7.2.3 Stabilisers

For n qubits, one has 4×4^n Pauli operators. The first factor 4 accounts for the four relevant phases $\{+1, -1, +i, -i\}$, while the rest count the operators being the basis of the operators acting on a 2^n dimensional space. We define as stabilisers the elements of an abelian subgroup which is responsible of the division of $\mathbb{H}'$ in subspaces. Namely, there will be one code space $\mathbb{H}_\text{C}$ and the other are the error spaces.

Example 7.1
Consider the case of $n = 3$. The stabilisers are

$$\text{(Stabilisers for } n = 3\text{)} = \{\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z, \hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z\}. \tag{7.99}$$

Consider $\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$. It determines the parity of the first and the second qubit. Thus, it divides the Hilbert space $\mathbb{H}'$ in two parts, one associated to its $+1$ eigeinvalue and to its -1 eigeinvalue:

$$\begin{array}{c|c} +1 & -1 \\ \hline |000\rangle & |100\rangle \\ |111\rangle & |011\rangle \\ |001\rangle & |010\rangle \\ |110\rangle & |101\rangle \end{array} \tag{7.100}$$

A similar division can be done considering the operator $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$, for which we have

$$\begin{array}{c|c} +1 & -1 \\ \hline |000\rangle & |001\rangle \\ |111\rangle & |110\rangle \\ |100\rangle & |010\rangle \\ |011\rangle & |101\rangle \end{array} \tag{7.101}$$

We notice that there are no other possible partitions of $\mathbb{H}'$. Indeed, the last non-trivial stabilisers is $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$ can be expressed as the product of the other two:

$$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z = (\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z)(\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}). \tag{7.102}$$

To be specific, the operators $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$ and $\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$ are the generators of the abelian subgroup of the stabilisers.

Now, we can define the code space $\mathbb{H}_C$ as that associated to the $+1$ eigeinvalues for all the stabilisers. Namely, this is the subspace of $\mathbb{H}'$ which is spanned by the +1 eigeinstates of all the generators of the abelian subgroup:

$$\mathbb{H}_C = \text{span}(|000\rangle, |111\rangle). \tag{7.103}$$

The partitioning of $\mathbb{H}'$ is represented graphically in Fig. 7.11.

Now, consider $\hat{E}_i$ being one of the 4×4^n Pauli operators not being one of the stabilisers S_k. Now, since it is constructed as the product of single qubit Pauli operators, $\hat{E}_i$ can only commute or anticommute with the stabilisers S_k.

- Assume that it commutes: $\left[\hat{E}_i, \hat{S}_k\right] = 0$. Then, we have that for any $|\psi\rangle \in \mathbb{H}_\text{C}$, it holds

$$\hat{S}_k\hat{E}_i|\psi\rangle = \hat{E}_i\hat{S}_k|\psi\rangle = \hat{E}_i|\psi\rangle, \tag{7.104}$$

where the last equality follows from the fact that S_k is a stabiliser and thus acts as an identity on $\mathbb{H}_\text{C}$. Then, if $\hat{E}_i$ commutes stabiliser $\hat{S}_k$, it is associated to the eigeinvalue $+1$ of the latter. Indeed, the state $|\phi_i\rangle = \hat{E}_i|\psi\rangle$ is associated to the $+1$ eigeinvalue of $\hat{S}_k$.

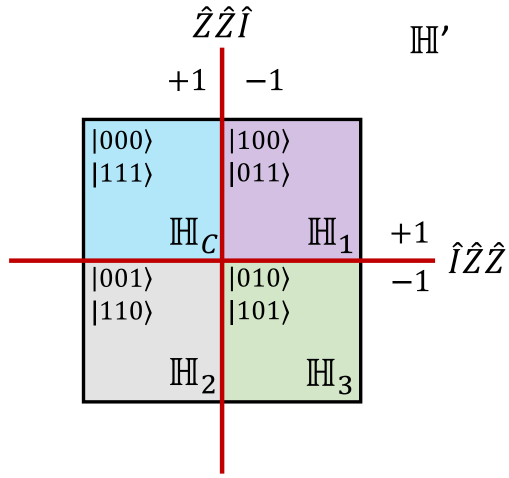

Fig. 7.11: Division of the Hilbert space $\mathbb{H}'$ with respect to the subspaces defined by the eigeinvalues of $\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$ and $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$.

- Conversely, if it anticommutes: $\left\{\hat{E}_i, \hat{S}_k\right\} = 0$, then

$$\hat{S}_k\hat{E}_i\left|\psi\right\rangle = -\hat{E}_i\hat{S}_k\left|\psi\right\rangle = -\hat{E}_i\left|\psi\right\rangle. \tag{7.105}$$

In such a case, one says that $\hat{E}_i$ is associated to the -1 eigeinvalue of S_k.

Example 7.2
In the case of $n = 3$ one has that the operators $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ and $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$ are associated to the eigenvalues of $\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$ and $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$ as

	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$
$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	-1	$+1$
$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$+1$	$+1$

(7.106)

Now, $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$ commutes with both the (generators of the subgroup of) stabilisers. Thus, it means that if

$$\left|\psi\right\rangle \in \mathbb{H}_C, \quad \textit{then} \quad \hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\left|\psi\right\rangle \in \mathbb{H}_C. \tag{7.107}$$

Namely, it is a normaliser (see below) and it acts as a logical $\hat{\sigma}_x$.
Conversely, $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ anticommutes with $\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$. This means that given a state

$$\left|\psi\right\rangle \in \mathbb{H}_C, \quad \textit{then} \quad \hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}\left|\psi\right\rangle \in \mathbb{H}'_1 \cup \mathbb{H}'_3, \tag{7.108}$$

where $\mathbb{H}'_1$ and $\mathbb{H}'_3$ are respectively associated to the eigeinvalues $(-1, +1)$ and $(-1, -1)$ of $(\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z)$. However, since $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ commutes with $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$, then $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}\left|\psi\right\rangle \in \mathbb{H}'_1$.
Namely, the operator $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ is one of the $\hat{V}_i$ errors that maps the states from the code space to the corresponding $\mathbb{H}'_i$.

If we have k qubits that are encoded in n qubits, with $n > k$, then we need $(n - k)$ generators from the stabilisers to define the partitions. By starting with $\dim(\mathbb{H}') = 2^n$, since each generator divides the Hilbert space in two parts, we have 2^{n-k} different subspaces of dimension 2^k.

7.2.4 Normalisers and Centralisers

There are Pauli operators that commute with all the elements of the stabilisers, but do not appartain to the stabilisers subgroup. These are the normalisers $\hat{N}_k$. They respect the partition of $\mathbb{H}$, meaning that they do not map states from the code space $\mathbb{H}_C$ to an error space $\mathbb{H}'_i$, and act non-trivially in the code space. They are defined via

$$\hat{N}_k \hat{S}_i \hat{N}_k^\dagger = \hat{S}_j, \tag{7.109}$$

where $\hat{S}_i$ are the stabilisers. If $i = j$ they are called centralisers, while for $i \neq j$ they are normalisers. In the particular case of the Pauli algebra, meaning that all the operators are generated by the product of Pauli operators, one has that the normalisers are centralisers. Indeed,

$$\hat{N}_k \hat{S}_i \hat{N}_k^\dagger = \pm \hat{N}_k \hat{N}_k^\dagger \hat{S}_i = \pm \hat{S}_i, \tag{7.110}$$

since Pauli operators can only commute or anticommute. However, given the stabilisers $\hat{S}_i$, the operator $-\hat{S}_i$ does not stabilise the code space. Thus, the $-$ sign cannot be accepted and one gets

$$\hat{N}_k \hat{S}_i \hat{N}_k^\dagger = \hat{S}_i = \hat{S}_j. \tag{7.111}$$

Thus, they are all centralisers.

Example 7.3

Consider the case of $n = 3$. The operator $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$ acts as follows:

$$\begin{aligned} |000\rangle &\xrightarrow{\hat{\sigma}_x} |111\rangle\,, \\ |111\rangle &\xrightarrow{\hat{\sigma}_x} |000\rangle\,. \end{aligned} \tag{7.112}$$

Thus, it acts as a logical $\hat{\sigma}_x$. The same happens for $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{S}_k$ for any stabiliser $\hat{S}_k$. Indeed, for $|\psi\rangle \in \mathbb{H}_C$, we have that

$$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{S}_k\,|\psi\rangle = \hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\,|\psi\rangle\,, \tag{7.113}$$

since $\hat{S}_k$ acts as a logical identity on $\mathbb{H}_C$. Suppose we take the stabiliser $\hat{S}_k = \hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$, then

$$(\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x)(\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}) = -\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x, \tag{7.114}$$

which also acts as a logical $\hat{\sigma}_x$.

The normalisers contain $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$, $-\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$, $\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$, and all the products of these with all the stabilisers $\hat{S}_k$, which act as a logical identity.

Physical operation	*Logical operation*
$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}$
$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}$
$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}$
$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\mathbb{1}}$
$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x$
$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{S}_k$	$\hat{\sigma}_x$
$-\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y$
$-\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{S}_k$	$\hat{\sigma}_y$
$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z$
$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{S}_k$	$\hat{\sigma}_z$

(7.115)

7.2.5 Stabiliser code

Consider the three qubits encoding a single logical qubit. The stabilisers are:

$$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}, \quad \hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}, \quad \hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z, \quad \hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z. \tag{7.116}$$

Among the possible errors, there are some that are more and less likely to occur. Under the assumption of errors that act independently on the qubits, the error $\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$ is more likely to occur than $\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$. The first has weight 1 (only one operator different from the identity), while the second has weight 3.

Now, the question is which are the errors that can be corrected, and eventually how they can be corrected. As we already saw, the errors where only one of the qubits is modified can be corrected (see bit-flip, phase-flip and 9-qubit Shor QEC codes). These can be corrected via the application of the recovery operator $\hat{R}_k = \hat{V}_k^\dagger$, so that

$$\hat{R}_k\hat{V}_k = \hat{V}_k^\dagger\hat{V}_k = \hat{\mathbb{1}}. \tag{7.117}$$

However, the operator $\hat{R}_k$ can correct for a much wider class of operators. Indeed, given a state $|\psi\rangle \in \mathbb{H}_\mathrm{C}$ and a stabiliser $\hat{S}_i$, one has

$$\hat{R}_k(\hat{V}_k\hat{S}_i)\,|\psi\rangle = \hat{R}_k\hat{V}_k\,|\psi\rangle = |\psi\rangle\,. \tag{7.118}$$

Thus, $\hat{R}_k$ can correct also errors of the form of a correctable error multiplied by a stabiliser, i.e. $\hat{V}_k\hat{S}_i$.

Conversely, an error in the class of normalisers which is not a stabiliser is a non-correctable error. Indeed, it acts non-trivially on the code space.