Extensionality regained

- The problems of the loss of extensionality can be fixed.

- Modal discourse can be treated in an extensional way by complicating the semantics we usually use for classical logic (which is extensional).

- A quick reminder of propositional classical logic and its semantics is provided below,

Formal language of classical propositional logic

- Our formal language L is built by specifying:

  a vocabolary

  a syntax (well formation rules)

  a semantics

Vocabolary of language L

1. A set **At** of **non** logical symbols:

‘P’,’Q’,’R’,…

→ Sentential letters.

- A set **C** of logical symbols:

‘-’,’v’,’&’,’ → ’,’ ↔ ’

→ The connectives. Logical operators

- A set of auxiliary symbols, Par:

  ‘(‘ , ’)’

  → Parentheses.

Syntax of language L

- We determine the set of **well formed formulas** (WFF) of the language L, by giving the syntactical rules they must respect.

(1) Any sentential letter (At) is a WFF of L.

(2) If φ is a WFF of L, then $-\varphi$ is a WFF of L.

(with no parentheses!)

(3) if φ and ψ are WFFs of L, then also $(\varphi \vee \psi)$, $(\varphi \,\&\, \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are WFFs of L. (Between parentheses!)

(4) (Closure) Nothing else is a WFF of L.

- External parentheses sometimes can be omitted.

To avoid confusion in this case, we adopt the convention that v and & are stronger than → and↔ , but weaker than -.

Write some examples of WFFs.

Semantics of L

- The semantics gives meaning.

- To give meaning, we specify an interpretation function I, that gives meaning to all the basic expression in the vocabulary and to their (admissible) combinations.

- For the propositional part of the language meaning is given just by attributing a **truth value** to sentential letters.

- Truth values are True (1) and False (0).

- Note that:

  1. The semantics is **extensional**, because truth values are extensions of sentences.

  2. We do not need (for now) other extesions, because in L the simplest elements are sentences. We cannot analise their parts, like names and predicates.

- There are many interpretations of the language L, depending on what truth values are given to the sentential letters.

- For example, the interpretation, I, could be one that gives:

  I(P) = True (1)

  I(Q)=True (1)

  I(R)=False (0)

  ...

- The meaning of more complex WFF is given by extending I form sentential letters to all formulas,

  by considering the contribution of the logical connectives.

- This is given by **Truth tables.**

Truth Tables

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

- These notions are enough to define the notion of logical validity in propositional logic.

- An argument is composed of a finite number of premises and a conclusion.

- **Logical validity** is a **relation** between premises and conclusion of an argument.

- An argument is logically valid iff:

  in any case in which the premises are true,

  also the conclusion is true.

- In other words,

  a valid argument is an argument in which we never go from truth to false.

  Truth is never lost from premises to conclusion.

- To check whether an argument is valid, truth tables are used.

- In this case, we check whether in all cases (lines) in which the premises are true, also the conclusion is true.

- Note!

  An argument can be **valid** even if its premises and conclusion are **false**!

  and an argument can be **not valid** even if its premises and conclusion are **true**!

- What matters is only that in **no line we pass from truth to false**.

  Example:

  “Trieste is a city and Trieste is a city, so the moon is made of gold.”

  This is a logically valid argument!

Back to Modal discourse and extensionality

- Since the extensions of sentential letters are truth values,

  and the values of complex WFFs is determined by syntax and truth tables,

  it is easy to see that propositional logic is extensional.

- Suppose, for example, P =1, Q=1. So the extensions are the same.

- Suppose that R=0.

- Then (P v R) = 1, and so is (Q v R).

- But, as we know, modal discourse is not, apparently, extensional.

- A problematic example was the following:

- In English, we have:

  “The number 8 is prime” is true.

  “Rome is the capital of Italy” is true.

  But: “Necessarily, the number 8 is prime” is true

  “Necessarily, Rome is the capital of Italy” is false.

- However, it is possible to regain extensionality for modal logic by complicating the semantics.

- In particular we extend the language of propositional logic, introducing an operator symbol for necessity and one for possibility.

- And then extend the semantics of the propositional language L.

The language of Modal Propositional Logic

MPL

- Again, to define the language of modal propositional logic, we need to specify:

  - Vocabulary
  - Syntax
  - Semantics

- **Vocabulary** of MPL

- The vocabulary is the same of L plus two symbols for **necessity** and **possibility**.

- The new symbols are:

- Necessity: □

  (□ A means *It is necessary that A*)

- Possibility: ◊

  (◊A means *it is possible that A*)

- **Syntax** of MPL

- The syntax of MPL is like that of L, plus clauses for the modal operators:

  - If φ is a WFF, □φ is also a WFF

  - If φ is a WFF, ◊φ is also a WFF

- Give examples of modal formulas in MPL.

- Now the crucial step: we complicate the **semantics** of propositional logic.

- The complication allows to give an **extensional semantics** for (propositional) **modal discourse**.

- The complication consists in

1.introducing the notion of **possible world**.

2. Relativizing the interpretation to **possible worlds**.

Possible worlds

- Intuitively, **possible worlds** are like invented stories (fictional scenarios), or parallel universes.

- The idea was already used by earlier philosophers (Leibniz in particular).

- More precisely, what are *possible worlds*?

  This is a metaphysical question. We leave it open for now. We will discuss it in the part on the metaphysics of possible worlds.

- For now, we just use our intuitions.

- Note that the term “world” could be misleading.

- Possible worlds are not planets, but alternative **universes**. They can even include a great number fo galaxies.

- In some worlds, natural laws are violated. According to the idea that natural worlds are not necessary in the absolute sense.

- But in no worlds bachelors are married, or

  2+2=5.

→ Possible worlds are possible in the absolute sense.

- Intuitively, there is a great variety of possible worlds.

  (Examples...)

- In particular,

  In some possible worlds w, Milan is the capital of Italy.

  In some possible worlds w`, Rome is the capital of Italy (so w` is like the actual world).

  And so on.

- But then, the truth value of a claim is relative to a the possible world.

- A claim can be true or false, depending on what possible world is considered.

- For example:

  “Rome is the capital of Italy” is False in w, but it is True in w`.

- Thus, the semantic interpretation that assigns a truth value to sentence (letters) must be relativized to worlds.

- The interpretation of MPL should take worlds into account, and give different truth values in different worlds.

I(P,w) = 1	I’(P,w) = 0
I(Q,w) = 0	I’(Q,w) = 0
I(R,w) = 0	I’(R,w) = 1
...	

- The idea can then be generalized easily to complex WFF of MPL.

- Namely to WFF in which connectives occur.

- Also the semantic clauses for connectives (the truth tables) are now relativized to worlds.

- For example, a conjunction (A&B) <u>is true **in w**</u> iff A <u>is true **in w**</u> and B <u>is true **in w**</u>.

→ Note the dame world is considered. Different worlds are not mixed.

- Indeed, the specification of a world does not change much for the non modal part of the propositional language.

  If necessity and possibility are not involved, the world is redundant.

- We specify a world, but that world remains inert.

- This is not surprising, since we do not need possible worlds when there is no modality involved.

- We did not need possible worlds for L.

  We could already give an extensional semantics to L.

Possibility and necessity

- Possible worlds are crucial, however, for the semantic clauses of necessity and possibility.

- The underlying idea is very simple, but extremely powerful.

- What does it mean that *Florence could be the capital of Italy*?

  Namely, that *it is possible that Florence is the capital of Italy*?

- It means that **there is at least one possible world**, say w, in which Florence is the capital of Italy.

- w is not the actual world.

  It is a different, non actual possible world.

- Note that our world is also a possible world. It is the actual possible world.

- Other worlds, instead, are not actual.

- So, “It is possible that Rome is the capital of Italy”

  means: “**there is at least a possible world**, say n, in which Rome is the capital of Italy”

  In this case, n is just be the actual world.

- Give other examples of possibility and their reading in terms of possible worlds.

- Similarly, that something is **necessary** means that it is the case in every world.

- For example:

  “It is necessary that every bachelor is unmarried”

  means that **in every possible world**, bachelors are unmarried.

  There is not even a single world in which bachelors are married.

- “In every world” includes our worlds.

- “It is necessary that 2+2=4” means that in **every possible world** 2=2=4.

- “It is necessary that salt solves in water”

  means that in **every possible world** salt solves in water.

- Since we are considering absolute modality, this claim is false.

- It is false, because there is at least one possible world with different chemical laws and in which salt does not solve in water.

- So the statement is false, but its meaning is given in terms of all possible worlds.

- Give other examples of necessity.

The semantic interpretation is thus the following:

- It is **possible** that p =

  *there is **at least one** possible world in which p is true*

- It is **necessary** that p =

  ***in all** possible worlds p is true*

- Notice that possible worlds are not used (so far) to establish when modal claims are true or false.

  If not in some intuitive sense.

- They are used to clarify what such modal claims mean.

Semantics of MPL

- Semantics of MPL

  Now we can give the semantics also for the modal operators: □ and ◊.

- The semantics is:

$\Box\varphi$ is true in w, iff in **every** possible world v, φ is true in v.

Namely (more formally):

$I(\Box\varphi, w)=1$ iff in **every** possible world v, $I(\varphi, v)=1$

$\Diamond\varphi$ is true in w iff in there is **at least one** possible world v such that φ is true in v.

Namely:

$I(\Diamond\varphi,w)=1$ iff in there is **at least one** possible world v such that $I(\varphi,v)=1$

- Remarkably, this is enough to give an extensional treatment of (propositional) modal discourse.

Extensionality regained

- We can finally show how we got extensionality also for modal discourse.

- Since we introduced possible worlds, the notion of **extension** must be adapted.

  Extension is relativized to possible worlds.

- Consider, our original example but now relativized to worlds. Suppose that w is the actual world.

  “The number 8 is even” is true in w.
  “Rome is the capital of Italy” is true in w.

  → The two sentences have the same extension (truth) in w.

- However, in other worlds, they might have different extensions.

- Suppose that z is the possible world in which Florence, not Rome, is the capital of Italy.

  Then “Rome is the capital of Italy” is false in z.

  but “The number 8 is prime” is still true in z.

- Now take:

  “Necessarily, the number 8 is prime”

- Now we know that “Necessarily” can be taken to mean “In every possible world”

- So we have that

  “In every possible world, the number 8 is prime”

- Since it must be true in every world, in particular we can take world z.

So, we get: “the number 8 is prime” is true in z.

- But in z, “Rome is the capital of Italy” is false. So, in z, it has a different extension.

- So the two sentences have different extensions in different possible worlds.

- But if they have different extensions, the counter example is blocked.

- We thus obtain **a modal version of co-extensionality**:

Modal co-extensionality:

Two expressions are co-extenional if they have the same extension in the same possible worlds.

- Reframed in terms of possible worlds, the semantic of modal sentences respect substitution of modally co-referential expressions *salva veritate*.

- We have given an acceptable, extensional treatment of modal discourse, by using possible worlds.

- Note that we no longer have circularity.

- We broke the circularity by introducing possible worlds.

  Using possible worlds, we refined the extension, and we give an extensional semantics for modal discourse.

- Namely, now we understand modal discourse in terms of possible worlds.

- We no longer understand modal talk directly in terms of intensions, and intensions in terms of modal talk. Which was circular.

- Indeed, using possible worlds, we can also define in a clear way intensional notions.

- So, as a basis, we have possible worlds, and using them we can understand both modal discourse and intensional notions.

  And we do both things in an extensional way.

Intensional notions defined

- Another virtue of possible worlds, is that, in terms of possible worlds and extensions, we can give a rigorous, non circular, definition of intensional and seemingly problematic entities like properties and propositions.

- **Propositions** are the content of (declarative) sentences.

- For example, “Andrea is a teacher” and “Coltrane is a musician” have different contents.

  They are then taken to **express different propositions**.

- How can propositions be identified?

  By using possible worlds.

- A proposition is **the set of possible worlds in which the proposition is true**.

  (Or, quite equivalently, it is **a function from possible worlds to truth values.**)

- Clearly, if two propositions are different (as in our example), they are true in different possible worlds.

  So they consists in different sets of possible worlds.

  Note: we can say when two propositions are the same or not. We have clear identity conditions.

- Consider properties.

  A property is taken to be not just the set of elements that have that property in the actual world (as we did for cordates and renates)

  But **the variance of tha set in different worlds**.

- For example, being cordate and being renate have the same extension in the actual world.

- But not in every world!

  In some possible worlds, animals with a different biological structure have kidneys without having a heart and vice versa.

- Thus, we can define the intension of a predicate p (namely a property) to be the function I_P from possible worlds to elements in that world.

- Namely, a function that, for a predicate p and a world w, gives as value a set of elements in w.

  (Namely, the elements that are p in w)

- In this way, also the other initial problem is solved:

  We have a rigorous, non circular extensional definition of intensional notions lime propositions and properties.

- By using possible worlds,

  we have brought light into the darkness of the intensional realm.

What’s next?

- If modal discourse can be made sense of, we can reason about modal notions.

- We need to develop our logic more, however.

- Our modal semantics was enough to show the theoretical the legitimacy of modal notions.

  But it is still weak under other respects.

- For example, try, by using logical tools, to define whether the following is logically true.

  Namely, if it is true in every case:

$$(\Box P \rightarrow \Diamond P)$$

- You might be tempted to draw the truth table.

  But how can we handle □ and ◊?

- We know that they are about possible worlds.

  And, to be evaluated, we need to consider all possible worlds.

- But possible worlds, are greatly many. They are infinitely many. And we cannot draw infinitely many truth tables.

- Perhaps, a finite number of truth tables may be enough, because only some combinations of truth values are relevant.

- But, in some cases (if the sentence is complex) this can be non practical.

- Moreover, truth tables do no extend to Quantified Logic, and thus to Quantified Modal Logic.

- Thus, it is better if we consider a more practical and flexible tool to evaluate logically valid modal reasoning and claims.

- This tool is provided, for example, by refutation trees.

- We will study them next.