Statistical Methods with Application to Finance
a.y. 2023/2024

Exercises

1. The prices of a stock at times 1, 2 and 3 are $P_1 = 81, P_2 = 92, P_3 = 98$. Find the net return R_3, the log return r_3, the 2-period log return $r_3(2)$, the multiperiod return $R_3(2)$.

2. The monthly log returns on an asset within a quarter are 6.7%, 8.4%, 4.9%. Compute the quarterly net return (in %).

3. Suppose $X_1, X_2, ...$ are iid $N(0.06, 0.47)$
   - What is the mean of $X_1 + X_2 + X_3$?
   - What is the standard deviation of $X_1 + X_2 + X_3$?
   - What is the distribution of $\exp(X_1 + X_2)$?
   - What is the probability $P(X_1 < 1.5)$?
   - What is $Cov(X_1, X_1 + X_2)$?

4. The stock prices and dividends for a company for a 1-year period are given below:

t	P_t	D_t
06/16/11	30,44	
09/15/11	30,22	0,23
12/15/11	31,34	0,22
03/15/12	33,45	0,25
06/15/12	32,22	0,25

   What is the simple net return over the period from 09/15/11 to 12/15/11?

5. The price per share and outstanding shares for stocks A, B, and C, are given below P_0 and Q_0 are prices and quantities in the base period, respectively):

Stock	P_0	P_1	Q_0	Q_1
A	80	109	50	51
B	60	53	80	97
C	20	53	180	264

   - What is the Laspeyres price index? (base year index is 100).
   - What is the Paasche price index? (base year index is 100).

6. Let Φ be the standard normal cumulative distribution function:
   - What is the value x such that $\Phi(-x) = 0.025$?
   - What is the 0.975-quantile of the normal distribution with mean 2 and variance 0.49?

7. The log returns on a stock are normally distributed with mean 0.1 and standard deviation 0.13.

- What is the distribution of one-period gross returns?
- What is the probability that the gross return is 1.2 or more?
- Assume the log-returns are also independent, what is the 0.85-quantile of the 2-period log return?

8. Suppose that the daily log returns r_t on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015.

   - What is $P(r_t < 0.02)$?
   - What is the standard deviation of $r_1 + r_2 + r_3$?
   - What is the distribution of the five-day log return?
   - Suppose you buy \$1000 worth of this stock, what is the probability that after five trading days yout investment is worth less than \$990?

9. The prices on a stock market are assumed to follow a lognormal geometric random walk with parameters $\mu = 0,15$ and $\sigma^2 = 0,04$.

   - What is the expected log return for 5 years?
   - What is the median log return for 10 years?
   - What is the median gross 10-year return?
   - If the stock price starts at \$100, what is the expected price after 10 years?

10. Consider the random walk model
$$X_t = \delta + X_{t-1} + w_t$$
with drift δ, $x_0 = 0$, and w_t a white noise series with variance σ_w^2.

    - Show that the model can be written as $X_t = \delta t + \sum_{k=1}^{t} w_k$.
    - Find the mean function and the autocovariance function of X_t.
    - Suggest a transformation to make the series stationary and prove that the transformed series is stationary.

11. Consider the AR(1) model
$$Y_t = 5 + 0.7Y_{t-1} + a_t$$
where $a_t \sim \mathrm{WN}(0, \sigma_a^2)$, and assume that $\sigma_a^2 = 2$.

    - Is this process stationary?
    - What is the mean of this process?
    - What is the variance of this process?
    - What is the covariance between Y_1 and Y_3?
    - Sketch the theoretical ACF for this model.

12. The series $\{w_t\}$ is white noise with zero mean and variance σ_w^2. For the following moving average models find the autocorrelation function and determine whether they are invertible.

$$(a) \quad Y_t = w_t + \frac{1}{2} w_{t-1}$$
$$(b) \quad Y_t = w_t + 2 w_{t-1}$$

13. Consider the MA(2) process

$$X_t = a_t + 0.7 a_{t-1} - 0.2 a_{t-2}$$

where a_t is a Gaussian white noise with mean 0 and variance $\sigma_a^2 = 1$.

- Derive the mean and the variance function of the process.
- Is this process invertible?
- Show that the ACF of the process is

$$\rho(k) = \begin{cases} 1 & k = 0 \\ 0.37 & k = \pm 1 \\ -0.13 & k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

14. Suppose that $X_1, X_2, \ldots$ is a lognormal geometric random walk with parameters μ and σ^2:

$$X_k = X_0 \exp(r_1 1 + \cdots + r_k)$$

where X_0 is a fixed constant and $r_1, r_2, \ldots$ are iid $\mathcal{N}(\mu, \sigma^2)$.

- Find $Pr(X_2 > 1.3 X_0)$ for $\mu = 0.1$ and $\sigma = 0.15$.
- Find the first quartile of X_k, for all k.
- What is the expected value of X_k?

15. The log prices on a stock are assumed to follow a random walk model

$$p_t = p_0 + \sum_{i=1}^{t} r_i$$

where $p_0 = \log(P_0)$ is the log price at time 0 and r_1, $r_2, \ldots$ are iid $\mathcal{N}(0.1, 0.2^2)$.

- What is the expected log return after 20 years?
- If the stock price starts at \$100, what is the median price after 20 years?
- Find the 0.05-quantile and 0.95-quantile of the 20-year gross return.

16. Assume that the simple gross return $1 + R_t$ is lognormal with log-mean zero and log-variance 0.2. Also, assume that the returns are an independent sequence.

- Find the probability that a gross two-period return is less than 0.8.
- What is the third quartile of the log return r_t?

- Compute the probability $Pr(r_t(3) \geq 0.7)$.
- What is the covariance between $r_2(1)$ and $r_2(2)$?

17. Suppose that there are two risky assets, A and B, with expected returns equal to 2% and 5%, respectively. Suppose that the standard deviations of the returns are $\sqrt{6}\%$ and $\sqrt{11}\%$ and that the returns on the assets have a correlation $\rho = 0.1$.

    - What portfolio of A and B achieves a 3% rate of expected return?
    - What portfolios of A and B achieve an $\sqrt{5}\%$ standard deviation of return? Among these, which has the largest expected return?
    - Is is true that

    $$R_P = \omega R_A + (1 - \omega) R_B$$

    if R_P, R_A, R_B are log returns on the portfolio, on asset A and on asset B, respectively?

18. Suppose that Y_t is an AR(1) process with $\mu = 1$, $\phi_1 = 0.3$, and $\sigma_a^2 = 2$.

    - What is the variance of Y_1?
    - What is the covariance between Y_1 and Y_3?
    - What is the lag-3 autocorrelation of Y_t?

19. An AR(1) is fit to time series data and the papameter estimates are $\hat{\phi}_0 = 3$, $\hat{\phi}_1 = 0.7$. Moreover the last two observed values where $y_n = 7.88$, $y_{n-1} = 9.61$. Find the one- and two-steps ahead forecasts and the 95% prediction interval for y_{n+1}.

20. The following ARMA(1,1) model has been fit to a time series:

    $$X_t = 0.2X_{t-1} + a_t - 0.5a_{t-1}$$

    where $\{a_t\}$ is white noise.

    - Establish whether the process is stationary and/or invertible.
    - Find the one- and two steps ahead forecasts.
    - Suppose you observe $x_{n+1} = 0.32$ ($\hat{a}_{n+1} = 0.02$). Update the prediction for x_{n+2}.
    - Assume $\sigma_a^2 = 1.2$. Find the 99% prediction interval for x_{n+1} (recall that, using a large sample approximation, $e_n(1) \approx a_{n+1}$).