Question 1

Prices : $P_1 = 81$; $P_2 = 92$; $P_3 = 98$

Find $R_3, r_3, r_3(2), R_3(2)$

$$R_3 = \frac{P_3}{P_2} - 1 = 0.065$$

$$r_3 = \log(1+R_3) = \log(1.065) = 0.063$$

$$r_3(2) = r_3 + r_2 = \log\left(\frac{P_3}{P_1}\right) = 0.191$$

$$R_3(2) = e^{r_3(2)} - 1 = 0.21$$

Question 2

Monthly log returns : $r_1 = 0.067$ (6.7%)
$r_2 = 0.084$ (8.4%)
$r_3 = 0.049$ (4.9%)

Compute quarterly net return (%).

$$R_3(3) = e^{r_3(3)} - 1 = e^{r_1+r_2+r_3} - 1 = 0.2214 \ (22.14\%)$$

Question 3

$X_1, X_2, \dots$ iid $N(0.06, 0.47)$

- Find the mean and standard deviation of $X_1+X_2+X_3$:

$$E(X_1+X_2+X_3) = \sum_i E(X_i) = 3 \times 0.06 = 0.18$$

$$SD(X_1+X_2+X_3) = \sqrt{V(\textstyle\sum_i X_i)} \underset{\text{ind.}}{=} \sqrt{V(X_1)+V(X_2)+V(X_3)} = \sqrt{3 \times 0.47} = 1.187$$

- Distribution of $e^{X_1+X_2}$

Let $Y = e^{X_1+X_2}$; since $X_i \sim N(0.06, 0.47)$ $(i=1,2)$ and X_i are independent, $X_1+X_2 \sim N(0.12, 0.94)$

Recall that if $X \sim N(\mu, \sigma^2)$, then $e^X \sim LogN(\mu, \sigma^2)$.

Hence $Y = e^{X_1+X_2} \sim LogN(0.12, 0.94)$

- Find $P(X_1 < 1.5)$, with $X_1 \sim N(0.06, 0.47)$

$$P(X_1 < 1.5) = P\left(\frac{X_1 - \mu_{X_1}}{\sqrt{\sigma_{X_1}^2}} < \frac{1.5 - 0.06}{\sqrt{0.47}}\right) = \Phi(2.1) = 0.982$$

- Find $Cov(X_1, X_1+X_2)$. Using the covariance properties, we get

$$Cov(X_1, X_1+X_2) = Cov(X_1, X_2) + Cov(X_1, X_1) = 0 + V(X_1) = 0.47.$$

Question 4.

t	date	P_t	D_t
1	16 June 11	30.44	
2	15 Sept 11	30.22	0.23
3	15 Dec 11	31.34	0.22
4	15 Mar 12	33.45	0.25
5	15 June 12	32.22	0.25

Simple net return from t_2 to t_3

$$R = \frac{31.34 + 0.22}{30.22} - 1 = 0.044$$

Questions 5

Stock	(base y.) P_0	t_1 P_1	(base y.) Q_0	t_1 Q_1
A	80	109	50	51
B	60	53	80	97
C	20	53	180	264

Compute :

- the Laspeyres price index
- the Paasche price index

Using base-year quantities (Q_{0i}, $i = A, B, C$) we obtain

$$(\text{Laspeyres})\quad I^L_{t_1} = \frac{P_{1A}Q_{0A} + P_{1B}Q_{0B} + P_{1C}Q_{0C}}{P_{0A}Q_{0A} + P_{0B}Q_{0B} + P_{0C}Q_{0C}} \times 100$$

$$= \frac{109 \cdot 50 + 53 \cdot 80 + 53 \cdot 180}{80 \cdot 50 + 60 \cdot 80 + 20 \cdot 180} \times 100 = 155.081$$

Using current-year quantities (Q_{1i}) we get

$$(\text{Paasche})\quad I^P_{t_1} = \frac{P_{1A}Q_{1A} + P_{1B}Q_{1B} + P_{1C}Q_{1C}}{P_{0A}Q_{1A} + P_{0B}Q_{1B} + P_{0C}Q_{1C}} \times 100$$

$$= \frac{109 \cdot 51 + 53 \cdot 97 + 53 \cdot 264}{80 \cdot 51 + 60 \cdot 97 + 20 \cdot 264} \times 100 = 162.66$$

Question 6.

Let Φ be the cdf of $N(0,1)$.

- Find x such that $\Phi(-x) = 0.025$.

  $$\Phi(-x) = P(Z \leq -x) \Rightarrow -x = q_{0.025} = -q_{0.975}$$

  where q_α is the α-quantile of $Z \sim N(0,1)$.

  Hence, $x = q_{0.975} = 1.96$.

- Find the 0.975-quantile of $X \sim N(2, 0.49)$.

  $x_{0.975}$ is such that $0.975 = P(X \leq x_{0.975}) = \Phi\left(\frac{x_{0.975} - 2}{\sqrt{0.49}}\right)$

  $$\Rightarrow \frac{x_{0.975} - 2}{\sqrt{0.49}} = q_{0.975} = 1.96$$

  $$\Rightarrow x_{0.975} = 1.96\sqrt{0.49} + 2 = 3.37$$

Question 7.

log returns are normally distributed with mean 0.1 and standard deviation 0.13: $r_t \sim N(0.1, 0.13^2)$ h

- The distribution of one-period gross return is

$$1+R_t = e^{r_t} \sim \log N(0.1, 0.13^2)$$

- $P(1+R_t \geq 1.2) = 1 - P(1+R_t < 1.2)$

$$= 1 - P(\log(1+R_t) < \log(1.2))$$
$$= 1 - P(r_t < 0.18) = 1 - \Phi\left(\frac{0.18-0.1}{0.13}\right)$$
$$= 1 - \Phi(0.62) \approx 0.27$$

- What is the 0.85-quantile of $r_t(2)$?

$r_t(2) = r_t + r_{t-1} \Rightarrow r_t(2) \sim N(0.2, 0.034)$

since the $r_t \sim$ iid $N(0.1, 0.017)$

We want to find $q_{0.85}$: $\Phi\left(\frac{q_{0.85}-0.2}{\sqrt{0.034}}\right) = 0.85$

$$\Rightarrow q_{0.85} = \Phi^{-1}(0.85) \cdot \sqrt{0.034} + 0.2 = 1.032\sqrt{0.034} + 0.2$$
$$= 0.39$$

Question 8.

Daily log returns $r_t \overset{iid}{\sim} N(0.001, 0.015^2)$

- $P(r_t < 0.02) = P\left(\frac{r_t - \mu}{\sigma} < \frac{0.02 - 0.001}{0.015}\right) = \Phi(1.27) = 0.898$
- $SD(r_1+r_2+r_3) \overset{iid}{=} \sqrt{3V(r_t)} = \sqrt{3} \cdot 0.015 = 0.026$
- five-day log return: $r_t(5) = \sum_{i=0}^{4} r_{t-i}$

  Given that $r_t \overset{iid}{\sim} N(0.001, 0.015^2)$ we get

  $$r_t(5) \sim N(0.005, 0.001125)$$

- Let $X_0 = 1000$. What is $P(X_0(1+R_t(5)) < 990)$?

$$P(X_0(1+R_t(5)) < 990) = P(1+R_t(5) < 0.99)$$
$$= P(\log(1+R_t(5)) < \log(0.99))$$
$$= P(r_t(5) < -0.01) = \Phi(-0.45)$$
$$= 1 - \Phi(0.45) = 0.3264$$

Question 9.

Prices are assumed to follow a lognormal geometric random walk $(\mu = 0.15, \sigma^2 = 0.04)$

- 5-year expected log return : $E(r_5(5))$

  $r_5(5) = r_1 + r_2 + r_3 + r_4 + r_5$ and $r_i \overset{iid}{\sim} N(0.15, 0.04)$

  $\Rightarrow E(r_5(5)) = 5 \cdot 0.15 = 0.75$

- 10-year median log return $me(r_{10}(10))$

  $r_{10}(10) \sim N(1.5, 0.4) \Rightarrow me(r_{10}(10)) = 1.5$

- 10-year median gross return $me(1 + R_t(10))$

  $k=10$ gross return $1 + R_t(10) \sim \log N(1.5, 0.4)$

  Let $q_{0.5}$ be the 50th percentile (median) of $Y = 1 + R_t(10)$

  $$0.5 = P(Y \le q_{0.5}) = P(\log(Y) \le \log(q_{0.5}))$$
  $$= P(X \le \log(q_{0.5})) \quad , X \sim N(1.5, 0.4)$$
  $$= \Phi\left(\frac{\log(q_{0.5}) - \mu_X}{\sigma_X}\right)$$

  $$\Rightarrow 0 = \Phi^{-1}(0.5) = \frac{\log(q_{0.5}) - 1.5}{\sqrt{0.4}} \Rightarrow q_{0.5} = e^{1.5} = 4.48$$

  In general, $me(1 + R_k(k)) = e^{k\mu}$, where μ is the log-mean of the gross return $1 + R_t = e^{r_t}$.

- Let P_0 be the initial price at time $t=0$, $r_t \overset{iid}{\sim} N(\mu, \sigma^2)$.

  Given that $P_k = P_0 e^{r_1 + \dots + r_k} = P_0(1 + R_k(k))$

  - <u>median</u> price after k years $= \underline{P_0 e^{k\mu}}$
  - <u>mean</u> price after k years $= P_0 E(1 + R_k(k))$

  Since $1 + R_k(k) \sim \text{LogN}(k\mu, k\sigma^2)$ we have

  $$E(1 + R_k(k)) = \exp\left(k\mu + \frac{k\sigma^2}{2}\right)$$

  $\Rightarrow$ mean price of P_k is $P_0 E(1 + R_k(k)) = \underline{P_0 e^{k\mu + k\sigma^2/2}}$

The model for <u>log prices</u> is

$$p_t = p_0 + r_t + r_{t-1} + \dots + r_1 \quad , \text{ where } p_t := \log(P_t)$$

and $r_i \overset{iid}{\sim} N(\mu, \sigma^2)$ are log-returns.