Modal logic 3

Normal Modal Logics

- Modal rules of S5 are of three kinds:

  - MN (Interdefinability. Worlds have no role)
  - ◊S5 rule: it **generates** a new world
  - □S5 rule: it **fills** a world.

(MN)

$$\sim\Diamond\alpha \quad (\omega)$$
$$\vdots$$
$$\Box\sim\alpha \quad (\omega)$$

$$\sim\Box\alpha \quad (\omega)$$
$$\vdots$$
$$\Diamond\sim\alpha \quad (\omega)$$

(◇S5)

$$\Diamond\alpha \quad (\omega)$$
$$\vdots$$
$$\alpha \quad (\upsilon)$$

where υ is NEW *to this path of the tree*

(□S5)

$$\Box\alpha \quad (\omega)$$
$$\vdots$$
$$\alpha \quad (\upsilon)$$

where υ is ANY *index*

(Closure)

$$\alpha \quad (\omega)$$
$$\vdots$$
$$\sim\alpha \quad (\omega)$$
$$\times$$

(□T)

$$\Box\alpha \quad (\omega)$$
$$\vdots$$
$$\alpha \quad (\omega)$$

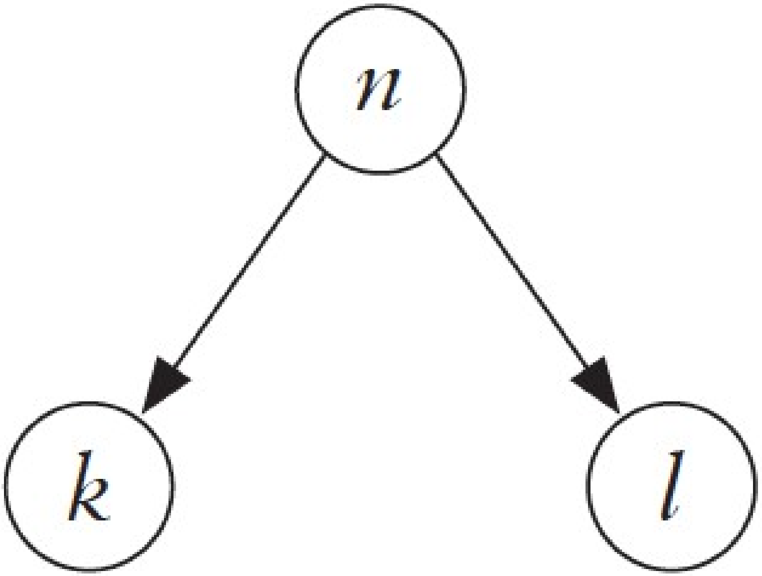
1.	$\sim((\Diamond p \,\&\, \Diamond q) \supset \Diamond(p \,\&\, q))$	(n)	NTF
2.	$(\Diamond p \,\&\, \Diamond q)$	(n)	1
3.	$\sim\Diamond(p \,\&\, q)$	(n)	1
4.	$\Box\sim(p \,\&\, q)$	(n)	3, **MN**
5.	$\Diamond p$	(n)	2
6.	$\Diamond q$	(n)	2
7.	p	(k)	5, ◊**S5**
8.	q	(l)	6, ◊**S5**
9.	$\sim(p \,\&\, q)$	(k)	4, □**S5**
10.	$\sim(p \,\&\, q)$	(l)	4, □**S5**
11.	$\sim p$ (k) × | $\sim q$ (k)		9
12.	(under $\sim q$ (k)): $\sim p$ (l) ↑ | $\sim q$ (l) ×		10

Accessibility

- In the tree above, worlds k and l are **generated** from world n.
- When one world generates another then it has **access** to it.

  - We write “wAv” for “w has access to v”
  - **Accessibility relations** (relative possibility)

In the tree above we have:



- One way of changing the logic is to restrict the world **filler** rule (□).

- The world filler rules are unrestricted in S5:

  If □A is in a world, then A can be put into **any world**.

So, in S5 every world has access to every world (including itself).

- A different, simple condition could be:
  - **Filling** (□ rule) **only** holds for **worlds generated** from the world in which the formula occurs.
    - → Only the directly generated worlds can be accessed

$(\Diamond R)$

$$\Diamond\alpha \qquad (\omega)$$

$$\vdots$$

$$\omega A \upsilon$$

$$\alpha \qquad (\upsilon)$$

where υ *is* NEW *to this path of the tree*

$(\Box R)$

$$\Box\alpha \qquad (\omega)$$

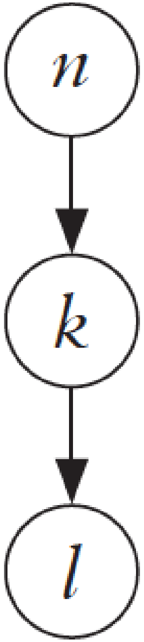
$$\omega A \upsilon$$

$$\vdots$$

$$\alpha \qquad (\upsilon)$$

1.	$\sim(\Box p \supset \Box\Box p)$	(n)	NTF
2.	$\Box p$	(n)	1
3.	$\sim\Box\Box p$	(n)	1
4.	$\Diamond \sim \Box p$	(n)	3, **MN**
5.	nAk		4, $\Diamond$R (to keep track of accessibility)
6.	$\sim\Box p$	(k)	4, $\Diamond$R
7.	$\Diamond \sim p$	(k)	6, **MN**
8.	p	(k)	2, 5, $\Box$R (*two* entries are needed to justify this)
9.	kAl		7, $\Diamond$R (see line 5 above)
10.	$\sim p$ $\uparrow$	(l)	7, $\Diamond$R

– Note that n has no access to l, so we cannot apply □R to close the tree.



There is no arrow from n to l, because there is no access nAl

From K to S5

(Normal modal logic. Logics built from K)

- Call **P**Tr the propositional rules for trees
- Call **MN** the modal equivalences
- Together they are the **SW** (single world) Ptr + MN
    - So: **S5**Tr = SW + (◊S5, □S5, □T)

K

K

- The logic obtained with the restricted rules above is **K**.

KTr = SW + (◊R, □R)

Consider (p→ □◊P) in K

1.	$\sim(p \supset \Box\Diamond p)$	(n)	NTF
2.	p	(n)	1
3.	$\sim\Box\Diamond p$	(n)	1
4.	$\Diamond \sim \Diamond p$	(n)	3, MN
5.	nAk		4, ◊R
6.	$\sim\Diamond p$	(k)	4, ◊R
7.	$\Box \sim p$	(k)	6, MN

↑

- In **K** we cannot go any further.

  Because the world k has access to no other world.

  (No world has been generated from k).

- Note:

  A world does not even have generated access to itself!

- **K** is a very limited logic.
- Richer logics can be built by adding new filler rules (□ rules).

- We consider some of these logics:

  T

  S4

  Br

  S5

- How good is K as a logic for possibility?

→For example, intuitively, $(p \rightarrow \Diamond p)$ should be valid.

Can it be proved in **K**?

- What formula is K valid? Can you find one?

- For example, the necessitation of tautologies, like:

$\Box(P \vee -P)$

- Or formulas such as:

$-(\Diamond P \,\&\, \Box -P)$

T

T

- The logic **T** is obtained by adding a filler rule to **K,** the rule □T.

  □T is the same as in S5.

- **T**Tr = SW + (◊R, □R, □**T**)

Consider (□P → P).
It is **T**-Valid, but not **K**-valid.

(Crucial formula for the difference)

1.	$\sim(\Box p \supset p)$	(n) NTF
2.	$\Box p$	(n) 1
3.	$\sim p$	(n) 1
4.	p	(n) 2, $\Box$T
	$\times$	

- Also $(p \rightarrow \Diamond p)$ can be proved now.

- Can T prove $(\Box p \rightarrow \Box\Box p)$?

S4

S4

Logic **S4** is obtained adding another filler rule to **T**, the Rule: □□R.

S4Tr = SW + (◊R, □R, □T, □□**R**)

The rule □□**R** makes you move an entire formula □p in another accessible world (not just the formula p).

$$(\Box\Box\mathrm{R}) \qquad \begin{array}{ll} \Box\alpha & (\omega) \\ \omega A \upsilon & \\ \vdots & \\ \Box\alpha & (\upsilon) \end{array}$$

The crucial formula, distinguishing **S4** from **T** is: $(\Box p \rightarrow \Box\Box P)$

1.	$\sim(\Box p \supset \Box\Box p)$	(n)	NTF
2.	$\Box p$	(n)	1
3.	$\sim\Box\Box p$	(n)	1
4.	$\Diamond\sim\Box p$	(n)	3, MN
5.	nAk		4, $\Diamond$R
6.	$\sim\Box p$	(k)	4, $\Diamond$R
7.	$\Diamond\sim p$	(k)	6, MN
8.	p	(k)	2, 5, $\Box$R
9.	kAl		7, $\Diamond$R
10.	$\sim p$	(l)	7, $\Diamond$R

Applying □□R we can proceed:

11.	$\Box p$	(k)	2, 5, □□**R**
12.	p	(l)	9, 11, □**R**
	×		

- Can T prove $(p \rightarrow \Box\Diamond p)$?

- Can S4 prove it?

Br

- The logic **Br** is obtained by adding a different filler rule to **T** (not to S4!)*, the rule □**SymR.**

* There is a mistake in Girle's book.

BrTr = SW + (◊R, □R, □T, □**SymR**)

- □**SymR** allows the exemplification of □p (like □R**)**, but *backwards* with respect to accessibility.

$$(\Box\text{SymR}) \qquad \begin{array}{ll} \Box\alpha & (\upsilon) \\ \omega A\upsilon & \\ \vdots & \\ \alpha & (\omega) \end{array}$$

The crucial formula distinguishing **Br** from **T** is: (P→□◊P).

In the last line we need □SymR

1.	$\sim(p \supset \Box\Diamond p)$	(n)	NTF
2.	p	(n)	1
3.	$\sim \Box\Diamond p$	(n)	1
4.	$\Diamond\sim\Diamond p$	(n)	3, **MN**
5.	nAk		4, **◊R**
6.	$\sim\Diamond p$	(k)	4, **◊R**
7.	$\Box\sim p$	(k)	6, **MN**
8.	$\sim p$	(k)	7, **□T**
9.	$\sim p$	(n)	5, 7, **□SymR**

- Can T, Br, or S4 prove $(\Diamond p \rightarrow \Box\Diamond p)$?

S5

- We already know **S5**.
- It can be obtained by the rules we gave at the beginning, or by adding a new rule to **S4**.

- **S5** is obtained (in a new way) by adding a different filler rule (a new □ rule) to **S4**.

- **S5**Tr = SW + (◊R, □R, □T, □□R, □□**SymR**).

The rule: □□**SymR** is similar to □□R (S4) but backwards.

(as □SymR (Br) is □R backwards)

$$(\square\square\text{SymR}) \qquad \begin{array}{ll} \square\alpha & (\upsilon) \\ \omega A\upsilon & \\ \vdots & \\ \square\alpha & (\omega) \end{array}$$

The crucial formula of **S5** is $(\Diamond P \rightarrow \Box\Diamond P)$

1.	$\sim(\Diamond p \supset \Box\Diamond p)$	(n)	NTF
2.	$\Diamond p$	(n)	1
3.	$\sim \Box\Diamond p$	(n)	1
4.	$\Diamond\sim\Diamond p$	(n)	3, MN
5.	nAk		4, $\Diamond$R
6.	$\sim\Diamond p$	(k)	4, $\Diamond$R
7.	nAl		2, $\Diamond$R
8.	p	(l)	2, $\Diamond$R
9.	$\Box\sim p$	(k)	6, MN
10.	$\sim p$	(n)	5, 9, $\Box$SymR
11.	$\Box\sim p$	(n)	5, 9, $\Box\Box$SymR
12.	$\sim p$	(l)	5, 11, $\Box$R
	$\times$		

The orthodox strategy

- **Hintikka** strategy (used so far) generates logics adding filler **rules**.

- The **orthodox** strategy generates logics by adding properties to the **accessibility relation** between worlds.

- So far, given worlds w, n, we had that wAn only if w **generates** n.

- Now we are going to enrich the accessibility relation A with formal properties, so that wAn even in cases in which n is not generated from w.

- For example, A can be reflexive (Refl):

  for every world: wAw

- This gives, in another way, the logic T:

  Ttr = SW + (◊R, □R, Refl)

- Take (□P →P) (characteristic of T)

1.	$\sim(\Box p \supset p)$	(n) NTF
2.	$\Box p$	(n) 1
3.	$\sim p$	(n) 1
4.	nAn	Refl
5.	p	(n) 2, 4, □R
	$\times$	

- If we also add transitivity (Trans) to T, S4 is obtained.

Trans = if wAn and nAk, then wAk

- S4Tr = SW + (◊R, □R, Refl, Trans)

Consider (□p → □□P) (characteristic of S4)

1.	$\sim(\Box p \supset \Box\Box p)$	(n)	NTF
2.	$\Box p$	(n)	1
3.	$\sim\Box\Box p$	(n)	1
4.	$\Diamond\sim\Box p$	(n)	3, MN
5.	nAk		4, ◇R
6.	$\sim\Box p$	(k)	4, ◇R
7.	$\Diamond\sim p$	(k)	6, MN
8.	p	(k)	2, 5, □R
9.	kAl		7, ◇R
10.	$\sim p$	(l)	7, ◇R
11.	nAl		5, 9, Trans
12.	p	(l)	2, 11, □R
	×		

- If *symmetry* (Sym) is added to T we obtain Br

  (In Br we have reflexivity, but not transitivity)

  Sym = if wAn, then nAw

- BrTr = SW + (◊R, □R, Refl, Sym)

- S5 is obtained by having all these properties: *reflexivity*, *transitivity*, *and symmetry*.

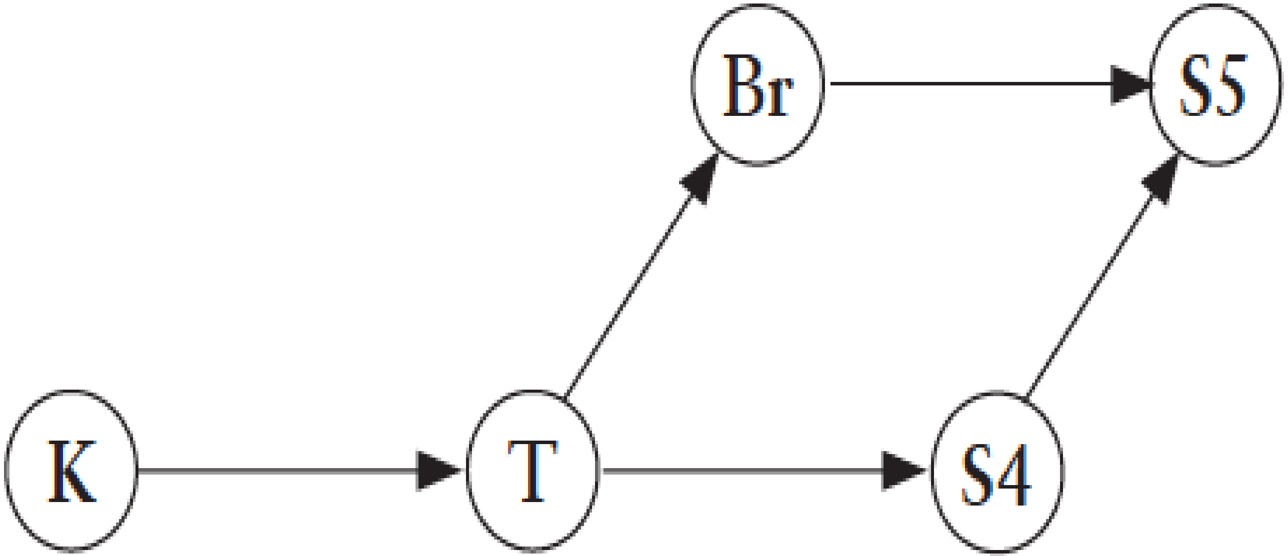
  (equivalence relation)

- S5Tr = SW + (◊R, □R, Refl,Trans, Sym)

Consider (◊P → □◊P) (Characteristics of S5)

1.	$\sim(\Diamond p \supset \Box\Diamond p)$	(n)	NTF
2.	$\Diamond p$	(n)	1
3.	$\sim\Box\Diamond p$	(n)	1
4.	$\Diamond\sim\Diamond p$	(n)	3, MN
5.	nAk		4, ◊R
6.	$\sim\Diamond p$	(k)	4, ◊R
7.	nAl		2, ◊R
8.	p	(l)	2, ◊R
9.	$\Box\sim p$	(k)	6, MN
10.	kAn		5, Sym
11.	kAl		10, 7, Trans
12.	$\sim p$	(l)	11, 9, □R
	×		

Relations among the main normal modal logics:



Finite modalities

- Consider a formula O...OP

  Where O...O stand for a finite sequence of modal operators, for example: ◊□□◊□◊◊ P

- IN S4 there are equivalences (see the book for details) that allow to simplify any of those sequences to just **seven modalities** plus their negation (14 in total):

/ □ ◊ □◊ ◊□ ◊□◊ □◊□

Modal equivalences in S4

$$\Box\Box p \equiv \Box p$$
$$\Diamond\Diamond p \equiv \Diamond p$$
$$\Box\Diamond\Box\Diamond p \equiv \Box\Diamond p$$
$$\Diamond\Box\Diamond\Box p \equiv \Diamond\Box p$$

- In S5 there are only **three modalities** plus their negations (6 in total):

/ □ ◊

- Given the equivalences, in S5, in any sequence of modal operators we *<u>may delete all but the **last**</u>* to gain an equivalent formula.

For example: ◊□□◊□◊◊□◊P is equivalent to ◊P

Counterexamples

- Counterexamples in S5 are as before.
- The definition is the same:

  A system is a counter-example to a formula's being valid iff the formula is ***false in at least one world in the system****.*

- But for the other logics we need to change it considering **accessibility** relations.

Counterexamples in K

- Consider (□P → P). It has an open tree in K.

1.	$\sim(\Box p \supset p)$	(n)	NTF
2.	$\Box p$	(n)	1
3.	$\sim p$	(n)	1

- There is only one world in this system: n
- n **accesses** no world (not even itself).
- P(n) = 0 (because we have -P in n)

- *What is the value of □P in n?*

□P is true in n, if P is true in all worlds to which n has access.

- **if n has no access to worlds, then *□P (n) = 1***

(namely, ***□P*** is true in n).

- Why?
- Because □*P can be read, intuitively, as:

“you cannot even see a single w world in which P is false”

- *If there is access o no world, then this is the case. So* □*P is true.*

- By contrast,

◊P reads:

“*You can see at least one world in which P is true.*”

If no world can be accessed, then this is false.

- *So we obtain:*

  □P is true in n

  and P is false in n.

- So we have a counterexample to (□P→P)

→Accessibility is crucial to establish that *□P is true in n*

1.	$\sim((\Diamond p \,\&\, \Diamond q) \supset \Diamond(p \,\&\, q))$	(n)	NTF
2.	$(\Diamond p \,\&\, \Diamond q)$	(n)	1
3.	$\sim\Diamond(p \,\&\, q)$	(n)	1
4.	$\Box\sim(p \,\&\, q)$	(n)	3, **MN**
5.	$\Diamond p$	(n)	2
6.	$\Diamond q$	(n)	2
7.	p	(k)	5, **◊S5**
8.	q	(l)	6, **◊S5**
9.	$\sim(p \,\&\, q)$	(k)	4, **□S5**
10.	$\sim(p \,\&\, q)$	(l)	4, **□S5**
11.	$\sim p$ (k) × ⎮ $\sim q$ (k)		9
12.	(under $\sim q$:) $\sim p$ (l) ↑ ⎮ $\sim q$ (l) ×		10

Remember that if n has access to no world, then ◊P is false in n.

So the system (and counterexample in K) is:

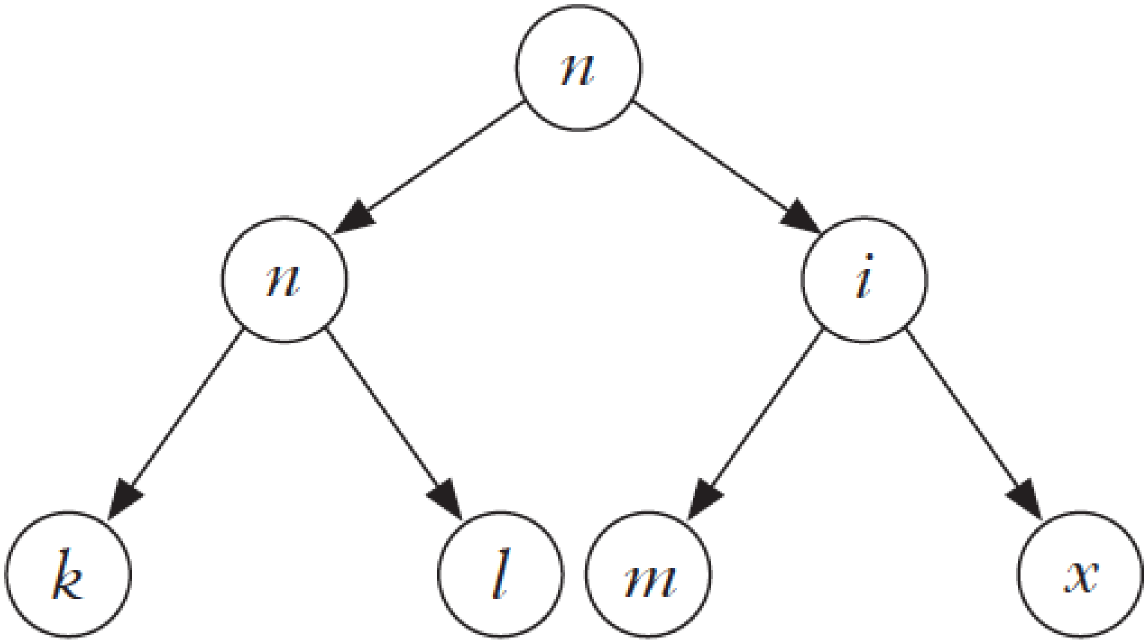
	nAk		nAl
	n	k	l
p	0	1	0
q	0	0	1
$(p \,\&\, q)$	0	0	0
$\Diamond p$	1	0	0
$\Diamond q$	1	0	0
$\Diamond(p \,\&\, q)$	0	0	0

Counterexamples in T

- Consider the formula $\Box\Diamond P \rightarrow \Diamond\Box P$. *Its tree in T is:*

1.	$\sim(\Box\Diamond p \supset \Diamond\Box p)$	(n)	NTF
2.	$\Box\Diamond p$	(n)	1
3.	$\sim\Diamond\Box p$	(n)	1
4.	$\Box\sim\Box p$	(n)	3, MN
5.	$\Diamond p$	(n)	2, $\Box$T
6.	$\sim\Box p$	(n)	4, $\Box$T
7.	$\Diamond\sim p$	(n)	6, MN
8.	nAk		7, $\Diamond$R
9.	$\sim p$	(k)	7, $\Diamond$R
10.	$\Diamond p$	(k)	2, 8, $\Box$R
11.	$\sim\Box p$	(k)	4, 8, $\Box$R
12.	$\Diamond\sim p$	(k)	11, MN
13.	kAl		12, $\Diamond$R
14.	$\sim p$	(l)	12, $\Diamond$R
15.	kAj		10, $\Diamond$R
16.	p	(j)	10, $\Diamond$R
17.	nAi		5, $\Diamond$R
18.	p	(i)	5, $\Diamond$R
19.	$\Diamond p$	(i)	2, 17, $\Box$R
20.	$\sim\Box p$	(i)	4, 17, $\Box$R
21.	$\Diamond\sim p$	(i)	20, MN
22.	iAm		21, $\Diamond$R
23.	$\sim p$	(m)	21, $\Diamond$R
24.	iAx		19, $\Diamond$R
25.	p	(x)	19, $\Diamond$R

These are the accessibility relations among the worlds.



- Building a T-counterexample based on this tree is difficult.

- But we can proceed differently:

  we can try to directly build a system making the antecedent of the formula true in n, and the consequent false in n.

With accessibility relations defined accordingly.

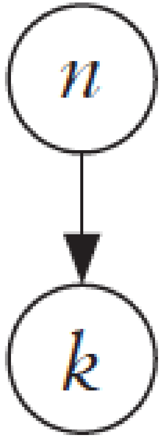
- Note:

  When there are too many worlds (more than 3), it is better to proceed directly with a system of worlds, without the tree.

Counterexample to $\Box\Diamond P \rightarrow \Diamond\Box P$

$nAk \quad nAn \quad kAk$

	n	k
p		
$\Diamond p$		
$\Box\Diamond p$	1	
$\Box p$		
$\Diamond\Box p$	0	



◊P must be true in both worlds if □◊P is true in n.
□P must be false in both worlds, if ◊□P is false in n.

$nAk \quad nAn \quad kAk$

	n	k
p		
$\Diamond p$	1	1
$\Box\Diamond p$	1	
$\Box p$	0	0
$\Diamond\Box p$	0	

- Since □P is false in k, P must also be false in K (K has access only to itself)

(This makes □P false also in n, since n accesses both n and k)

$nAk \quad nAn \quad kAk$

	n	k
p		0
$\Diamond p$	1	1
$\Box\Diamond p$	1	
$\Box p$	0	0
$\Diamond\Box p$	0	

- *But there is a problem:*

  since k has only access to itself, and P is false in k, ◊*P should also be false in k.*

  Against the model above.

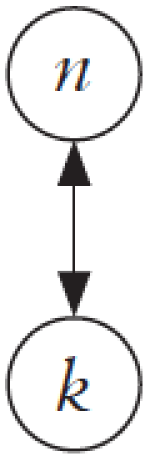
  - If we change it, and put ◊P = 0 in k, then □◊P is false in n (since nAk).

    But then we do not have a counterexample.

The solution is making the accessibility relations more complex, adding kAn, and putting P true in n.

$nAk \quad kAn \quad nAn \quad kAk$

	n	k
p	1	0
$\Diamond p$	1	1
$\Box\Diamond p$	1	
$\Box p$	0	0
$\Diamond\Box p$	0	



- *This gives a counterexample in T*.

  – In general, T requires reflexivity, and we have it.

    Although it is **not obligatory** for the accessibility relation to be symmetric in T, it is **permissible**.

    → *Remember what we said about frames!*

- Since the relation in this system is also transitive, reflexive and symmetric,

  this is also a countermodel also for S4 and for S5.

- Indeed, in general:

  a counterexample in a stronger system is also a counter example for a weaker system, but not always *vice versa*.

  → For example, an S4 counterexample is always also a K.counterexample.

  The converse is not always false (but sometimes it can).

  → Remember what we said about frames!

The end