Cosmology and Particle Physics

DARK MATTER

Dark matter

Good reasons to believe ∃ a dark (ie non-luminous) component in our Universe w/

$$\Omega_{DM} h^2 \approx 0.12$$

That complements the NR SM particles $\Omega_b h^2 \approx 0.02$.

1) How do we know? What evidences do we have?

2) What is it made of?
   - dark, astrophysical objects?
   - new particles?
   - modified laws of gravitation?

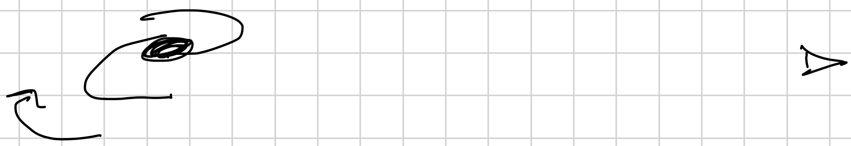
3) Astrophysical consequences?

4) if particles: what interactions?

5) How do we look for it? Experimental searches

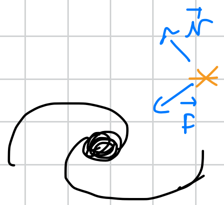
Observational evidences

1) Rotation curves of galaxies



Measure the tion velocity of a spiral galaxy using the redshift of 21 cm hyperfine transition of neutral hydrogen.

Newtonian dynamics:



$$m_* \frac{v^2(r)}{r} = \frac{G M(r) m_*}{r^2}$$

$M(r)$: from counting stars + luminous gas

Assume for simplicity

$$M(r) = \begin{cases} \rho \, \frac{4}{3}\pi r^3 & r < r_c \\ M & r > r_c \end{cases}$$

w/ $r_c \sim O(10\,\text{kpc})$ (eg we live 8kpc away from the MW centre)

$$\Rightarrow \quad v \propto \begin{cases} r & r < r_c \\ r^{-1/2} & r > r_c \end{cases}$$ valid for luminous and non luminous matter

Observations:

at large radii : $v(r) \sim \text{const}$

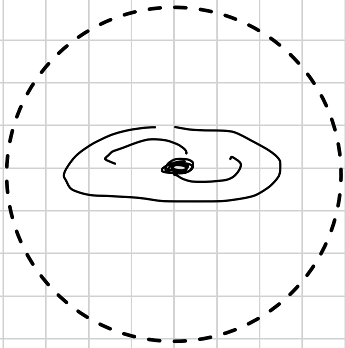
achieved for $M(r) \sim r$ $\qquad \frac{dM}{dr} = 4\pi r^2 \rho(r) = \text{const}$

$$\Rightarrow \rho(r) \propto r^{-2}$$

Rubin & al, Astrophys. J. 238 (1980) 471

Assumption: baryons interact ⇒ dissipation ⇒ collapse into a disk

DM is "collisionless" ⇒ spherical "halo"



MW: from stellar kinematics we have

$$M_{halo} \sim 10^{12} M_\odot \qquad \rho_0 \sim 0.3 \frac{GeV}{cm^3}$$

$$M_{halo} \sim 4\pi \int_0^{R_{halo}} dr\, r^2 \rho(r) \Rightarrow R_{halo} \sim 100\,\text{kpc} \gg R_{disk}$$

Can also estimate the average velocity using the virial theorem

$$2\langle K\rangle = -\langle V\rangle \Rightarrow \langle v^2\rangle \sim \frac{G M_{halo}}{R_{halo}} \sim (200\,\text{km/s})^2$$

$\frac{v}{c} \sim 10^{-8}$ ⇒ DM is non-relativistic

(Virial theorem: for a stable system bound by a conservative force

$$V = \alpha r^n \Rightarrow 2\langle T\rangle = n\langle V_{TOT}\rangle$$)

2) Dynamics of galaxy cluster

First historical evidence (consequences not 100% understood)

Coma cluster: ~ 800 galaxies, from luminosity we estimate each galaxy contained $\sim 10^9 M_\odot$

$$\Rightarrow M^{vis}_{Coma} \sim 800 \times 10^9 M_\odot \simeq 10^{42}\,\text{kg}$$

Method 2: from virial theorem

Redshift: $v \sim 10^3 \text{ km s}^{-1}$

$$2K + V = 0$$

$$K = \sum_i \frac{1}{2} m_i v_i^2 \qquad V = -\sum_{i<j} G_N \frac{m_i m_j}{r_{ij}} = -\frac{3}{5} \frac{G_N M^2_{clus}}{R_{clus}}$$

$$\langle v^2 \rangle = \frac{\sum_i v_i^2}{N_{gal}} \approx \frac{\sum m_{gal} v_i^2}{m_{gal} N_{gal}} = \frac{2K}{M_{clus}} = \frac{V}{M_{clus}} = \frac{3}{5} \frac{G_N M_{clus}}{R_{clus}}$$

(assume $m_i = m_{gal} \; \forall i$) $R_{clus} \sim 10^6$ pc

$$\Rightarrow M^{(grav)}_{clus} = \frac{5}{3} \frac{R_{clus}}{G_N} \langle v^2 \rangle \simeq 10^{44} \text{ kg}$$

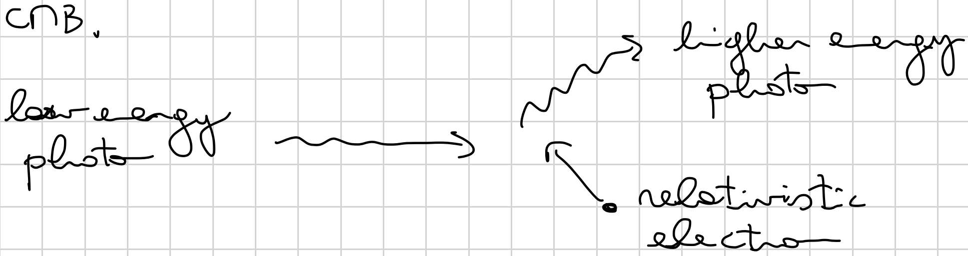
(Zwicky '30s)

Value is wrong (he used the wrong value of H to infer the distance of galaxies) but the point was correct.

3) More on galaxy clusters

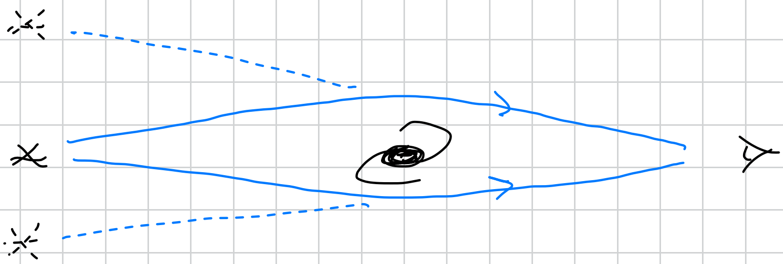
other ways to measure the total mass of the cluster

- emitted light → temperature profile of the gas → → mass profile (from hydrodynamic equilibrium)
- Sunyaev–Zel'dovich effect: inverse Compton scattering of CMB photons. Scales as n_{gas}. Measure n_{gas} from the CMB.



4) gravitational lensing (both at the scale of galaxies and galaxy clusters)

light from a distant quasar is bended by a galaxy or a cluster along the line of sight, resulting in distortions and/or multiple images

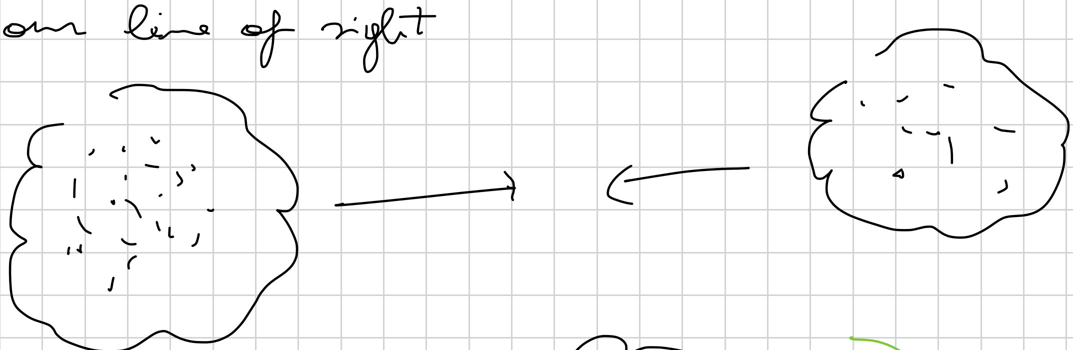


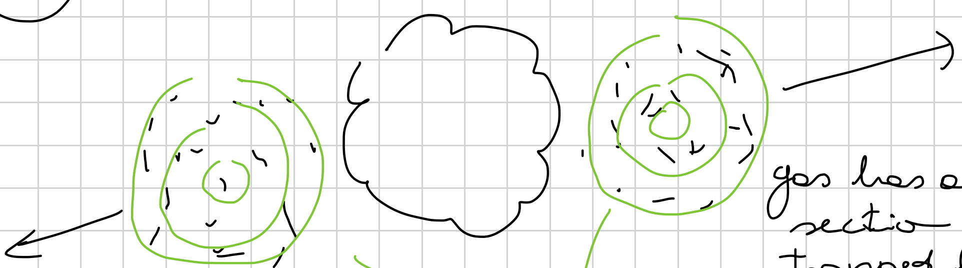
Can be used to measure the mass of the object

5) Bullet cluster ("direct evidence"? 2000's)

First examples of a space separation between the different components

Two clusters collided ~ 150 million yrs ago, perpendicular to our line of sight





gas has a large cross section and was trapped by its viscosity

galaxies are "pointlike" and kept moving almost undisturbed

gravitational lensing allows us to reconstruct the mass distribution

$\Rightarrow$ the main component is collisionless DM

The bullet cluster provides an upper bound on the self-interactions of DM particles:

$$\frac{\sigma}{m_\chi} \lesssim 1.25 \text{ cm}^2\, g^{-1} = 2\times 10^{-24} \text{ cm}^2\, \text{GeV}^{-1}$$

6) Large scale structures formation

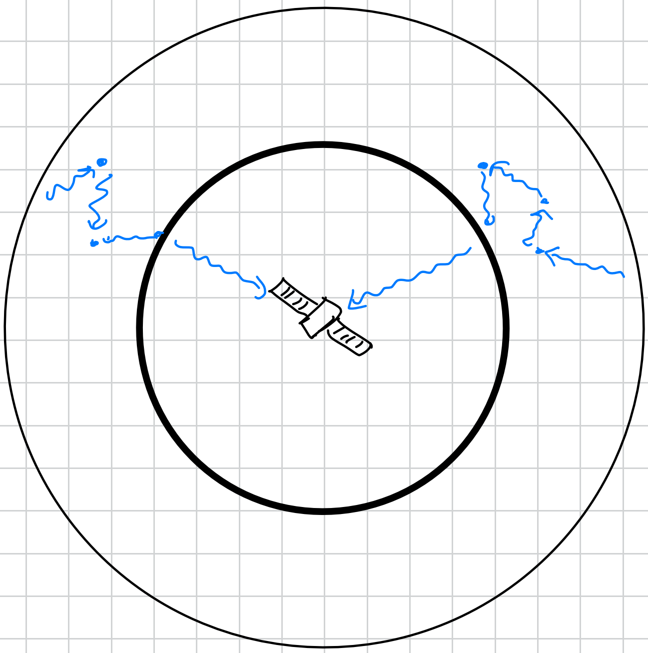
DM is a key ingredient of our understanding of structure formation

- initial density fluctuations are very small
- DM decouples quite early from the thermal bath, and can collapse to form the first halos
- baryons follow
- small objects form first, and organize themselves into larger and larger structure

7) Cosmic Microwave Background

DM can not only be studied by obs of gravitationally bounded objects

CMB: after recombination ($e^- + p^+ \to H + \gamma$) there are no charged particles around. Photons travel freely



CMB is a picture of the last scattering surface

It's a perfect black-body spectrum with $\mathcal{O}(10^{-5})$ special fluctuations

$$n(f,T)\,df = \frac{2}{c^3}\,\frac{4\pi f^2}{e^{hf/k_BT}-1}\,df$$

Fluctuations can be expanded in spherical harmonics

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\theta, \phi)$$

$l=0$: monopole (black body)

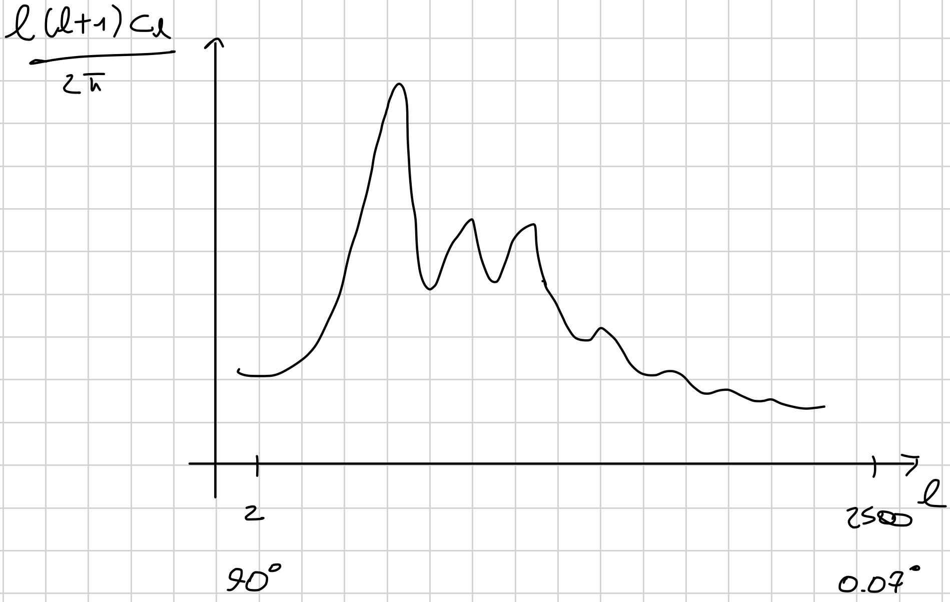
$l=1$: dipole (Earth proper motion → Doppler shift)

Suppose a_{lm} come from a gaussian random distribution. We have one sky only, but we can measure the expectation value of $\langle a_{lm} a_{l'm'} \rangle$ by measuring in different directions in the sky. Defining

$$\langle a_{lm} a_{l'm'} \rangle = C_l \, \delta_{ll'} \, \delta_{mm'} \quad \text{(isotropy)}$$

the temperature power spectrum is

$$\langle \delta T(\theta_i \phi_i) \, \delta T(\theta_j \phi_j) \rangle = \sum_l (2l+1) C_l P_l(\cos\theta_{ij})$$



- baryons are pushed by gravity from underdense to overdense regions
- photon pressure contrast this motion : oscillatory effect
- DM feels gravity only ⇒ different behaviour wrt baryons

Through a fit of the power spectrum w/ 6 parameters one infers

$$\Omega_{b} h^2 \simeq 0.022$$

$$\Omega_{DM} h^2 \simeq 0.12$$

NB: DM $\exists$ at the time of CMB $\Rightarrow$ must be non-baryonic
(cannot be formed in stars)

8) Growth was initiated by DM

Overdensities grow due to gravitational collapse as

$$\frac{\delta\rho}{\rho} \propto \begin{cases} \log a & RD \\ a & MD \end{cases}$$

Baryonic overdensities do not grow until recombination because of photon pressure. In a Universe w/o dark matter we would have

$$\left.\frac{\delta\rho}{\rho}\right|_0 \approx \left.\frac{\delta\rho}{\rho}\right|_{rec} \frac{a_0}{a_{rec}} \approx \left.\frac{\delta\rho}{\rho}\right|_{CMB} z_{rec} \approx 10^{-5} \times 10^3 = 10^{-2}$$

But galaxies are overdensities of order 10^5 $\Rightarrow$ we need DM to collapse first

9) Large scale structure surveys

They give us similar information as the CMB. Next generation (EUCLID) has been launched recently

DM properties

→ from now on: assume DM is made of one new, neutral, stable particle

Yield

$$\Omega_{DM} h^2 = \frac{\rho_{DM}}{\rho_{crit}/h^2} = 0.12$$

both constant

define $Y_{DM} = \frac{n_{DM}}{s}$

$$\xi_{DM} = \frac{\rho_{DM}}{s} = m_{DM} \frac{n_{DM}}{s}$$

$$\left.\begin{aligned} \rho_{crit}(t_0) &= 1.05 \times 10^{-5}\, h^2 \text{ GeV/cm}^3 \\ s(t_0) &= 2891.2 \text{ cm}^{-3} \end{aligned}\right\} \Rightarrow \xi_{DM} = 4.35 \times 10^{-10} \text{ GeV}$$

Distribution in the galaxy

Self gravitating isothermal sphere:

$$\rho(r) \propto r^{-2} \qquad f(v) \propto e^{-v^2/\sigma^2}$$

But $M_{halo} = 4\pi \int_0^\infty dr\, r^2 \rho(r)$ diverges ⇒ need to deviate from r^{-2} as for flat rotation curves at large radius

"DM only" simulations

"universal" profile Navarro-Frenk-White $\quad \rho_{NFW} = \dfrac{\rho_0}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2} \qquad r_s = 20\,\text{kpc}$

Einasto $\quad \rho_{Ein} = \rho_0 \exp\left[-\dfrac{2}{\gamma}\left(\left(\dfrac{r}{r_s}\right)^\gamma - 1\right)\right] \qquad r_s = 20\,\text{kpc}, \quad \gamma = 0.17$

"cuspy" profiles: peaked at the centre

Cored profiles

Burkert $\rho_{Burk} = \dfrac{\rho_0}{\left(1+\frac{r}{r_h}\right)\left[1+\left(\frac{r}{r_h}\right)^2\right]}$

Core vs cusp

- most detailed simulations (DM only) obtain cuspy profiles
- observation of dwarf galaxies suggest maybe cored (evidence still inconclusive) (dwarf satellite galaxies are good because DM rich, and close enough to distinguish individual stars)
- baryon feedback: adiabatic contraction leads to condensation of gas in the center, that may gravitationally pull the DM enhancing the central density. But the energy injection may produce a core
- DM self interaction leads to the formation of cores?

DM velocity distribution

relevant for direct searches

Standard halo model

$$f(\vec{v}) = \begin{cases} \dfrac{1}{N_{esc}}\left(\dfrac{3}{2\pi\sigma_v^2}\right)^{3/2} e^{-\frac{3v^2}{2\sigma_v}} & v < v_{esc} \\ 0 & \text{otherwise} \end{cases}$$

(truncated maxwellian)

Substructures

$\exists$ out of equilibrium objects

clumps : localized overdensities
streams : locally $f(\vec{v}) \approx \delta(\vec{v} - \vec{v}_{\text{stream}})$ (related to stellar streams?)

understood as remnants of the merger process

dark disks

- - - -

Mass of particle DM

* upper bound $m_{DM} \lesssim M_P \simeq 10^{18}$ GeV

* lower bound : bosons

dwarf galaxies $R_d \simeq 1$ kpc $\approx 3 \times 10^{19}$ m

smallest possible halo $\Longleftrightarrow$ de Broglie wavelength

for very light fields, large occupation number

From $\frac{\lambda}{2\pi} = \frac{\hbar}{mv} < \frac{GM}{v^2}$ "virial radius"

$v < \frac{GMm}{\hbar} \Rightarrow v_{vir}^2 = \frac{2GM}{R} < \frac{G^2M^2m^2}{\hbar^2} \Rightarrow R > 2\frac{\hbar^2}{GMm^2}$

found by imposing
$\langle T \rangle = -\frac{1}{2}\langle U \rangle$
$\frac{1}{2}M\sigma^2 = \frac{1}{2}\frac{GM^2}{r} \Rightarrow r = \frac{GM}{\sigma^2}$

the smallest halo has radius

$$R \simeq 0.3 \frac{\hbar^2}{GMm^2}$$ 1610.08297

$\gtrsim$ smallest dwarf-galaxies observed ($R_d \sim 1$ kpc)

$\Rightarrow m \gtrsim 10^{-22}$ eV

* lower bound: fermions (Tremaine - Gunn bound)

Galactic halo w/ radius R_h, assume spherical symmetry

Requirement: phase space density does not exceed that of degenerate Fermi gas

$$\langle n_{\vec{k}} \rangle = g \begin{cases} 1 & \text{for } k < k_F \\ 0 & \text{for } k > k_F \end{cases}$$

$$N < g V \left(\frac{1}{\hbar}\right)^3 \int \frac{d^3k}{(2\pi)^3} = g \frac{V}{6\pi^2} \left(\frac{k_F}{\hbar}\right)^3$$

Halo mass is

$$M_h = N m$$

Take $k = m v$ (non-relativistic gas)

$$\Rightarrow \quad M_h < m g \frac{V}{6\pi^2} \frac{m^3 v_F^3}{\hbar^3} = m g \frac{\frac{4}{3}\pi R_h^3}{6\pi^2} \frac{m^3 v_F^3}{\hbar^3}$$

$\frac{2}{9\pi}$

$$\Rightarrow \quad v_F > \left(\frac{9\pi \hbar^3}{2 g m^4} \frac{M_h}{R_h^3}\right)^{1/3} =$$

Now impose $v_F < v_{esc}$

$$\left(\frac{1}{2} m v_{esc}^2 = \frac{G M_h m}{R_h} \qquad v_{esc}^2 = \frac{2 G M_h}{R_h}\right)$$

$$\Rightarrow \left(\frac{9\pi \hbar^3}{2 g m^4} \frac{M_h}{R_h^3}\right) < \left(\frac{2 G M_h}{R_h}\right)^{3/2}$$

$$m^4 > \frac{9\pi}{4\sqrt{2}} \frac{\hbar^3}{g} M_h^{-1/2} R_h^{-3/2} G^{-3/2}$$

$$\Rightarrow \begin{cases} m > 30 \text{ eV} \left(\frac{2}{g}\right)^{1/4} \left(\frac{10^{11} M_\odot}{M_h}\right)^{1/2} \left(\frac{10 \text{ kpc}}{R_h}\right)^{3/2} & \text{(Large Magellanic Cloud)} \\ m > 0.8 \text{ keV} \left(\frac{2}{g}\right)^{1/4} \left(\frac{10^{6} M_\odot}{M_h}\right)^{1/2} \left(\frac{70 \text{ pc}}{R_h}\right)^{3/2} & \text{(Coma Berenices)} \end{cases}$$

Summary of Tremaine - Gunn:

- Fermionic DM lighter than $\sim O(10^2\,eV)$ is disfavoured
- Lighter (M_{eV}) and smaller (R_{eV}) galaxies give the most stringent bounds
- Uncertainties: exact determination of M_{e}, R_{e}, spherical symmetry assumption, uniform density (need some refinement)

- avoid the bound: self-interactions or degenerate particles ($g \gg 2$)

* "Statistical" Tremaine - Gunn bound

Liouville's theorem: flow in phase space of the halo seen as a N-body system is "incompressible"

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} = 0 \quad \text{ie} \quad \frac{\partial f}{\partial t} + \sum_i \left(\dot{\vec{x}}_i \cdot \frac{\partial f}{\partial \vec{x}_i} + \dot{\vec{p}}_i \, \frac{\partial f}{\partial \vec{p}_i} \right) = 0$$

$\Rightarrow$ the maximum of f remains constant in time

Therefore at any time $\bar{f} < f_{max}$

$\bar{f}$ is the "coarsed-grain" distribution, ie some average distrib. that we may have the chance to measure

$$\bar{f}(t, x, v) = \frac{1}{vol(\Delta\pi)} \int_{\Delta\pi} d\pi' \; f(t, x', v')$$

If I know the initial and final distribution I can set a bound on m

eg: initial: FD $\quad \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{p/(k_B T)}+1} \leq \frac{1}{2}\frac{g}{(2\pi\hbar)^3}$

final: isothermal sphere $\quad \bar{f}_{iso,max} = \frac{9\sigma^2}{4\pi G_N (2\pi^2\sigma^2)^{3/2} r_c^2}$

$$\bar{f}_{iso,max} < \frac{g}{2(2\pi\hbar)^3}$$

$$\Rightarrow \quad m > \frac{9(2\pi\hbar)^3}{(2\pi)^{5/2}\, g\, G \sigma r_c^2}$$

Use virial theorem for σ $\Rightarrow$ same bound as before (up to a factor $2^{1/4}$)

NB: I can apply this to bosons as well, if I know that they don't populate the small p region

→ ULA axions are the exception

Refs: 0808.3902 reviews Tremaine-Gunn for DM

hep-ph/0404239 mention bosons

Binney & Tremaine, Galactic Dynamics

↳ 4.33 ∋ isothermal sphere

7.2 Liouville's theorem

Charge neutrality

DM is dark. Most stringent limit comes for DM being decoupled at recombination

(remember: for CMB it is crucial that baryons $-\gamma$ are coupled, DM is not)

$$\left(q \gtrsim e \times \begin{cases} 3.5 \times 10^{-7} \left(\frac{m}{1\,\mathrm{GeV}}\right)^{0.58} & m > 1\,\mathrm{GeV} \\ 4.0 \times 10^{-7} \left(\frac{m}{1\,\mathrm{GeV}}\right)^{0.35} & m < 1\,\mathrm{GeV} \end{cases} \right)$$

numbers from the PDG review

10.11.2907

$q \gtrsim 10^{-6}\, e$ at $1\,\mathrm{GeV}$

$q \gtrsim 10^{-4}\, e$ at $10\,\mathrm{TeV}$

Self-interactions

limits from merger of clusters and ellipticity of certain galaxies

$$\frac{\sigma}{m} \lesssim 1 \frac{\mathrm{cm}^2}{g} \sim 1 \frac{\mathrm{barn}}{\mathrm{GeV}}$$

(pretty mild: barn is typical of U nuclear scattering)

Numerically, this limit means DM interacts once in the galactic lifetime

$$n_{DM}\, \sigma\, v\, \tau \simeq \frac{\rho}{m} \sigma\, v\, \tau \sim 10^{-2}$$

$\rho_{DM} \simeq 0.3\,\mathrm{GeV\,cm^{-3}}$

$v \approx 220\,\mathrm{km/s}$

$\tau \approx 10^{8}\,\mathrm{yr}$

(more for the larger ρ in the center)

Stability

DM was around at CMB formation and is still around today

$\Rightarrow$ it must be stable (or long-lived)

* $\tau > \tau_U \simeq 14\ \text{Gyr} \gtrsim 14 \times 10^9 \times \pi\, 10^7\, s = 4 \times 10^{17}\, s$

* If DM decays into relativistic particles, energy is transferred from the "matter" to the "radiation" component, thus altering both the expansion history and structure formation

$$\Rightarrow \quad \tau_{DM} \gtrsim 160\ \text{Gyr} \simeq 5 \times 10^{18}\, s \qquad (1407.2418)$$

* Model-dependent bounds: if the decay products are charged SM particles they can affect the CMB (depending on the branching ratios) and be looked for in indirect detection.

Small scale problems

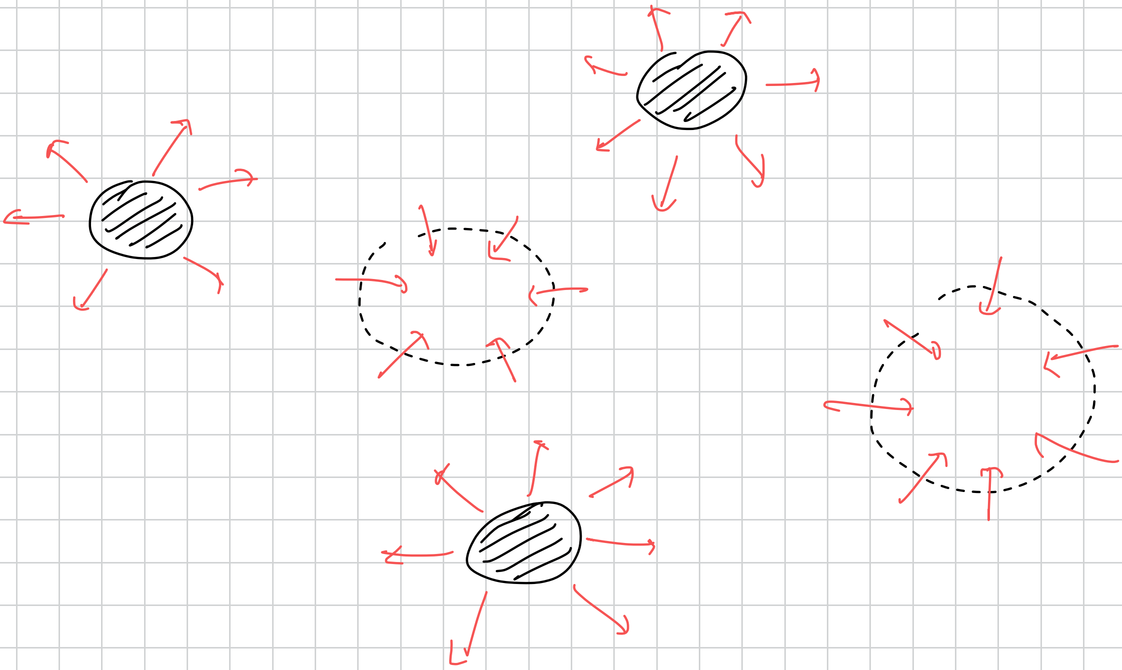
Astrophysical observations seem to be in tension with the pure CDM picture. All tensions are related with "small" scales, where observations are complicated and simulations as well

- smooth profile at the center of galaxies
- abundance of dwarf galaxies
- ...

Can be solved by WDM, self-interactions but also by baryonic effects not properly modelled in simulations.

Coldness: DM is non-relativistic

Suppose DM is produced non-relativistic. It is also collisionless. If there are overdensities, DM will freely stream from the overdense to the underdense regions



This is called collisionless (Landau) damping: perturbations are smoothed out.

The process is active until matter-radiation equality. After that inhomogeneities start to grow.

Perturbations are smoothed up to a scale ≈ distance travelled by DM from their formation to t_{eq} (free-streaming length)

$$\lambda_{FS} = \Delta r = \int_{t_x}^{t_{eq}} \frac{v(t')}{a(t')}\, dt'$$

Assume DM is produced relativistic. Becomes NR at t_{NR}

$$\Rightarrow \quad \lambda_{FS} = \int_{t_x}^{t_{NR}} \frac{1}{a(t')}\, dt' + \int_{t_{NR}}^{t_{eq}} \frac{v(t')}{a(t')}\, dt'$$

After t_{NR} : $p = mv$, $p \propto a^{-1}$ $\Rightarrow$ $v \propto a^{-1}$

$$\Rightarrow \quad v(t') = v(t_{NR})\, a(t_{NR}) / a(t') = a_{NR}/a$$

In radiation dominance $a \propto t^{1/2}$ $\Rightarrow$ $a = a_{NR} \left(\frac{t}{t_{NR}}\right)^{1/2}$

and $v/a = a_{NR}/a^2$

$$\Rightarrow \quad \lambda_{FS} = \int_{t_x}^{t_{NR}} \frac{t_{NR}^{1/2}}{a_{NR}} \frac{dt'}{t'^{1/2}} + \int_{t_{NR}}^{t_{eq}} \frac{a_{NR}}{a_{NR}^2} \frac{t_{NR}}{t'} dt'$$

$$= 2 \frac{t_{NR}^{1/2}}{a_{NR}} \Big(t_{NR}^{1/2} - \overbrace{t_x^{1/2}}^{\approx 0} \Big) + \frac{t_{NR}}{a_{NR}} \log \frac{t_{eq}}{t_{NR}}$$

$$\approx \frac{t_{NR}}{a_{NR}} \left(2 + \log \frac{t_{eq}}{t_{NR}} \right)$$

Assume initially DM has a thermal distribution with temperature T_x (often $T_x = T_\gamma$, sometimes $T_x < T_\gamma$).

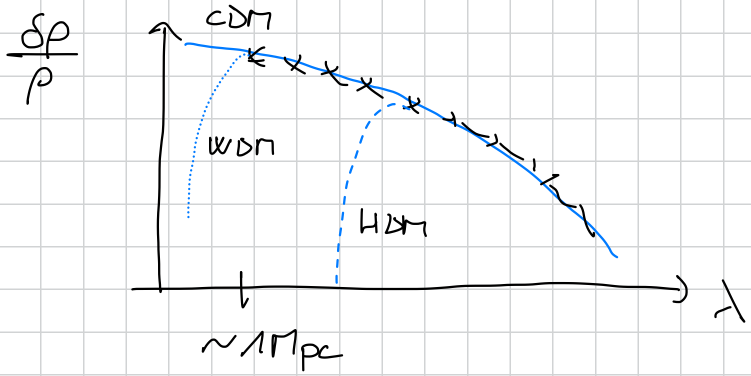
t_{NR} corresponds to $T_x = m_x/3$

$$\Rightarrow \begin{cases} t_{NR} = 1.2 \times 10^{7} \left(\frac{m_x}{1\,\mathrm{keV}}\right)^{-2} \left(\frac{T_x}{T}\right)^2 \,\mathrm{s} \\ a_{NR} = 7.1 \times 10^{-7} \left(\frac{m_x}{1\,\mathrm{keV}}\right)^{-1} \left(\frac{T_x}{T}\right) \end{cases}$$

$$\rightarrow \quad \lambda_{FS} \simeq 0.2\,\mathrm{Mpc} \left(\frac{m_x}{1\,\mathrm{keV}}\right)^{-1} \left(\frac{T_x}{T}\right) \Big(2 + \underbrace{\log \frac{t_{eq}}{t_{NR}}}_{\hookrightarrow 3}\Big)$$

$$\lambda_{FS} \approx 1\,\mathrm{Mpc} \left(\frac{m_x}{1\,\mathrm{keV}}\right)^{-1}$$

Study of large scale structures give us information on the "matter power spectrum"



$\Rightarrow m_{DM} \gtrsim 1\,\mathrm{keV}$ (if $T_X = T_\gamma$)

(*) For $t > t_{eq}$, λ_{FS} saturates. (In MD $a \propto t^{2/3}$)

$$\lambda_{FS} = \lambda_{FS}^{eq} + \int_{t_{eq}}^{t} \frac{a_{NR}}{a^2}\, dt' = \lambda_{FS}^{eq} + \int_{t_{eq}}^{t} a_{NR} \left[a_{eq} \left(\frac{t'}{t_{eq}} \right)^{2/3} \right]^{-2} dt'$$

$$= \lambda_{FS}^{eq} + \frac{a_{NR}}{a_{eq}^2} t_{eq}^{4/3} \int_{t_{eq}}^{t} \frac{dt'}{t'^{4/3}} = \lambda_{FS}^{eq} + a_{NR} \frac{t_{eq}^{4/3}}{a_{eq}^2}\, 3 \left(\frac{1}{t_{eq}^{1/3}} - \frac{1}{t^{1/3}} \right)$$

$$= \lambda_{FS}^{eq} + 3 a_{NR} \frac{t_{eq}}{a_{eq}^2} \left(1 - \frac{t_{eq}^{1/3}}{t^{1/3}} \right)$$

$\frac{t_{eq}^{1/3}}{t^{1/3}} \rightarrow (a_{eq}/a)^{1/2}$

$\frac{t_{NR}}{a_{NR}^2} = \frac{t_{eq}}{a_{eq}^2}$

$$= \lambda_{FS}^{eq} + 3 \frac{t_{NR}}{a_{NR}} \left(1 - \left(\frac{a_{eq}}{a} \right)^{1/2} \right)$$

$\Rightarrow$ max value: $\lambda_{FS} = 2 \frac{t_{NR}}{a_{NR}} \left(\frac{5}{2} + \log \frac{a}{a_{NR}} \right)$

[see notes by Riotto]

DARK MATTER CANDIDATES

What kind of particle can DM be?

Almost anything: stable, electrically neutral, broad mass

Other interactions w/ the SM? If not, that's it.

Construct a DM model

- top-down: we start from a theory, maybe SUSY or GUT, and identify a stable, neutral relic.
- bottom-up: take the SM and add the minimal extra content: a DM particle, maybe a mediator

Some examples:

- WIMPs: (Weakly Interacting Massive Particles) interact through EW interactions (Z, H)
  $1\,\text{GeV} \lesssim m_\chi \lesssim 10\,\text{TeV}$ $\sigma \simeq 1\,\text{pb}$ at small v
  for extension: anything in this ranges is a WIMP
- asymmetric DM: $\rho_{DM} \simeq 5\,\rho_B$ instead of $\gg$ or $\ll$.
  Maybe related? Some models of baryogenesis predict a stable neutral relic w/ abundance $\sim$ to Ω_B.
- sterile neutrinos: related to neutrino masses, interesting mass scale is keV
- axions: very light scalars, behave as a classical field

Thermal production of DM and freeze-out

DM in thermal eq w/ the SM through interactions

$$\begin{cases} \chi\chi \longleftrightarrow SM + SM & \text{number-changing} & \text{chemical eq} \\ \chi + SM \longleftrightarrow \chi + SM & \text{elastic interactions} & \text{kinetic eq} \end{cases}$$

Consider DM initially in thermal (chemical and kinetic) eq

$$n_\chi^{eq} = \begin{cases} g_{eff} \dfrac{\zeta(3)}{\pi^2} T^3 & T \gg m_\chi \\ g_\chi \left(\dfrac{m_\chi T}{2\pi}\right)^{3/2} \exp\left(-\dfrac{m_\chi}{T}\right) & T \ll m_\chi \end{cases}$$

$$g_{eff} = \begin{cases} g_\chi & (\text{bosons}) \\ \frac{3}{4} g_\chi & (\text{fermions}) \end{cases}$$

If number-changing interactions are not efficient $\Rightarrow$ $n_\chi a^3 = \text{const.}$
The interesting part is understanding what happens at decoupling

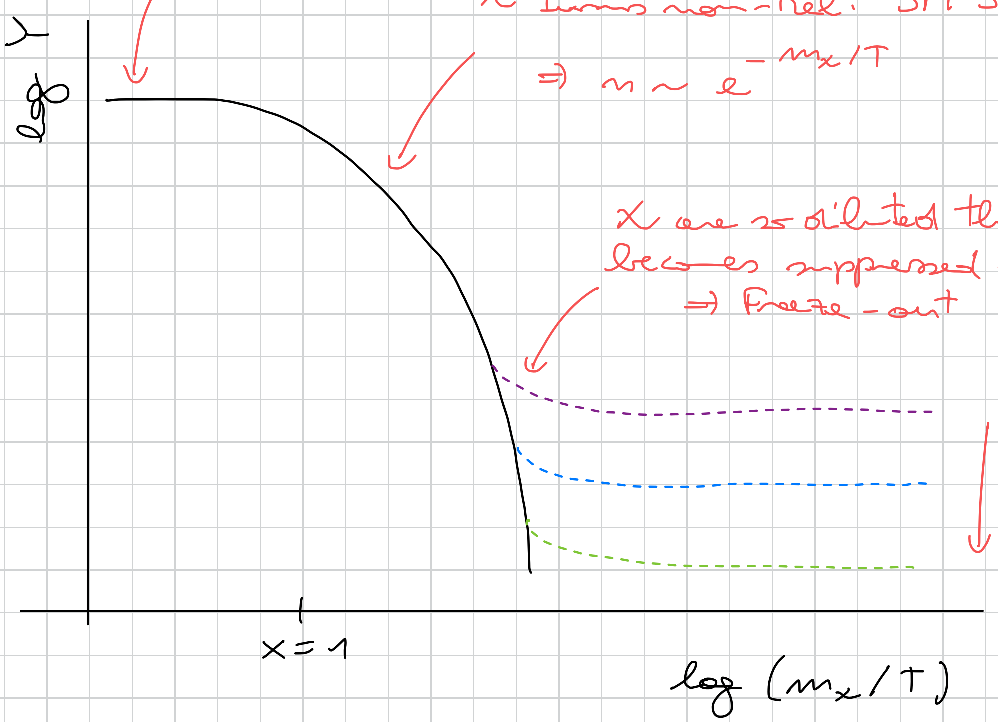
Recall that we defined the yield $Y_\chi = \dfrac{n_\chi}{s}$ $\left(\sim n_\chi a^3\right)$

χ relativistic: Y constant

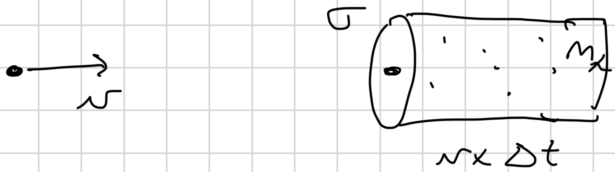
χ turns non-rel: $SM\ SM \to \chi\chi$ suppressed
$\Rightarrow n \sim e^{-m_\chi/T}$

χ are so diluted that $\chi\chi \to SM\ SM$ becomes suppressed as well
$\Rightarrow$ Freeze-out

increasing $\Gamma_{ann.}$



Interaction rate <u>per particle</u>: how many times does one particle interact in a unit time



$$\# = \Gamma \cdot \Delta t = n_x \sigma v \Delta t \quad \Rightarrow \quad \Gamma = n_x \sigma v$$

v: relative velocity (in the frame of the target particle)

σ: depends on v $\Rightarrow$ take a thermal average

$$\Rightarrow \quad \Gamma = n_x \langle \sigma v_{rel} \rangle$$

$\downarrow$

(NB: this is $\langle \sigma v_{Møl} \rangle$ in the cosmic frame or $\langle \sigma v_{lab} \rangle^{lab}$ in the lab frame (which is the same), but <u>not</u> $\langle \sigma v_{cm} \rangle^{cm}$.)

Departure from thermal eq

Departure from thermal eq is necessary: otherwise the Universe would be very boring

Thermalization is achieved through collisions

In order to preserve thermal eq, we need a large collision rate

Count # of occurrences of a given process in the time interval $[t_1, t_2]$:

$$N_{coll} = \int_{t_1}^{t_2} dt\ \Gamma(t) = \int_{T_1}^{T_2} dT\ \frac{dt}{dT}\ \Gamma(T)$$

neglecting T dependence of g_{*S} $\quad Ta = \text{const} \Rightarrow \frac{dT}{dt} = -HT$

$$\Rightarrow N_{coll} = \int_{T_2}^{T_1} \frac{dT}{T}\frac{\Gamma}{H} \approx \frac{\Gamma(T)}{H(T)}\frac{\Delta T}{T}$$

In the time it takes to change the temperature by $\Delta T \sim T$, a particle undergoes Γ/H interactions: if $\Gamma \gg 1$, it can track the evolution of the temperature, otherwise it cannot.

$$\Rightarrow N_{coll} \approx \frac{\Gamma(T)}{H(T)}$$

Typical dependence: $\Gamma(T) \propto T^n \qquad \frac{\Gamma}{H} \propto T^{n-2}$

For $n > 2 \Rightarrow \Gamma/H$ decreases: if a particle is out of eq at some point, it will also be at lower temperature

General criterion:

$$\frac{\Gamma(T)}{H(T)} : \begin{cases} \gtrsim 1 & \text{species coupled} \\ \lesssim 1 & \text{species decoupled} \end{cases}$$

Hot freeze-out

A particle may decouple when it is still relativistic. Freeze-out:

$$n_x \langle \sigma v \rangle = H$$

$$g_{eff} \frac{\zeta(3)}{\pi^2} T_f^3 \langle \sigma v \rangle = \sqrt{\frac{\pi^2}{90}} \frac{T_f^2}{M_P}$$

$$\Rightarrow T_f = \underbrace{\frac{\pi^3}{3\zeta(3)\sqrt{10}}}_{\approx 2.7} \underbrace{\frac{g_*^{1/2}(T_f)}{g_{eff}}}_{\sim \frac{10.33}{2}} \frac{1}{\langle \sigma v \rangle M_P}$$

Self-consistency: $T_f \gg m_x$ for small m_x or small σ

DM yield: $$Y_x^f = \frac{n_x(T_f)}{s(T_f)} \approx \frac{n_x^{eq}(T_f)}{s(T_f)} = \dots$$

$$= \frac{45\zeta(3)}{2\pi^4} \frac{g_{eff}}{g_{*s}(T_f)} = 0.0026\, g_{eff} \left(\frac{106.75}{g_{*s}(T_f)} \right)$$

At present day $Y_x^0 = Y_x^f$ and $\rho_x = m_x n_x^0 = m_x Y_x^f s_0$

we know $\rho_{crit,0} = 1.05 \times 10^{-5}\, h^2\, \mathrm{GeV\, cm^{-3}}$

$s_0 = 2891.2\ \mathrm{cm^{-3}}$

$$\Rightarrow \Omega_x h^2 = 0.076 \left(\frac{g_{eff}}{g_{*s}(T_f)} \right) \left(\frac{m_x}{1\,\mathrm{eV}} \right)$$

From $\Omega_x h^2 \lesssim 0.12$ we obtain the Cowsik–McClelland bound:

$$m_x \lesssim 168\,\mathrm{eV} \frac{1}{g_{eff}} \left(\frac{g_{*s}(T_f)}{106.75} \right)$$

Distribution after decoupling (relativistic particles)

After decoupling a particle does not interact $\Rightarrow$ free particle

$$E(t) \approx |\vec{p}(t)| = |\vec{p}(t_f)| \left(\frac{a(t_f)}{a(t)}\right)$$

$$E(T) = E(T_f)\left(\frac{a(T_f)}{a(T)}\right)$$

At $T = T_f$: particles w/ energy $[E, E+dE]$:

$$dn(E, t_f) = f(E, t_f)\frac{d^3p}{(2\pi)^3} = \frac{1}{2\pi^2} f(E, t_f)\, E^2\, dE$$

At $T < T_{dec}$, 1) their energy will be $\frac{a(T_f)}{a(T)}$ smaller

2) their number density will be $\left(\frac{a(T_f)}{a(T)}\right)^3$ smaller

$$\Rightarrow f(E,t) = f\left(\frac{a(T)}{a(T_f)}E,\, t_f\right)$$

The distribution is thermal, with a temperature that scales as

$Ta = \text{const}$ (thermalized particles)

as opposed to $T a\, g_{*S}^{1/3} = \text{const}$ (decoupled particles)

Number density:

$$n(t) = \int_0^\infty \frac{1}{2\pi^2} f(E,t)\, E^2\, dE = \frac{1}{2\pi^2}\int_0^\infty f\left(\frac{a(T)}{a(T_f)}E,\, t_f\right) E^2\, dE =$$

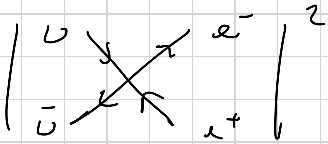
$$= \frac{1}{2\pi^2}\int_0^\infty f(E', t_f)\left(\frac{a(T_f)}{a(T)}\right)^3 E'^2\, dE' = \left(\frac{a(T_f)}{a(T)}\right)^3 n(t_f)$$

Application: neutrino decoupling

Neutrinos freeze-out when relativistic

$$\nu_e + \bar{\nu}_e \longleftrightarrow e^+ + e^-$$

Cross section: comes from



$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\left[\bar{e}\,\gamma^\mu (c_V - c_A \gamma_5)\, e\right]\left[\bar{\nu}\,\gamma_\mu (1-\gamma_5)\,\nu\right] \rightsquigarrow \sigma \sim G_F^2 T^2$$

$$\Rightarrow \Gamma \sim G_F^2 T^5 \qquad \text{w/} \quad G_F \approx 1.17 \times 10^{-5}\ \mathrm{GeV}^{-2}$$

Decoupling:

$$\frac{\Gamma}{H} \sim \left(\frac{T}{\mathrm{MeV}}\right)^3 \Rightarrow T_f \sim 1\,\mathrm{MeV}$$

at this temperature $g_{*s} = 10.75$

and for neutrinos $g_{eff} = 3/2$

$$\Rightarrow \Omega_\nu h^2 \approx 0.001 \left(\frac{m_\nu}{0.1\,\mathrm{eV}}\right)$$

"Cosmic neutrino background" → similar to CMB, but much harder to detect. The temperature is lower by a factor

$$T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3} = 1.96\,\mathrm{K}$$

because, around $T \sim m_e$, e become non-rel $\Rightarrow g_{*S}$

below MeV $\qquad a T_\nu = \text{const}$

$$a\, g_{*S}^{1/3}\, T_\gamma = \text{const}$$

Cold freeze-out

Assume at freeze-out $T \ll m_x$. $\Rightarrow v \ll 1$.

Define $x = \frac{m_x}{T}$

I can expand the cross section in v

$$\sigma v = a + b v^2 + \dots$$

one can show $\langle \sigma v \rangle = a + b \langle v^2 \rangle \simeq a + 6b/x$

Here assume for simplicity $\langle \sigma v \rangle \simeq a \neq 0$ (may break down)

Freeze-out: $\Gamma = H$

$$g_x \left(\frac{m_x T_f}{2\pi} \right)^{3/2} e^{-\frac{m_x}{T_f}} \langle \sigma v \rangle = \left(\frac{\pi^2}{90} g_*(T_f) \frac{T_f^4}{M_p^2} \right)^{1/2}$$

use $T_f = m_x / x_f$

$$\rightsquigarrow \quad \frac{e^{x_f}}{x_f^{1/2}} = \frac{3\sqrt{5}}{2\pi^{5/2}} \frac{g_x}{g_*^{1/2}(x_f)} m_x M_p \langle \sigma v \rangle$$

Take a log

$$x_f = \frac{1}{2} \log x_f + \log\left(\frac{3\sqrt{5}}{2\pi^{5/2}} \right) + \log \frac{g_x}{g_*^{1/2}(x_f)} + \log\left(m_x M_p \langle \sigma v \rangle \right)$$

Take some number typical of weak interactions:

$$m_x \approx 100\,\text{GeV} \qquad \langle \sigma v \rangle \simeq 1\,\text{pb} = 2.6 \times 10^{-9}\,\text{GeV}^{-2}$$

$$\Rightarrow \quad x_f \approx 25 + \frac{1}{2}\log x_f + \log \frac{g_\chi}{g_*(x_f)^{1/2}}$$

$$x_f = \frac{m_\chi}{T_f} \approx 25 \quad \Rightarrow$$ non-rel approx is self-consistent

Relic density

$$Y_\chi(T_f) = \frac{n_\chi(T_f)}{s(T_f)}$$

$$n_\chi = \frac{\Gamma}{\langle\sigma v\rangle} = \frac{H(T_f)}{\langle\sigma v\rangle} = \left(\frac{\pi^2}{30}\, g_*(T_f)\, \frac{T_f^4}{3M_p^2}\right)^{1/2} \frac{1}{\langle\sigma v\rangle}$$

$$s = \frac{2\pi^2}{45}\, g_{*s}\, T_f^3$$

$$\Rightarrow Y_\chi(T_f) = \frac{45\, \pi\, g_*^{1/2}(T_f)\, T_f^2}{2\pi^2\, g_{*s}(T_f)\, T_f^3 \quad 3\sqrt{10}\, M_p \,\langle\sigma v\rangle^{1/2}}$$

$$= \frac{3\sqrt{5}}{2\sqrt{2}\,\pi}\; \frac{g_*^{1/2}(T_f)}{g_{*s}(T_f)}\; \frac{x_f}{M_p\, m_\chi \langle\sigma v\rangle^{1/2}}$$

Today's abundance:

$$\Omega_\chi h^2 = \frac{\rho_\chi}{\rho_{crit,0}} = \frac{m_\chi Y_\chi(T_f)\, s_0}{\rho_{crit,0}} \approx 2\times 10^8\, \frac{g_*^{1/2}(T_f)}{g_{*s}(T_f)}\, \frac{x_f}{M_p\langle\sigma v\rangle}\; \mathrm{GeV}^{-1}$$

$$\boxed{\Omega_\chi h^2 \approx 0.12 \left(\frac{106.75}{g_*(T_f)}\right)^{1/2} \left(\frac{0.7\,\mathrm{pb}}{\langle\sigma v\rangle}\right)}$$

pb is typical of weak interactions: "WIMP miracle"

WIMP miracle

Example: Dirac fermion DM, coupled to SM fermions via a vector mediator

$$\mathcal{L} = \mathcal{L}_{SM} + \bar{\chi}(i\partial\!\!\!/ - m_\chi)\chi + \frac{1}{2} M^2 V_\mu V^\mu - g_\chi V_\mu \bar{\chi}\gamma^\mu \chi - g_q V_\mu \bar{q}\gamma^\mu q$$

DM annihilation: $\chi\bar{\chi} \to q\bar{q}$ $\qquad q \in SM$

$$(\sigma v) \approx \frac{N_c\, g_q^2 g_\chi^2}{2\pi} \frac{\sqrt{1 - m_q^2/m_\chi^2}}{(M^2 - 4m_\chi^2)^2 + \Gamma^2 M^2} \left[(m_q^2 + 2m_\chi^2) + v^2 (\ldots\ldots) \right]$$

for simplicity: $\mathcal{O}(v^2) \to 0$; $m_q \ll m_\chi \ll M^2$

$$\Rightarrow \langle \sigma v \rangle \approx \frac{N_c\, g_q^2 g_\chi^2}{\pi} \frac{m_\chi^2}{M^4}$$

assume $g_q \approx g_\chi \approx 0.1$, $N_c = 3$

$$\langle \sigma v \rangle \approx 10^{-4} \frac{m_\chi^2}{M^4}$$

Convert GeV to length

$$1 = \hbar c \approx 197\ \text{MeV fm} \approx 2 \times 10^2 \times 10^{-3}\ \text{GeV fm}$$

$$\Rightarrow 200\ \text{GeV} = 10^3\ \text{fm}^{-1}$$

Take $M = 200$ GeV, $m_\chi = 2$ GeV

$$\Rightarrow \langle \sigma v \rangle \approx 10^{-4} \frac{10^2\ \text{fm}^{-2}}{10^{12}\ \text{fm}^{-4}} \approx 10^{-14}\ \text{fm}^2 = 1\ \text{pb}$$

WIMP miracle: a stable neutral particle with weak scale mass and interactions naturally accounts for the observed DM abundance. We do have one "natural" candidate: SUSY's LSP (name χ comes from the neutralino)

No miracle: it's not so special after all: $\exists$ can have eg a much lighter sector with $\alpha \ll 1$.

Nevertheless WIMPs are probably the best motivated DM targets and the best to be searched for.

Chemical vs Kinetic Decoupling

For non-relativistic particles, kinetic decoupling typically takes place well after the chemical one

$$\chi\chi \to f\bar{f}$$ ← the rate drops first, because of the scarcity of χ particles

$$\chi + f \to \chi + f$$ ← Keeps $T_\chi = T$ until kinetic decoupling

Typical momentum: $\frac{p_\chi^2}{2m_\chi} \sim k_B T \Rightarrow p_\chi \sim (m_\chi T)^{1/2}$

(light) thermal bath particles have $p \sim T$, thus the momentum transferred to a particle in each collision is $\delta p \sim T$.

In order to change appreciably the momentum of a particle I need

$$N_{kicks} = \left(\frac{p}{\delta p}\right)^2 = \frac{m_\chi}{T}$$

I take the square because it's a random walk, $\Delta p \propto \sqrt{N}$

Kinetic freeze-out happens when

$$\Gamma_{elastic}(T_{kf}) = N_{kicks} H(T_{kf})$$

$$\frac{T_{kf}}{m_\chi} n_{SM} \langle \sigma_{elastic} v \rangle_{kf} = H_{kf}$$

Assuming that the annihilation and elastic cross sections are of the same order of magnitude, kf is retarded

$$H = \left(\frac{\pi^2}{90} g_* \frac{T^4}{M_p^2} \right)^{1/2} \qquad n_{SM} = g_i \frac{\zeta(3)}{\pi^2} T^3$$

$$\langle \sigma v \rangle_{kf} = \frac{\pi^3 g_*^{1/2}}{3\sqrt{10}\, g_i\, \zeta(3)} \frac{m_x}{M_p T_{kf}^2}$$

Assume $\langle \sigma v \rangle \sim \frac{g^4 T^2}{M^4}$

$$\Rightarrow T_{kf} \approx \left(\frac{\pi^3}{3\sqrt{10}\,\zeta(3)} \frac{g_*^{1/2}}{g_i} \frac{m_x M^4}{M_p^2 g^4} \right)^{1/4}$$

$$\approx 150 \text{ MeV} \left(\frac{g_*}{106.75}\right)^{1/8} \left(\frac{2}{g_i}\right)^{1/4} \left(\frac{m_x}{100\text{GeV}}\right)^{1/4} \left(\frac{M}{100\text{GeV}}\right) \left(\frac{0.1}{y}\right)$$

$$x_{kf} = \frac{m_x}{T_{kf}} \approx 645 \left(\frac{g_*}{106.75}\right)^{-1/8} \left(\frac{2}{g_i}\right)^{-1/4} \left(\frac{m_x}{100\text{GeV}}\right)^{3/4} \left(\frac{M}{100\text{GeV}}\right)^{-1} \left(\frac{0.1}{y}\right)^{-1}$$

Summary: x_f is easily of order $100 - 1000$. During freeze-out, we can safely assume that the distribution of x is $\propto n_x^{eq}$, with no spectral distortion.

Distribution after decoupling (non-relativistic)

[skip]

$$E \approx m + \frac{p^2}{2m}$$

$$f = \frac{1}{\exp\left[\left(m + \frac{p^2}{2m} - \mu\right)/T\right] \pm 1}$$

n must scale as a^{-3}. p scales as a^{-1} $\Rightarrow$ $\frac{p^2}{2m} \sim a^{-2}$

If a particle has momentum p at t_f and the chemical pot is μ, at some later time t' $p \to p' = p\,\frac{a_f}{a}$,

$$\Rightarrow f(p', t') = f(p, t_f) = f\left(p'\frac{a}{a_f}, t_f\right)$$

$$= \frac{1}{\exp\left[\left(m + \frac{p'^2}{2m}\left(\frac{a}{a_f}\right)^2 - \mu_f\right)/T_f\right] \pm 1}$$

$$= \frac{1}{\exp\left[\frac{m-\mu_f}{T_f} + \frac{p'^2}{2m}\left(\frac{a}{a_f}\right)^2 / T_f\right] \pm 1}$$

Define $T = \left(\frac{a_f}{a}\right)^2 T_f$

$$\frac{m-\mu_f}{T_f} = \frac{m-\mu_f}{T}\left(\frac{a_f}{a}\right)^2 \equiv \frac{m-\mu}{T}$$

$\Rightarrow$ I define μ such that $m - \mu = (m-\mu_f)\left(\frac{a_f}{a}\right)^2$

$$\Rightarrow \mu = m - \left(\frac{a_f}{a}\right)^2 (m-\mu_f)$$

Boltzmann equation

The evolution of the phase space distribution f is governed by the Boltzmann equation

$$\mathcal{L}[f] = \mathcal{C}[f] \longrightarrow$$ collision term: interactions between particles

Liouville operator

in the non-rel world

$$\mathcal{L} = \frac{d}{dt} + \frac{d\vec{x}}{dt} \cdot \vec{\nabla}_x + \frac{d\vec{v}}{dt} \vec{\nabla}_v = \frac{d}{dt} + \underbrace{\vec{v} \cdot \vec{\nabla}_x}_{\text{diffusion}} + \underbrace{\frac{\vec{F}}{m} \cdot \vec{\nabla}_v}_{\text{external forces}}$$

Relativistic generalization:

$$\mathcal{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$$

This can be seen as a derivative wrt the affine parameter.
Give a look to Francesco's book and try to understand it

Homogeneity: $p^\alpha \frac{\partial}{\partial x^\alpha} f = E \frac{\partial}{\partial t} f \qquad (\vec{\nabla} f = 0)$

Isotropy: $f(\vec{p}, t) = f(|\vec{p}|, t) \Rightarrow \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} = H |\vec{p}|^2 \frac{\partial f}{\partial E}$

$$\Rightarrow \mathcal{L}[f] = E \frac{\partial f}{\partial t} - H |\vec{p}|^2 \frac{\partial f}{\partial E}$$

Integrated Liouville operator

Integrate both sides of the eq with the measure $g_x \dfrac{d^3 p_x}{(2\pi)^3 E_x}$

$$g_x \int \frac{d^3 p_x}{(2\pi)^3 E_x} \mathcal{L}[f] = g_x \int \frac{d^3 p}{(2\pi)^3} \left(\frac{\partial f}{\partial t} - H \frac{|\vec{p}|^2}{E} \frac{\partial f}{\partial E} \right)$$

$$= \dot{n}_x - \frac{g_x H}{(2\pi)^3} \int dp \, 4\pi \frac{p^4}{E} \frac{\partial f}{\partial E} \qquad E^2 = p^2 + m_x^2$$

$$= \dot{n}_x - \frac{g_x H}{(2\pi)^3} \int dE \, 4\pi \, E^3 \frac{\partial f}{\partial E} \qquad E\,dE = p\,dp$$

$$= \dot{n}_x - \frac{g_x H}{(2\pi)^3} 4\pi \left(E^3 f \right)_m^\infty + \frac{g_x H}{(2\pi)^3} \int dE \, 4\pi \, 3 E^2 f$$

$$= \dot{n}_x - g_x H \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{E} \frac{dp}{dE} \frac{\partial f}{\partial p} \qquad E^2 = p^2 + m^2 \Rightarrow p\,dp = E\,dE$$

$$= \dot{n}_x - g_x H \int \frac{d^3 p}{(2\pi)^3} \, p \frac{\partial f}{\partial p}$$

$$= \dot{n}_x - g_x H \int \frac{dp \, 4\pi}{(2\pi)^3} \, p^3 \frac{\partial f}{\partial p}$$

$$= \dot{n}_x - \underbrace{g_x H \frac{4\pi}{(2\pi)^3} \left(p^3 f \right)_0^\infty}_{0} + g_x H \int dp \frac{4\pi}{(2\pi)^3} 3 p^2 f$$

$$= \dot{n}_x + 3 H g_x \int \frac{d^3 p}{(2\pi)^3} f = \dot{n}_x + 3 H n_x$$

$$g_x \int \frac{d^3 p_x}{(2\pi)^3 E_x} \mathcal{L}[f] = \dot{n}_x + 3 H n_x$$

NB: this is nothing else than

$$\frac{1}{a^3} \frac{d}{dt} \left(n_x a^3 \right) = \frac{1}{a^3} \left(a^3 \dot{n}_x + 3 \frac{\dot{a} a^3}{a} n_x \right) = \dot{n}_x + 3 H n_x$$

Collision operator

Suppose a particle χ undergoes scattering of the type

$$\chi + a + b \ldots \longleftrightarrow i + j + \ldots$$

(I need to sum a term for each process)

sum over spins and all internal dof, eg color

$$C[f] = -\frac{1}{2} \sum_{ab\ldots ij\ldots} \int d\Pi_a \, d\Pi_b \ldots d\Pi_i \, d\Pi_j \ldots \times$$
$$\times (2\pi)^4 \delta^4(p_\chi + p_a + p_b + \ldots - p_i - p_j - \ldots)$$
$$\times \Big[|\mathcal{M}|^2_{\chi+a+b+\ldots \to i+j+\ldots} f_\chi f_a f_b \ldots (1 \pm f_i)(1 \pm f_j) \ldots$$
$$- |\mathcal{M}|^2_{i+j+\ldots \to \chi+a+b\ldots} f_i f_j \ldots (1 \pm f_\chi)(1 \pm f_a)(1 \pm f_b) \ldots \Big]$$

$\pm$ boson / fermion

$$d\Pi_i = g_i \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i} \qquad E_i^2 = p_i^2 - m_i^2$$

In most cases, for the freeze-out of DM I can make two approx:

* neglect CP violation

$$\Rightarrow |\mathcal{M}|^2_{\chi+a+b\ldots \to i+j+\ldots} = |\mathcal{M}|^2_{i+j+\ldots \to \chi+a+b+\ldots} \equiv |\mathcal{M}|^2$$

* small occupation numbers (no BE condensation, no Fermi degeneracy)

$$\Rightarrow \begin{array}{l} 1 \pm f_i \approx 1 \\ f_i \approx \exp\left(-\frac{E_i - \mu_i}{T}\right) \end{array}$$

$$C[f] = -\frac{1}{2} \sum_{\substack{ab\ldots \\ ij\ldots}} \int d\Pi_a \, d\Pi_b \ldots d\Pi_i \, d\Pi_j \ldots (2\pi)^4 \delta^4(\quad) \times$$
$$\times |\mathcal{M}|^2 \left(f_\chi f_a f_b \ldots - f_i f_j \ldots \right)$$

Integrated collision term

$$\sum_x \int \frac{d^3p_x}{(2\pi)^3 E_x} C[f] = -\sum_{x a b \ldots i j \ldots} \int d\Pi_x \, d\Pi_a \, d\Pi_b \ldots d\Pi_i \, d\Pi_j \ldots$$

$$\times (2\pi)^4 \delta^4 (\quad) \, |\mathcal{M}|^2 \left(f_x f_a f_b \ldots - f_i f_j \ldots \right)$$

NB: elastic processes give 0 after integration:

Consider a process that preserves x particles, eg

$$x(p) + a(q) \rightarrow x(p') + a(q')$$

i is now x:

$$\Rightarrow \; - \int \frac{d^3p}{(2\pi)^3 E_p} \frac{d^3q}{(2\pi)^3 E_q} \frac{d^3p'}{(2\pi)^3 E_{p'}} \frac{d^3q'}{(2\pi)^3 E_{q'}} \, g_x^2 g_a^2 \, (2\pi)^4 \delta(\ldots)$$

$$\times |\mathcal{M}|^2 \left[f_x(p) f_a(q) - f_x(p') f_a(q') \right]$$

Since I integrate over d^3p, d^3p', the integral vanishes

$\Rightarrow$ elastic terms do not contribute. This was expected, as they don't change particle numbers.

(*) seems to be more complicated for inelastic interactions that conserve n_x. I expect the thing to hold even when $|\mathcal{M}_\rightarrow|^2 \neq |\mathcal{M}_\leftarrow|^2$

Some references in Kolb & Turner or in Gondolo & Gelmini.

WIMP pair annihilation

$$a + b \longrightarrow c + d$$ $\leftarrow$ Note the change of notation

$$C_{\chi b \to cd} = - \sum \int (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)\, d\Pi_a\, d\Pi_b\, d\Pi_c\, d\Pi_d$$
$$|\mathcal{M}_{ab \to cd}|^2 \left(f_a f_b - f_c f_d\right)$$

we can assume thermal eq for c and d (assuming that they interact very fast after being produced) and kinetic equilibrium for a

$$f_i = \frac{n_i(t)}{n_i^{eq}(t)}\, f_i^{eq}(E,t)$$

with $n_i^{eq} = g_i \int \frac{d^3p}{(2\pi)^3}\, f_i^{eq}(|\vec{p}|)$

Assume no Fermi degeneracy or BE condensation, $\mu \simeq 0$

$$\Rightarrow f_i^{eq} \approx e^{-E_i/T}$$

Energy conservation: in the reaction $E_\chi + E_b = E_c + E_d$

$$\Rightarrow f_a^{eq} f_b^{eq} = \exp\left[-(E_a + E_b)/T\right] = \exp\left[-(E_c + E_d)/T\right] = f_c^{eq} f_d^{eq}$$

$$C = - \sum \int (2\pi)^4 \delta^4(\quad)\, d\Pi_a\, d\Pi_b\, d\Pi_c\, d\Pi_d\, |\mathcal{M}_{ab \to cd}|^2$$
$$\left(\frac{n_a n_b}{n_a^{eq} n_b^{eq}} f_a^{eq} f_b^{eq} - \frac{n_c n_d}{n_c^{eq} n_d^{eq}} f_c^{eq} f_d^{eq}\right)$$

I can use the energy conservation eq above to rewrite C in two forms

Collect $\frac{f_a f_b}{n_a^{eq} n_b^{eq}}$ or $\frac{f_c f_d}{n_c^{eq} n_d^{eq}}$

$$\mathcal{C} = -\left(n_a n_b - \frac{n_a^{eq} n_b^{eq}}{n_c^{eq} n_d^{eq}} n_c n_d\right) \times$$

$$\times \underbrace{\frac{1}{n_a^{eq} n_b^{eq}} \sum \int d\Pi_a\, d\Pi_b\, d\Pi_c\, d\Pi_d\, f_a^{eq} f_b^{eq} |\mathcal{M}_{ab\to cd}|^2 (2\pi)^4 \delta^4(p_a+p_b-p_c-p_d)}_{\equiv \langle \sigma v\rangle_{ab\to cd}}$$

$$= -\left(\frac{n_c^{eq} n_d^{eq}}{n_a^{eq} n_b^{eq}} n_a n_b - n_c n_d\right) \times$$

$$\times \underbrace{\frac{1}{n_c^{eq} n_d^{eq}} \sum \int d\Pi_a\, d\Pi_b\, d\Pi_c\, d\Pi_d\, f_c^{eq} f_d^{eq} |\mathcal{M}_{cd\to ab}|^2 (2\pi)^4 \delta^4(p_a+p_b-p_c-p_d)}_{\equiv \langle \sigma v\rangle_{cd\to ab}}$$

$$\Rightarrow \frac{dn_a}{dt} + 3Hn_a = -\langle \sigma v\rangle_{ab\to cd}\left(n_a n_b - \frac{n_a^{eq} n_b^{eq}}{n_c^{eq} n_d^{eq}} n_c n_d\right)$$

Assume c, d are in eq! $\frac{n_c n_d}{n_c^{eq} n_d^{eq}} = 1$

$$\frac{dn_a}{dt} + 3Hn_a = -\langle \sigma v\rangle_{ann}\left(n_a n_b - n_a^{eq} n_b^{eq}\right)$$

Interesting cases:

* $a = X$, $b = \bar{X}$ $\rightarrow$ assume $n_X = n_{\bar{X}}$
* $a = b = X$ $\rightarrow$ add a factor 2 because 2 particles disappear, another $\frac{1}{2}$ to avoid double counting

$$\Rightarrow \dot{n}_X + 3Hn_X = -\langle \sigma v\rangle_{ann}\left(n_X^2 - n_X^{eq\,2}\right)$$

If I have multiple processes I can sum the collision terms

$$C_x = \sum_{b,c,d} C_{a+b \to c+d} \qquad \langle \sigma v \rangle = \sum_{b c d} \langle \sigma v \rangle_{ab \to cd}$$

Now it is useful to reintroduce the yield $\quad Y_x = \frac{n_x}{s} = \frac{n_x}{S} a^3$

$$\dot{n}_x = \dot{Y}_x \frac{S}{a^3} - 3 \frac{\dot{a}}{a} Y_x \frac{S}{a^3}$$

$$\Rightarrow \quad \dot{Y}_x s - 3H Y_x s + 3H Y_x s = - \langle \sigma v \rangle s^2 \left(Y_x^2 - Y_x^{eq\,2} \right)$$

$$\dot{Y}_x = - s \langle \sigma v \rangle \left(Y_x^2 - Y_x^{eq\,2} \right)$$

Now I want to rewrite it in terms of x instead of t

$$\frac{d}{dt} = \frac{dT}{dt} \frac{dx}{dT} \frac{d}{dx}$$

Neglect variation of g_{*s} :

$$T a = \text{const} \Rightarrow \frac{dT}{dt} a + H T a = 0 \Rightarrow \frac{dT}{dt} = -HT$$

$$\frac{dx}{dT} = - \frac{m}{T^2} = - \frac{x}{T}$$

$$\Rightarrow \dot{Y}_x = HT \frac{x}{T} \frac{dY}{dx}$$

$$\Rightarrow x H T \frac{dY_x}{dx} = - s \langle \sigma v \rangle Y_x^{eq\,2} \left[\left(\frac{Y_x}{Y_x^{eq}} \right)^2 - 1 \right]$$

$$\Rightarrow \quad \frac{x}{Y_x^{eq}} \frac{d Y_x^{eq}}{dx} = - \frac{\Gamma}{H} \left[\left(\frac{Y}{Y_{eq}} \right)^2 - 1 \right]$$

Comments: - the rate of change of Y is controlled by Γ/H as expected

- annihilation and production are balanced for $n = n_{eq}$, or $Y = Y_{eq}$

Cross section and Møller velocity

The usual definition of a cross section is:

Cross section averaged over the initial spin states. If I don't do that I have to sum at the end

$$\sigma_{ab\to cd} = \frac{1}{g_a}\sum_a \frac{1}{g_b}\sum_b \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1-\vec{v}_2|}\times$$

$$\times\sum_c\sum_d\int\frac{d^3p_c}{(2\pi)^3 2E_c}\frac{d^3p_d}{(2\pi)^3 2E_d}(2\pi)^4\delta^4(p_a+p_b-p_c-p_d)\,|\mathcal{M}|^2$$

Transforms as an area for transverse boosts

Alternative definition (Lorentz invariant)

$$\hat{\sigma} = \frac{1}{g_a}\sum_a\frac{1}{g_b}\sum_b \underbrace{\frac{1}{4\sqrt{(p_a\cdot p_b)^2-m_a^2m_b^2}}}_{\to\ 1/(4F)}\ \ldots$$

$$F^2=(p_a\cdot p_b)^2-m_a^2m_b^2 = \left(E_aE_b-\vec{p}_a\cdot\vec{p}_b\right)^2-m_a^2m_b^2 =$$

$$= E_a^2E_b^2 + p_a^2p_b^2\cos^2\theta - 2E_aE_bp_ap_b\cos\theta - m_a^2m_b^2$$

$$= \cancel{m_a^2m_b^2} + m_a^2p_b^2 + m_b^2p_a^2 + p_a^2p_b^2(1+\cos^2\theta) - 2E_aE_bp_ap_b\cos\theta - \cancel{m_a^2m_b^2}$$

divide by $E_1^2E_2^2$ and use $\vec{v}=\vec{p}/E$ $\qquad m/E = 1/\gamma$

$$= E_1^2E_2^2\left(\frac{m_a^2}{E_a^2}v_b^2+\frac{m_b^2}{E_b^2}v_a^2+v_a^2v_b^2(1+\cos^2\theta)-2v_av_b\cos\theta\right)$$

$$= E_1^2E_2^2\left[(1-v_a^2)v_b^2+(1-v_b^2)v_a^2+v_a^2v_b^2(1+\cos^2\theta)-2v_av_b\cos\theta\right)$$

$$= E_1^2E_2^2\left[v_a^2+v_b^2-2v_av_b\cos\theta - v_a^2v_b^2(1-\cos^2\theta)\right]$$

$$= E_1^2E_2^2\left(|\vec{v}_a-\vec{v}_b|^2-|\vec{v}_a\times\vec{v}_b|^2\right) \equiv E_a^2E_b^2\,v_{Møl}^2$$

$$F = E_aE_b\,v_{Møl} \qquad \text{w/} \qquad v_{Møl} = \left(|\vec{v}_a-\vec{v}_b|^2-|\vec{v}_a\times\vec{v}_b|^2\right)^{1/2}$$

Thus we found

$$\sigma_{ab\to cd} = \frac{1}{g_a}\sum_a \frac{1}{g_b}\sum_b \frac{1}{4E_aE_b v_{rel}} \times$$

$$\times \sum_c \sum_d \int \frac{d^3p_c}{(2\pi)^3 2E_c} \frac{d^3p_d}{(2\pi)^3 2E_d} (2\pi)^4 \delta^4(p_a+p_b-p_c-p_d)\,|\mathcal{M}|^2$$

Now go back to !

$$C = -\left(n_a n_b - \frac{n_a^{eq} n_b^{eq}}{n_c^{eq} n_d^{eq}} n_c n_d\right) \frac{1}{n_a^{eq} n_b^{eq}} \int d\Pi_a\, d\Pi_b\, f_a^{eq} f_b^{eq}$$

$$\underbrace{\sum_{ab}\sum_{cd}\int d\Pi_c\, d\Pi_d\, |\mathcal{M}_{ab\to cd}|^2 (2\pi)^4 \delta^4(p_a+p_b-p_c-p_d)}_{= 4 g_a g_b E_a E_b v_{rel}\, \sigma}$$

$$d\Pi_a\, d\Pi_b\, 4E_aE_b = \frac{d^3p_a}{(2\pi)^3}\frac{d^3p_b}{(2\pi)^3}$$

$$C = -\left(n_a n_b - n_a^{eq} n_b^{eq} \frac{n_c n_d}{n_c^{eq} n_d^{eq}}\right) \frac{\int \frac{d^3p_a}{(2\pi)^3}\frac{d^3p_b}{(2\pi)^3} f_a^{eq} f_b^{eq}\, \cancel{g_a g_b}\, \sigma v_{rel}}{\int \frac{d^3p_a}{(2\pi)^3}\frac{d^3p_b}{(2\pi)^3}\, \cancel{g_a g_b}\, f_a^{eq} f_b^{eq}}$$

$$= -\left(n_a n_b - n_a^{eq} n_b^{eq}\right) \langle \sigma v_{rel} \rangle$$

Velocity and choice of reference frame

What reference frame should one use to compute $\langle \sigma v \rangle$?

Gondolo & Gelmini '91

I can prove that

$$\langle \sigma v_{\text{Møl}} \rangle = \langle \sigma v_{\text{lab}} \rangle^{\text{lab}} \neq \langle \sigma v_{\text{cm}} \rangle^{\text{cm}}$$

(1) (2) (3)

(1) It's computed in the cosmic reference frame. The momenta can be any, not necessarily back-to-back, but the distributions are simple

(2) "lab" means the rest frame of one of the two particles. Here $\vec{v}_{\text{lab}} = \vec{v}_2$, with $\vec{v}_1 = 0$.
It is the frame we used when we introduced $\Gamma = n \sigma v$.

(3) $v_{\text{cm}} = |\vec{v}_1 - \vec{v}_2|$. It's not a velocity: it can be $> c$.

Summary:

app B of 1410.7409

Low-velocity expansion:

$$\sigma v = a + b v^2 + \mathcal{O}(v^4)$$

$$\Rightarrow \langle \sigma v \rangle \approx a + b \langle v^2 \rangle = a + 6 \frac{b}{x_f}$$

(a: s-wave
b: p-wave)

The solution to the eq involves an integral of the rate from freeze-out until today

$$\Omega_\chi h^2 \approx (2\times) \frac{2 \times 10^8 \, \mathrm{GeV}^{-1} \, m_\chi}{M_P \int_{T_0}^{T_f} g_*^{1/2} \langle \sigma v \rangle \, dT}$$

$$\approx (2\times) \frac{2 \times 10^8 \, \mathrm{GeV}^{-1} \, x_f}{g_*^{1/2} \, M_P \, (a + 3b/x_f)}$$

average over $T_0 - T_f$

x_f: solution of

$$e^{x_f} = \frac{\sqrt{45} \, g_\chi \, m_\chi M_P \, c(c+2) \langle \sigma v \rangle}{2 \pi^{5/2} \, g_*^{1/2} \sqrt{x_f}}$$

$$\left(\frac{\sqrt{\frac{45}{8}} \sqrt{8\pi}}{2\pi^3} = \frac{\sqrt{45}}{2\pi^{5/2}} \right)$$

$$\rightsquigarrow x_f \sim 25$$

$c(c+2) = n+1$ w/ $\begin{cases} n = 0 & \text{s-wave} \Rightarrow c = \sqrt{2} - 1 \approx 0.4\ldots \\ n = 1 & \text{p-wave} \Rightarrow c = \sqrt{3} - 1 \approx 1.7\ldots \end{cases}$

A useful formula:

$$\Omega_\chi \approx \frac{3 \times 10^{-26} \, \mathrm{cm}^3 \, \mathrm{s}^{-1}}{\langle \sigma v \rangle}$$

← target for indirect detection experiments.

Cold relics mass window

* lower bound: we know that, at the time of BBN, there were only 3 neutrinos around. If there were a 4th, the abundance of light elements would be different

If $m_\chi \lesssim 1\,\mathrm{MeV}$, at the time of BBN it would behave like an additional neutrino $\Rightarrow m_\chi \gtrsim 1\,\mathrm{MeV}$

* upper bound: Griest & Kamionkowski

From optical theorem (sorry, no time for this)

$$\sigma v \lesssim \frac{4\pi}{m_\chi^2}$$

$$\Rightarrow \frac{4\pi}{m_\chi^2} \gtrsim 1\,\mathrm{pb}$$

$$\Rightarrow m_\chi^2 \lesssim \frac{4\pi}{10^{-12}\,(100\,\mathrm{fm}^2)} \approx \frac{4\pi}{10^{-12}\;100\;\frac{1}{(200\,\mathrm{MeV})^2}} \approx \frac{16\pi \quad \mathrm{MeV}^2}{10^{-12}\;10^2\;10^{-4}} \approx$$

$$\approx 16\pi\;10^{14}\;10^{-12}\;\mathrm{TeV}^2 \approx 10^4\,\mathrm{TeV}^2$$

$$\Rightarrow m_\chi \lesssim 100\,\mathrm{TeV}$$

Exceptions: the relic abundance can be more interesting

Resonances

Suppose annihilation involves a mediator w/ mass M, with

$$M \simeq 2 m_\chi$$

$$\sim \frac{1}{p^2 - M^2} \sim \frac{1}{4 m_\chi^2 - M^2 + m_\chi^2 v^2}$$

$$p = \begin{pmatrix} 2 m_\chi + \frac{1}{2} m_\chi v^2 \\ m_\chi \vec{v} \end{pmatrix} \Rightarrow p^2 \approx 4 m_\chi^2 + 2 m_\chi^2 v^2 - m_\chi^2 v^2 = 4 m_\chi^2 + m_\chi^2 v^2$$

lab frame

- if $M > 2 m_\chi$ v will want to be large enough to hit the resonance
- if $M < 2 m_\chi$ v will want to be 0

Thresholds

Suppose m_χ slightly smaller than m_t

$\chi \bar{\chi} \to t \bar{t}$ is forbidden at $v = 0$, but as soon as it is allowed the annihilation rate is enhanced $\Rightarrow$ in the thermal average, v will want to be right there.

Coannihilations

Suppose I have many particles w/ a similar mass χ_i, with $m_1 < m_2 < m_3 < \dots < m_N$

χ_i all decay to χ_1 at the end (much faster than the age of the universe)

$\Rightarrow$ Consider $n = \sum_i n_i$

and $\dot{n} + 3Hn = -\sum_{i,j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{eq} n_j^{eq} \right)$

w/ $\sigma_{ij} = \sigma(\chi_i \chi_j \to SM)$

$$v_{ij} = \frac{\sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}}{E_i E_j} \qquad \text{(Møller again)}$$

Sommerfeld enhancement / formation of bound states

Suppose DM has some self-interactions, through a light mediator

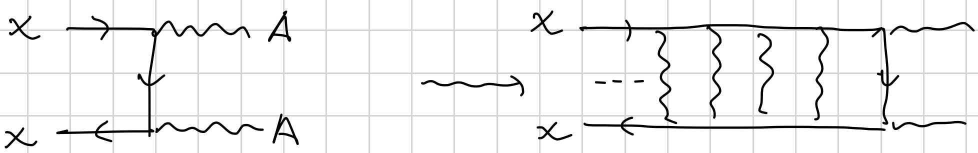
Most typical example: dark photon

$$\mathcal{L} = \bar{\chi}\,(i\not{\partial} - m_\chi)\,\chi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

Rapid exchange of many photons can lead to two exciting phenomena that can enhance $\langle \sigma v \rangle$ by $\sim 2-3$ orders of magnitude

[Petraki, talk at "Quarkonia meet DM" 2024]

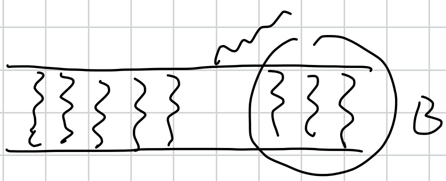
* Sommerfeld enhancement



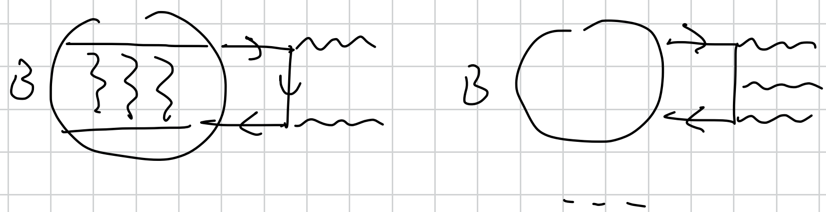
$$\sigma v = \frac{\pi \alpha_D^2}{m_\chi^2} \times \left(\frac{2\pi\zeta}{1 - e^{-2\pi\zeta}} \right) \qquad \left(\sim 2\pi\zeta \quad \text{for } \zeta \gg 1 \right)$$

$$\zeta = \frac{\alpha_D}{v_{rel}}$$

* Bound states formation (and decay)



and subsequently



Freeze - in

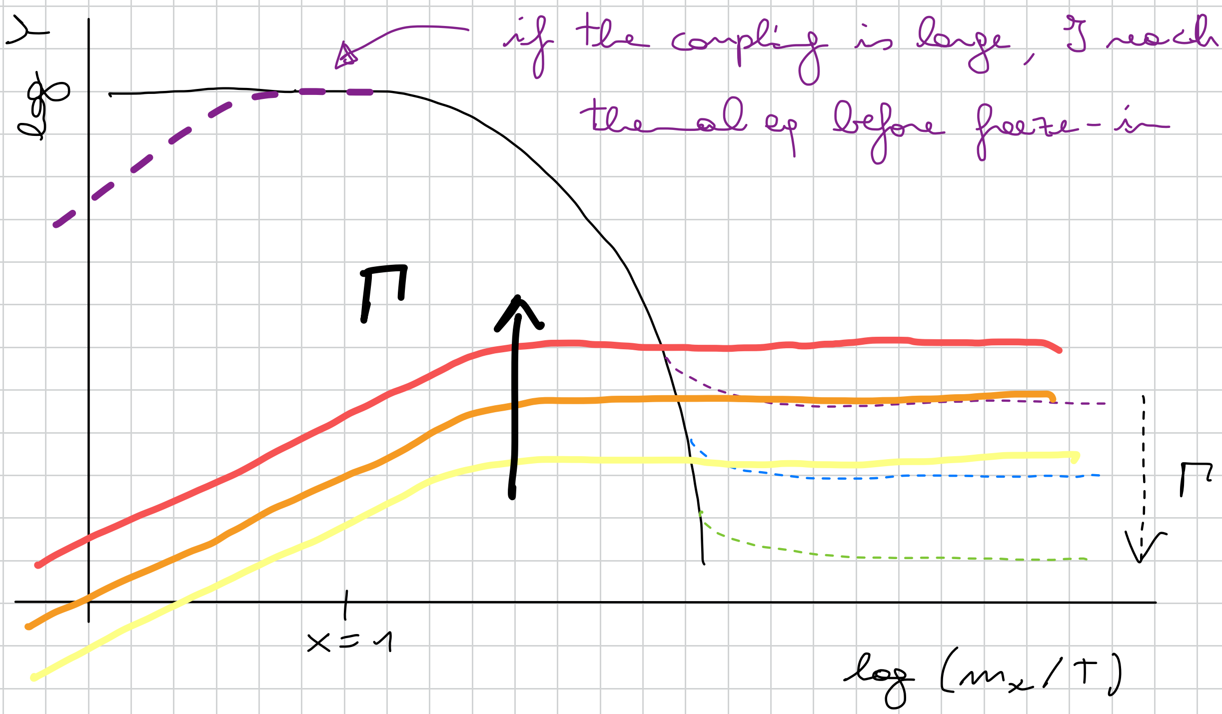
0911.1120

Consider a particle interacting so weakly w/ the SM to be called FIMP: feebly-interacting massive particle

Freeze-out: initially I produce χ's very efficiently, so that they are in thermal eq. After they become non-rel, they "freeze-out". The largest their cross section, the longest they remain in eq, the smallest the abundance

What if, instead, χ is not in thermal eq, and its initial abundance is 0?

I will start building up its abundance from $Y = 0$ to Y_0. I expect $Y_0 \propto \Gamma$



Freeze-in from decays & inverse decays

Consider 3 particles:

x DM, does not couple to the SM

A SM particle

B Does <u>not</u> belong to the SM, but it is in thermal eq.

$Y_B = Y_B^{eq} \Rightarrow Y_{B,0} \simeq 0$,   initially $Y_x = 0$

Eg in supersymmetric models, x can be the lightest susy particle, $R_x = -1$, no SM charge, while B is electrically charged and heavy.   CHECK W/ HALL+

Suppose B decays to A, x: $B \to A + x$ $(m_B > m_A + m_x)$

(B is the parent particle)

Collision operator:

$$C_x = \int d\Pi_A \, d\Pi_B \, d\Pi_x \, (2\pi)^4 \delta^4(p_B - p_A - p_x) \times$$

$$\times \left(|\mathcal{M}_{B\to Ax}|^2 f_B - |\mathcal{M}_{Ax\to B}|^2 f_A f_x \right)$$

$Ax \to B$: inverse decays are allowed in the thermal bath: given $\vec{p}_B, \vec{p}_A$, there $\exists$ a particle x w/ the right momentum to close the system of eq given by the $\delta^4(\dots)$ (with probability f)

Suppose $f_x \simeq 0$ $\quad (f_A, f_B \sim T^3 \gg f_x)$

$$\mathcal{C} \simeq \int d\Pi_A \, d\Pi_B \, d\Pi_x \, (2\pi)^4 \delta^4(p_B - p_A - p_x) \, |\mathcal{M}_{B\to Ax}|^2 f_B^{eq}$$

$$= \underbrace{\sum_B \int \frac{d^3 p_B}{(2\pi)^3} \frac{1}{2E_B} 2m_B \, f_B^{eq}(p_B, T)}\underbrace{\int d\Pi_A \, d\Pi_x \frac{|\mathcal{M}_{B\to Ax}|^2}{2m_B} (2\pi)^4 \delta^4(\ldots)}$$

$g_B \, \Gamma_{B\to xA}$ (independent of p_B: can go out of the integral)

$$= \Gamma_{B\to xA} \, g_B \int \frac{d^3 p_B}{(2\pi)^3} f_B^{eq}(p_B, T) \frac{m_B}{E_B}$$

$$= n_B^{eq}(T) \, \Gamma_{B\to xA} \left\langle \frac{m_B}{E_B} \right\rangle$$

where, in general, for any function $\mathcal{O}(p_B)$,

$$\langle \mathcal{O} \rangle = \frac{1}{n_B} \int \frac{d^3 p}{(2\pi)^3} f_B \, \mathcal{O}(p_B)$$

Without relying on a (non-) relativistic expansion:

$$n_B^{eq} = g_B \int \frac{d^3 p}{(2\pi)^3} f_B^{eq} \approx \int \frac{d^3 p_B}{(2\pi)^3} e^{-\sqrt{p_B^2 + m_B^2}/T}$$

$$= \frac{g_B}{2\pi^2} \int dp \, p^2 \exp(-E/T) \qquad\qquad p\,dp = E\,dE$$

$$= \frac{g_B}{2\pi^2} \int_m^\infty dE \, E \sqrt{E^2 - m^2} \, e^{-E/T}$$

$$= \frac{g_B}{2\pi^2} m_B \int_{m_B}^\infty dE \, E \sqrt{\left(\frac{E}{m}\right)^2 - 1} \, e^{-E/T} \qquad\qquad \frac{E}{m} = y \quad \frac{m}{T} = x$$

$$= \frac{g_B}{2\pi^2} m_B^3 \int_1^\infty dy \, y \sqrt{y^2 - 1} \, e^{-xy}$$

$$= \frac{g_B}{2\pi^2} m_B^3 (-1) \frac{d}{dx} \int_1^\infty dy \sqrt{y^2-1}\, e^{-xy} =$$

$$= - g_B \frac{m_B^3}{2\pi^2} \frac{d}{dx} \frac{K_1(x)\left(\frac{1}{2}!\right)}{\sqrt{\pi}\left(\frac{1}{2}x\right)}$$

$$K_n(z) = \frac{\sqrt{\pi}}{\left(n-\frac{1}{2}\right)!}\left(\frac{z}{2}\right)^n \int_1^\infty e^{-zt}(t^2-1)^{n-1/2}\, dt$$

$$= - g_B \frac{m_B^3}{2\pi^2} \frac{d}{dx} \frac{K_1(x)}{x}$$

$$\frac{1}{2}! = \frac{\sqrt{\pi}}{2} \qquad \left(z! = \Gamma(z+1) = \int_0^\infty e^{-t} t^z\, dt \right)$$

$$= g_B \frac{m_B^3}{2\pi^2}\left(\frac{K_1(x)}{x^2} - \frac{K_1'(x)}{x}\right)$$

$$= g_B \frac{m_B^3}{2\pi^2}\left(\frac{K_1(x)}{x^2} + \frac{K_0(x)}{2x} + \frac{K_2(x)}{2x}\right)$$

recurrence relation:

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z)$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K_\nu'(z)$$

$$= g_B \frac{m_B^3}{2\pi^2}\left(\frac{K_2(x)}{2x} - \frac{K_0(x)}{2x} + \frac{K_0(x)}{2x} + \frac{K_2(x)}{2x}\right)$$

$$= g_B \frac{m_B^3}{2\pi^2} \frac{K_0(x)}{x} = \frac{m_B^2 T}{2\pi^2} K_0\left(\frac{m}{T}\right)$$

$$\frac{2K_1(z)}{z} = K_2(z) - K_0(z)$$

$$\Rightarrow n_B^{eq} = g_B \frac{m_B^2 T}{2\pi^2} K_2\left(\frac{m_B}{T}\right)$$

$$\left\langle \frac{m_B}{E_B} \right\rangle = \frac{g_B}{n_B^{eq}} \int \frac{d^3p}{(2\pi)^3} e^{-E_B/T} \frac{m_B}{E_B} = \frac{g_B}{n_B^{eq}} \frac{1}{2\pi^2} \int_{m_B^2}^\infty dE_B\, E_B \sqrt{E_B^2 - m_B^2}\, e^{-E/T} \frac{m_B}{E_B}$$

$$= g_B \frac{m_B^3}{2\pi^2 n_B^{eq}} \int_1^\infty dy \sqrt{y^2-1}\, e^{-xy} =$$

$$= g_B \frac{m_B^3}{2\pi^2 n_B^{eq}} \frac{\frac{1}{2}!\, K_1(x)}{\sqrt{\pi}\ \frac{1}{2}x} = \frac{g_B m_B^2 T}{2\pi^2 n_B^{eq}} K_1(x) = \frac{K_1(x)}{K_2(x)}$$

Result: $$\tau = n_B^{eq}\, \Gamma_{B\to A x} \frac{K_1(m_B/T)}{K_2(m_B/T)}$$

Freeze-in from inverse decay

Suppose we are in the opposite regime.

$$m_x > m_A + m_B$$

thus B cannot decay into x. <u>But</u> x can be produced from inverse decays

$$B + A \longrightarrow x$$

which is effective as long as T is not much below m_x

$$\Rightarrow \quad C_x = \int d\Pi_B \, d\Pi_A \, d\Pi_x \, (2\pi)^4 \delta^4(p_A + p_B - p_x) \times$$

$$\times \left(|\mathcal{M}_{BA \to x}|^2 f_B^{eq} f_A^{eq} - \cancelto{\approx 0}{|\mathcal{M}_{x \to BA}|^2 f_x} \right)$$

energy conservation: $\quad f_A^{eq} f_B^{eq} = e^{-(E_A + E_B)/T} = e^{-E_x/T} = f_x^{eq}$

CP conservation: $\quad |\mathcal{M}_{AB \to x}| = |\mathcal{M}_{x \to AB}|$

$$\Rightarrow C \approx \int d\Pi_x \, f_x^{eq} \, 2m_x \underbrace{\int d\Pi_A \, d\Pi_B \, (2\pi)^4 \delta^4(p_x - p_A - p_B) \, \frac{|\mathcal{M}_{x \to AB}|^2}{2m_x}}_{\Gamma_{x \to AB} \text{ (it's a Lorentz scalar)}}$$

...

$$C = n_x^{eq} \, \Gamma_{x \to AB} \, \frac{K_1(m_x/T)}{K_2(m_x/T)}$$

Freeze in from scatterings

$A + B \to \chi + C$ (eg $e + \gamma \to e + a$)

$\to$ for the exam

Solve the Boltzmann eq

$$\frac{dY}{dt} = \frac{d}{dt}\frac{n a^3}{s a^3} = \frac{1}{s a^3}\frac{d(n a^3)}{dt} = \frac{1}{s a^3}\left(\dot{n} a^3 + 3 a^3 H n\right) = \frac{1}{s}\left(\dot{n} + 3Hn\right)$$

$$\Rightarrow \frac{dY_x}{dt} = \frac{C}{s} = \frac{n_B^{eq}}{s}\Gamma_{B\to Ax}\frac{K_1(m_B/T)}{K_2(m_B/T)}$$

Assume g_{*s} around the freeze-out temperature

$$\frac{dT}{dt} = -HT$$

$$\Rightarrow \frac{dY_x}{dt} = \frac{dT}{dt}\frac{dY_x}{dT} = -HT\frac{dx}{dT}\frac{dY_x}{dx} = Hx\frac{dY_x}{dx}$$

$$\Rightarrow \frac{dY_x}{dx} = \frac{\Gamma_{B\to Ax}}{H}\frac{Y_B^{eq}}{x}\frac{K_1(x)}{K_2(x)}$$

(for that for freeze-out)

Assume radiation dominance

$$H = \frac{\pi}{3\sqrt{10}} g_*^{1/2}\frac{T^2}{M_p} = \frac{\pi}{3\sqrt{10}} g_*^{1/2}\frac{m_B^2}{M_p x^2}$$

$$Y_B^{eq} = \frac{n_B^{eq}}{s} = \frac{g_B}{2\pi^2} m_B^2 T K_2(x)\left(\frac{2\pi^2}{45} g_{*s} T^3\right)^{-1}$$

$$= \frac{45}{4\pi^4}\frac{g_B}{g_{*s}} x^2 K_2(x)$$

$$\Rightarrow \frac{dY_x}{dx} = \Gamma_B \frac{45}{4\pi^4}\frac{g_B}{g_{*s}} x K_2(x)\frac{3\sqrt{10}}{\pi}\frac{1}{g_*^{1/2}}\frac{M_p x^2}{m_B^2}\frac{K_1(x)}{K_2(x)}$$

$$\frac{dY_x}{dx} = \frac{135\sqrt{10}}{4\pi^5}\frac{g_B}{g_{*s} g_*^{1/2}}\frac{M_p\Gamma_B}{m_B^2} x^3 K_1(x)$$

Integrate in dx

$$Y_\chi(x) - Y_\chi(0) = \frac{135\sqrt{10}}{4\pi^5} \frac{M_p \Gamma_B}{m_B^2} \int_0^x dx' \frac{g_B}{g_{*s}\, g_*^{1/2}} K_1(x')\, x'^3$$

↳ large T: $Y \approx 0$

$$\approx \frac{135\sqrt{10}}{4\pi^5} \frac{M_p \Gamma_B}{m_B^2} \frac{g_B}{g_{*s}\, g_*^{1/2}} \int_0^x dx' \, K_1(x')\, x'^3$$

Behaviour:

peak for $T \approx \frac{m_B}{2.4}$

$\Rightarrow$ freeze-in when B becomes non-relativistic and Boltzmann suppressed

$\Rightarrow$ IR-dominated: the production is dominated by the lowest temperature before the freeze-in

$\rightsquigarrow$ constant g_*, g_{*s} well justified

Present abundance $\leftrightarrow$ Y asymptotics:

$$\int_0^\infty dx' \, K_1(x')\, x'^3 = \frac{3\pi}{2}$$

$$\Rightarrow Y_{\chi,0} \approx Y_\chi(x\to\infty) = \frac{405\sqrt{10}}{4\pi^4} \frac{g_B}{g_{*s}\, g_*^{1/2}} \frac{M_p \Gamma_B}{m_B^2}$$

Present DM density: $\Omega_\chi h^2 = \dfrac{m_\chi Y_{\chi,0}\, s_0}{\rho_{cr,0}/h^2}$

$$\Omega_\chi h^2 = 0.12 \left(\frac{g_B}{2}\right)\left(\frac{106.75}{g_*}\right)^{3/2}\left(\frac{m_\chi}{100\,\mathrm{GeV}}\right)\left(\frac{300\,\mathrm{GeV}}{m_B}\right)\left(\frac{\Gamma/m_B}{0.9\times 10^{-25}}\right)$$

Consistency check: compare Y_χ^∞ with the equilibrium value for a relativistic χ, $Y_\chi^{eq,rel}$

$$\frac{Y_\chi^\infty}{Y_\chi^{eq,rel}} = \left(\frac{3}{4}\frac{g_\chi}{g_{*S}}\frac{45\,\zeta(3)}{2\pi^4}\right)^{-1}\frac{405\sqrt{10}}{4\pi^4}\frac{g_B}{g_{*S}\,g_*^{1/2}\,g_\chi}\frac{M_P\,\Gamma_B}{m_B^2}$$

$$= \frac{6\sqrt{10}}{\zeta(3)}\frac{g_B}{g_{*S}\,g_*^{1/2}\,g_\chi}\frac{M_P\,\Gamma_B}{m_B^2}$$

$\ll 1$ for $\quad \dfrac{\Gamma}{m_B} \ll 8\times 10^{-17}\left(\dfrac{g_B}{g_\chi}\right)\left(\dfrac{106.75}{g_*}\right)^{3/2}\left(\dfrac{300\,\mathrm{GeV}}{m_B}\right)$

ok

Decay rate.

* Yukawa theory: $\mathcal{L} \supset \lambda B \bar{q} \chi$

(B scalar, χ fermion)

$$\Gamma = \frac{\lambda^2}{8\pi} m_B \quad \Rightarrow \quad \Gamma / m_B = \frac{\lambda^2}{8\pi}$$

(* Triple scalar vertex $\mathcal{L} \supset \mu B A \chi$

$$\Rightarrow \Gamma_B \approx \frac{\mu^2}{8\pi m_B} \qquad (m_B \gg m_A, m_\chi)$$)

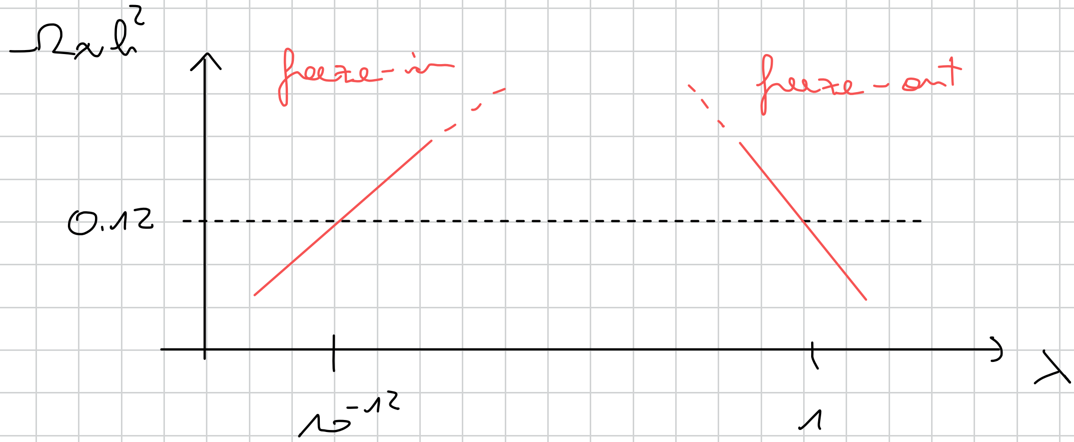
In a Yukawa theory:

$$\Omega_\chi h^2 = 0.12 \left(\frac{g_B}{2}\right) \left(\frac{106.75}{g_*}\right)^{3/2} \left(\frac{m_\chi}{100\ \mathrm{GeV}}\right) \left(\frac{300\ \mathrm{GeV}}{m_B}\right) \left(\frac{\lambda}{1.5 \times 10^{-12}}\right)^2$$

Freeze-in vs freeze-out

Freeze-in is in some sense the inverse of freeze-out.

It happens in the opposite regime of couplings



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Freeze-in:

$$Y_{FI} \sim \frac{M_P \Gamma_B}{m_B^2} \sim \lambda^2 \frac{M_P}{m_B}$$

Hubble time at freeze-in

$$t_{fi} \sim \frac{M_P}{m_B^2}$$

$$\Rightarrow Y_{fi} \sim \lambda^2 m \, t_{fi}$$

late freeze-in → largest Y

Freeze-out:

$$Y_{FO} \sim \frac{1}{\sigma v \, M_P m_x} \sim \frac{1}{\lambda^2} \frac{m_x}{M_P}$$

$$t_{fo} \sim \frac{M_P}{m_x^2}$$

$$Y_{fo} \sim \frac{1}{\lambda^2 m^2 t_{fo}}$$

early fo → largest Y

Common feature: can be computed solely from equilibrium quantities

If FI is governed by renormalizable couplings ⇒ both are IR dominated.