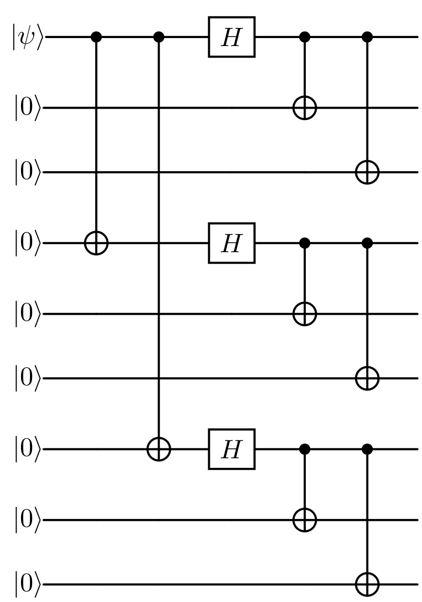
The encoding of the 9-qubit Shore code is given by

$$\begin{aligned} |0\rangle \to |0_\mathrm{L}\rangle &= \frac{1}{\sqrt{8}} \left(|000\rangle + |111\rangle\right) \left(|000\rangle + |111\rangle\right) \left(|000\rangle + |111\rangle\right) , \\ |1\rangle \to |1_\mathrm{L}\rangle &= \frac{1}{\sqrt{8}} \left(|000\rangle - |111\rangle\right) \left(|000\rangle - |111\rangle\right) \left(|000\rangle - |111\rangle\right) . \end{aligned} \tag{7.26}$$

This implies the following encoding for a generic state $|\psi\rangle$

$$|\psi\rangle \to \frac{\alpha}{\sqrt{8}} \left(|000\rangle + |111\rangle\right) \left(|000\rangle + |111\rangle\right) \left(|000\rangle + |111\rangle\right) + \frac{\beta}{\sqrt{8}} \left(|000\rangle - |111\rangle\right) \left(|000\rangle - |111\rangle\right) \left(|000\rangle - |111\rangle\right) . \tag{7.27}$$

The encoding is implemented via the following circuit



(7.28)

The action of the first two CNOT gates and three Hadamard in Eq. (7.28) is to map the qubits 1, 4 and 7 as follows:

$$\begin{aligned}|\psi 00\rangle &= \alpha\,|000\rangle + \beta\,|100\rangle\,, \\ &\to \alpha\,|000\rangle + \beta\,|110\rangle\,, \\ &\to \alpha\,|000\rangle + \beta\,|111\rangle\,, \\ &\to \alpha\,|+++\rangle + \beta\,|---\rangle\,.\end{aligned} \tag{7.29}$$

Namely, they perform the encoding for the phase-flip QEC code:

$$\begin{aligned}|0\rangle &\to |+++\rangle\,, \\ |1\rangle &\to |---\rangle\,.\end{aligned} \tag{7.30}$$

Then, every $|+\rangle$ and $|-\rangle$ state in these qubits is further encoded with the last CNOT gates. Specifically, one has

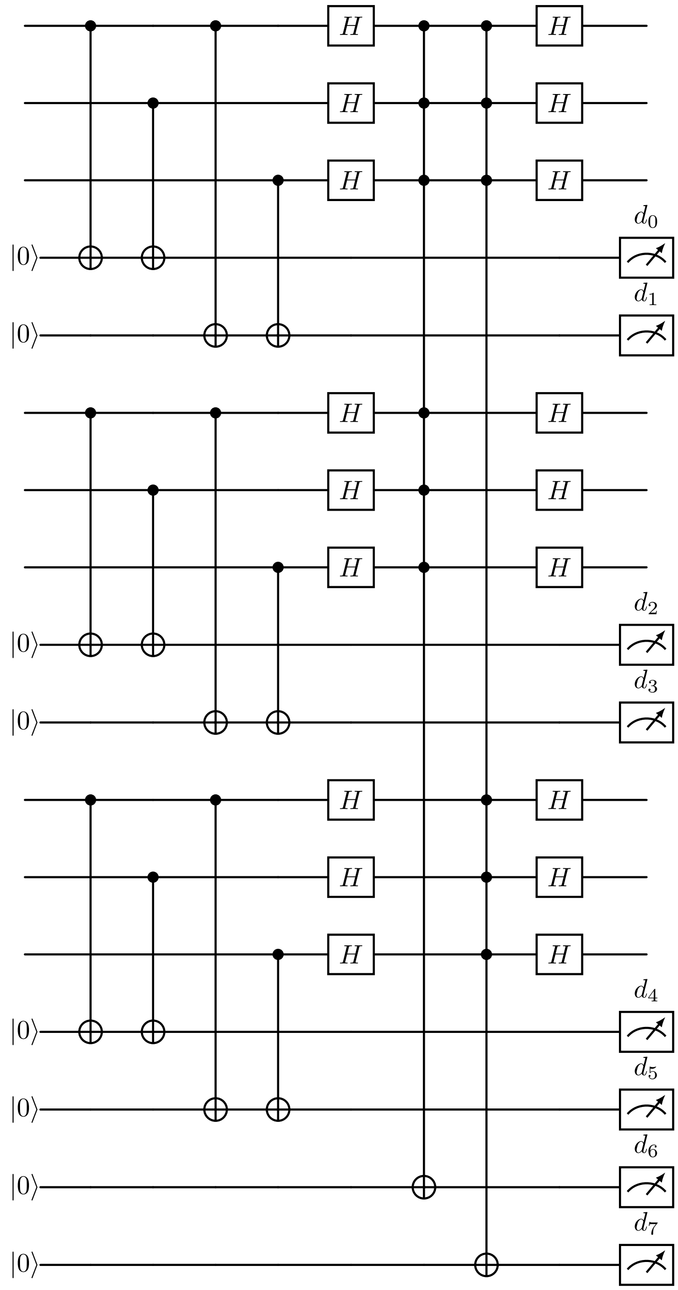
$$\begin{aligned}|+00\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right), \\ &\to \frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right), \\ &\to \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),\end{aligned} \tag{7.31}$$

and

$$\begin{aligned}|-00\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle - |100\rangle\right), \\ &\to \frac{1}{\sqrt{2}}\left(|000\rangle - |110\rangle\right), \\ &\to \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right).\end{aligned} \tag{7.32}$$

These, effectively perform the encoding for the bit-flip QEC code.

Such an encoding combines the phase-flip and the bit-flip encoding. To extract the error syndrome, one employs a collective measurement, similarly as for the bit-flip. In particular, 8 ancillary qubits are employed to construct the following circuit



(7.33)

Here, the outcomes (d_0, d_1), (d_2, d_3) and (d_4, d_5) respectively indicate bit-flip errors within the first, second and third block of three physical qubits. Specifically, for the first block, one employs exactly what described in Sec. 7.1.3.

The outcomes (d_6, d_7) are instead used to detect phase-flip errors of the logical state encoded with the three blocks. The collective measurements to do this are

$$\begin{aligned} &\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}\hat{\sigma}_x^{(3)}\hat{\sigma}_x^{(4)}\hat{\sigma}_x^{(5)}\hat{\sigma}_x^{(6)},\\ &\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}\hat{\sigma}_x^{(3)}\hat{\sigma}_x^{(7)}\hat{\sigma}_x^{(8)}\hat{\sigma}_x^{(9)}, \end{aligned} \tag{7.34}$$

which provide d_6 and d_7 respectively. If one gets, for example, $(d_6 = -1, d_7 = -1)$, then a phase flip occurred in the first block.

7.1.6 On the redundancy and threshold

As we saw, a fundamental step in the QEC codes is the redundancy of the state. Notably, there is no need in having exactly 3 copies. It can be extended to any k copies, as long as $k > 1$ is an odd number. What one wants is that the probability P_{fail} that the QEC code fails is smaller than the probability ϵ of an error occurring on a single physical qubit: $P_{\text{fail}} < \epsilon$.

Consider the case of k physical qubits encoding a single logical qubit. Given the probability ϵ of having an error on one of these qubits, that of having j qubits with errors is given by

$$p(j) = \epsilon^j (1-\epsilon)^{k-j}, \tag{7.35}$$

and there are

$$\binom{k}{j} = \frac{k!}{(k-j)!j!}, \tag{7.36}$$

different possible combinations. Then, P_{fail} is given by the sum over these when the faulty qubits are at least half of the total. This is

$$P_{\text{fail}} = \sum_{j=\frac{(k+1)}{2}}^{k} \binom{k}{j} \epsilon^j (1-\epsilon)^{k-j}. \tag{7.37}$$

Namely, for $k = 3$, one has

$$P_{\text{fail}} = \sum_{j=\frac{(3+1)}{2}}^{3} \binom{3}{j} \epsilon^j (1-\epsilon)^{3-j} = 3\epsilon^2(1-\epsilon) + \epsilon^3. \tag{7.38}$$

The behaviour of P_{fail} for different values of k is shown in Fig. 7.3.

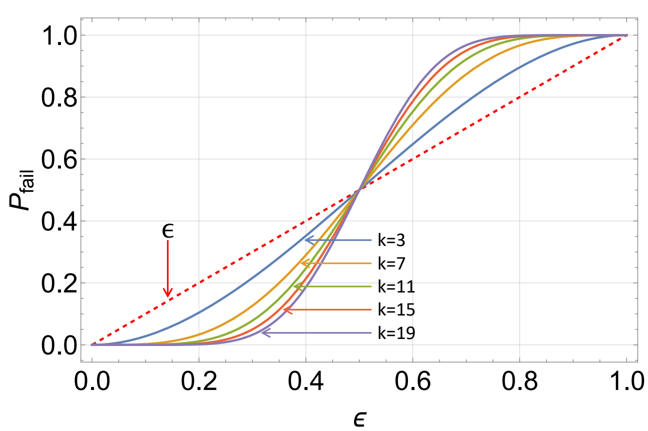


Fig. 7.3: Probability of failing P_{fail} for the single bit channel (dashed red line) with respect to a redundant encoding with k physical qubits (continuous lines).

However, one can consider an alternative approach. Instead of encoding a logical qubit just once in a large number of physical qubits, one can concatenate encodings. One encodes the logical qubit in different levels, where each level employs a small numeber of qubits. To be more explict, the following is the encoding of a single physical qubit in a 2 level encoding with three qubits each:

$$|0\rangle \xrightarrow{\text{first encoding}} |000\rangle \xrightarrow{\text{second encoding}} |000\rangle\,|000\rangle\,|000\rangle\,, \tag{7.39}$$

and similarly for $|1\rangle$. Now, with this encoding, the actual physical qubit is that in the highest level of encoding, and this is that directly suffering from the noise. Suppose there is a probability ϵ that an error occurs on this physical qubit. Then, at the level 1, the probability of failing, for example for the bit-flip QEC code, is

$$P_{\text{fail},1} = 3\epsilon^2 - 2\epsilon^3. \tag{7.40}$$

This quantity is the probability that the noise corrupts a qubit at the level 1. Thus, when computing the probability of failing for the qubit at level 0, the actual logical qubit, $P_{\text{fail},1}$ needs to be interpreted as the probability ϵ_1 that an error occurs on the qubit at the level 1. Then, at level 0, one has that the failing probability is

$$\begin{aligned} P_{\text{fail},0} &= 3P_{\text{fail},1}^2 - 2P_{\text{fail},1}^3, \\ &= 3[3\epsilon^2 - 2\epsilon^3]^2 - 2[3\epsilon^2 - 2\epsilon^3]^3, \\ &= 27\epsilon^4 - 36\epsilon^5 - 42\epsilon^6 + 108\epsilon^7 - 72\epsilon^8 + 16\epsilon^9. \end{aligned} \tag{7.41}$$

The question is then which is the best encoding. Figure 7.4 compares the failing probabilities P_{fail} for a single level encoding with 9 physical qubits (blue line), where Eq. (7.37) gives

$$P_{\text{fail}} = 126\epsilon^5 - 420\epsilon^6 + 540\epsilon^7 - 315\epsilon^8 + 70\epsilon^9, \tag{7.42}$$

and that for a 2 level encoding each with 3 qubits (red line). This is a fair comparison, as both the approaches are employing the same number of physical qubits, i.e. $n = 9$.

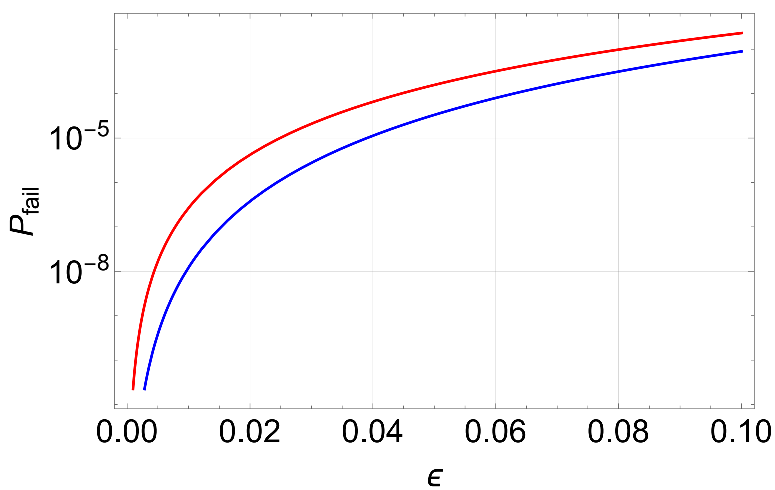


Fig. 7.4: Comparison of the failing probabilities P_{fail} for a single level encoding with 9 physical qubits (blue line) and that for a 2 level encoding each with 3 qubits (red line).

To keep the discussion more general, suppose p is the probability of failing for a qubit with no encoding (this is what we called ϵ until now). Then, the failing probability is

$$P_{\text{fail}}^{(0)} = p. \tag{7.43}$$

Suppose that after one encoding the failing probability is

$$P_{\text{fail}}^{(1)} = cp^2, \tag{7.44}$$

where c is some suitable constant. In the case of the 3-qubit encoding, one had

$$P_{\text{fail}} = 3p^2 - 2p^3 \sim 3p^2, \tag{7.45}$$

for small values of p. After 2 encodings, one has

$$P_{\text{fail}}^{(2)} = c(cp^2)^2 = \frac{1}{c}(cp)^{2^2}. \tag{7.46}$$

After k encodings, one has

$$P_{\text{fail}}^{(k)} = p_{\text{th}} \left(\frac{p}{p_{\text{th}}} \right)^{2^k}, \tag{7.47}$$

where we defined the threshold probability as

$$p_{\text{th}} = \frac{1}{c}. \tag{7.48}$$

Such a probability depends on varius parameters, among which the QEC code used, the physical components, the experimental implementation of the QEC protocol etc.

The threshold probability p_{th} is fundamental due to the following theorem.