Modal logic 5

Predicate Modal logic

- We extend the language of propositional S5 to modal predicate logic by adding **identity** and **quantifiers**.

- We also add formation rules
    - Namely, rules to get well formed formulas (but we skip this).

and **rules for refutation trees**.

- We focus on S5, but we could add quantifiers and identity to any of the propositional system.

- We will have S5QT (S5 Quantified Trees),

  and, in particular, S5QT= (S5QT with identity) in **two versions**.

  → Many philosophical issues.

Identity

- Identity limited to a single world is **contingent** identity (as in the next rule).

- Namely, t and k can be identical in one world, but not identical in other worlds.

  So identity is not necessary.

  → In the rules, worlds do not change.

Contingent SI

(CSI) $\kappa = \iota$ (ω)

Φ (ω)

⋮

Ψ (ω)

where Φ is a literal and where Ψ is the result of either *replacing one or more occurrences of κ in Φ with ι* or *replacing one or more occurrences of ι in Φ with κ*

(Closure) $\kappa \neq \kappa$ (ω)

⋮

×

1.	Fa	(n)	P
2.	$a = b$	(n)	P
3.	$\sim Fb$	(n)	NC
4.	Fb	(n)	1, 2, CSI

- Note that with the rule CSI this branch does not close:

1. a = b (n)
2. -(a = b) (k)

→ Since the worlds are **different** we cannot derive:

-(a=a) (k)

→ To close it we need “transworld” identity, namely **necessary identity**, as in the next rule.

Necessary SI:

(NSI) $\kappa = \iota \quad (\upsilon)$

$$\Phi \quad (\omega)$$

$$\vdots$$

$$\Psi \quad (\omega)$$

where υ *is* ANY *index and* Ψ *is the result of either replacing one or more occurrences of* κ *in* Φ *with* ι *or replacing one or more occurrences of* ι *in* Φ *with* κ

– We assume CSI for now.

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• Philosophical spin off:

- The philosophical discussion on the **necessity of identity** is one of the most important of the last century.

- The main contribution is by Saul Kripke, in the book *Naming and Necessity*.

- The main thesis of that book (on which there is general consensus) is that we should distinguish proper names (rigid) and descriptions (non rigid).

  (Descriptions are expressions like “the king of France”).

- **Identities involving proper names are necessary**.

  Namely: proper names refer to the same item in all world.

  Proper names are **rigid designators**.

Quantifiers

- We extend the language of propositional S5 to modal predicate logic by adding:

  - predicate logic symbols
  - formation rules for them.

Formulas like:

$\Box(\forall x)(Ax \supset Bx)$; $(\forall x)\Box(Ax \supset Bx)$; $(\exists x)(Ax \,\&\, \Diamond Bx)$

- The new symbols and formation rules are as usual.

(∀x) is abbreviated (x)

Symbols

Propositional letters operators and parentheses plus:

individual constants: $a, b, c, d, \ldots$

individual variables: $w, x, y, z, \ldots$

predicate letters $(n \geq 1)$: $F^n, G^n, H^n, J^n, \ldots$

quantifiers: $\forall, \exists$

- S5QT is S5 plus rules of the quantifiers (next slide).

(QN)

$$\sim(\exists\eta)\alpha \quad (\omega) \quad \checkmark$$
$$\vdots$$
$$(\forall\eta)\sim\alpha \quad (\omega)$$

$$\sim(\forall\eta)\alpha \quad (\omega) \quad \checkmark$$
$$\vdots$$
$$(\exists\eta)\sim\alpha \quad (\omega)$$

(EI)

$$(\exists\eta)\alpha \quad (\omega) \quad \checkmark\kappa$$
$$\vdots$$

$$\beta \quad (\omega)$$

where β *is the result of replacing every free occurrence of* η *in* α *with an individual constant, say* κ*, where* κ *is* NEW *to this path of tree*

(UI)

$$(\forall\eta)\alpha \quad (\omega) \quad \backslash\kappa$$
$$\vdots$$

$$\beta \quad (\omega)$$

where β *is the result of replacing every free occurrence of* η *in* α *with* ANY *individual constant, say* κ

- Test:

  $(x)\Box Fx \rightarrow \Box(x)Fx$

  (Barcan Formula)

$$((\forall x)(\Box Fx) \supset \Box(\forall x)Fx).$$

1.	$\sim((\forall x)\Box Fx \supset \Box(\forall x)Fx)$		(n)	NTF
2.	$(\forall x)\Box Fx$	\a	(n)	1
3.	$\sim\Box(\forall x)Fx$		(n)	1
4.	$\Diamond\sim(\forall x)Fx$		(n)	3 **MN**
5.	nAk			4, ◊R
6.	$\sim(\forall x)Fx$		(k)	4, ◊R
7.	$(\exists x)\sim Fx$		(k) ✓ a	5, QN
8.	$\sim Fa$		(k)	7, EI a
9.	$\Box Fa$		(n)	2, UI a
10.	Fa		(k)	9, 5, □R

X

Existence

- If an individual exists in some world, does it exist in every world?
- Namely, *is it possible that a certain individual i does not exist*?

Can $\Diamond{\sim}(\exists x)(x = i)$ be true (in some world)?

- The intuitive answer is “yes”.

  (This table might not exist.)

- But according to S5QT= the answer is “no”!

  - The negation of that claim, namely - ◊- (∃x)(x = a), is always true!

- Try a tree for $\Box(\exists x)(x = a)$,

  which is equivalent to $-\Diamond- (\exists x)(x = a)$.

  If you negate this you get the first line below by DN.

1.	$\Diamond \sim (\exists x)(x = a)$	(n)	*Formula*
2.	nAk		1, $\Diamond$R
3.	$\sim(\exists x)(x = a)$	(k)	1, $\Diamond$R
4.	$(x) \sim (x = a)$	$(k) \setminus a$	
5.	$\sim(a = a)$	(k)	

X

- If we want that some item might not exist,
  or we do not want the Barcan formulas,
  then the logic must be changed.

  - To falsify Barcan formulas, we will discuss **domains**,

  - To falsify the existence of individuals we will discuss **existential import**.

Barcan formulas

- **Barcan formula** (BF) is:

  $(x)\Box Fx \rightarrow \Box(x)Fx$

  → Note: the antecedent begins with $(x)\Box$...

- Its **converse** (CBF) is :

  $\Box(x)Fx \rightarrow (x)\Box Fx$

They can also be expressed by using the existential quantifier and the diamond.

By using inter-definability.

→ We focus on the formulas with the universal quantifier and box.

- So we have the following, which are all valid in S5QT=.

(And also in K)

1. $(\Diamond(\exists x)Fx \supset (\exists x)\Diamond Fx)$ (Barcan formula (BF))
2. $((\exists x)\Diamond Fx \supset \Diamond(\exists x)Fx)$ (converse of the BF (CBF))
3. $((x)\Box Fx \supset \Box(x)Fx)$ (BF)
4. $(\Box(x)Fx \supset (x)\Box Fx)$ (converse of the BF)

Barcan formulas and their converse express commutation of quantifiers/modalities.

$$((x)\Box Fx \equiv \Box(x)Fx)$$

$$((\exists x)\Diamond Fx \equiv \Diamond(\exists x)Fx)$$

Domains

- Quantification involves a domain of entities.

- (x)Fx means “all items **in a certain domain** are F”.

  The domain can be defined in different ways.

  (The domain of natural numbers, the domain of Chinese cities, the domain of books,...)

- A possible world is associated with a domain:

  the domain of items existing in that possible world.

- Domains in different worlds could contain or not the same elements.

  - The same thing could exist or not in different worlds.

- **Three options:**

1. (AE) each world has a different domain (no overlap).

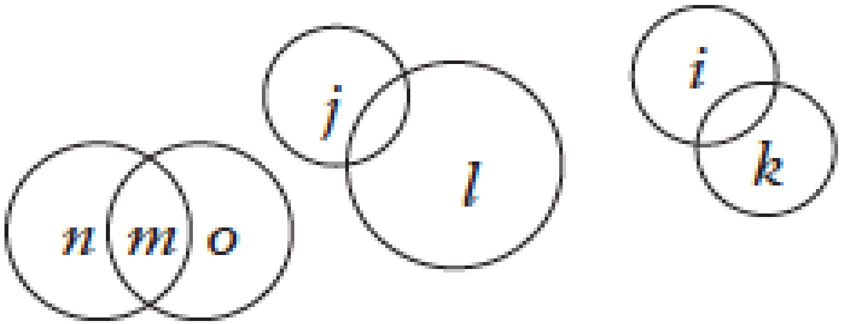
2. (BE) each world has the same domain.

3. (CE) in each worlds domain vary in arbitrary way (overlap is possible).

(AE)

(BE)

(CE)



Now we can go back to the Barcan formula.

$(x)\Box Fx \rightarrow \Box(x)Fx$

- How could we falsify the Barcan formula?

- Think of a counter-model to the Barcan formula.

Evaluations in variable domains CE

- Assume variable domains CE.

  So, some items exist in some worlds, but not in others.

  For example, i might exist in world n but not in world k.

- Suppose that i does not exist in world k.

- What is the value of a formula like F(i) in k?

→ This is a crucial question if we have variable domains and we want to evaluate formulas.

- There are two options:

1.

If i does not exist in world k, F(i) is **neither true nor false** in k.

F(i) has no value. F(i) is **gappy**.

2.

If i does not exist in world k, F(i) is **false** in k.

F(i) has a value: **falsity**.

- Both options could be defended.
- We focus on their effects on BF and CBF.
- We begin with the **gappy** option.

A (gappy) counter-model to the Barcan formula

- We use * for a truth value gap.

  * means neither true nor false.

    → This leaves a truth-value gap in the table of counter-example.

- A gappy counter example to BF is given by:

D(n) = {b}   D(k) = {a, b} (nAn,nAk,nAk,kAn)

	n	k
Fa	*	0
Fb	1	1
$(x)Fx$	1	0
$\Box(x)Fx$	0	0
$\Box Fa$	*	0
$\Box Fb$	1	1
$(x)\Box Fx$	1	0

- In the corresponding logical system the quantifiers are **relative to worlds**.

$(x)\Box Fx$ (n) is :

"*Everything* ***that exists in n*** *is, in all worlds to which n has access, F*."

$\Box(x)Fx$ (n) is:

"*In each world accessible from n, everything* ***that exist*** *is F*."

$((x)\Box Fx \supset \Box (x)Fx)$ (n) is:

“If everything **that exists in n** is, in all worlds to which n has access, F,

then in all worlds accessible from n everything **that exists** is F.”

- In the counter-example we have a system of two worlds, {n , k }.

- In world n the domain of quantification is just b, and b is F in world n.

- In world k the domain of quantification is b and a.

  In world k, b is F, but a is not F .

- The BF **antecedent** (x)□Fx states that everything in n (namely b) is F in every world to which n has access.

- n has access to both n and k, and in both worlds b is F.

So the antecedent is true.

- The BF **consequent** $\Box(x)Fx$ states that in every world to which n has access, everything is F .

- But, not everything in world k is F.

So the consequent is false.

Modal logic for Variable Domains

- To have a logic for CE, restrictions must be placed on the instantiation of quantifiers.

  S5**R**QT= (**R**elative domain) is the logic for CE.

- **A constant (k) now must be relativized to the** (domain of a) **world (w)** $\boldsymbol{\kappa_{\omega}}$**.** (The world in which k exists)

- UI must be changed.

(RUI)	$(\forall\eta)\alpha$	$(\omega) \setminus \kappa_{\omega}$
	β	(ω)

where β is the result of replacing every free occurrence of η in α with an individual constant, say κ, provided that, in this path of the tree, either κ occurs in any formula in ω or, if not, κ is NEW *to this path of the tree*

BF is **not** valid in S5RQT=

1.	$\sim((x)\Box Fx \supset \Box(x)Fx)$	(n)	NTF
2.	$(x)\Box Fx$	(n)	1
3.	$\sim\Box(x)Fx$	(n)	1
4.	$\Diamond\sim(x)Fx$	(n)	3, MN
5.	nAk		4, ◊R
6.	$\sim(x)Fx$	(k)	4, ◊R
7.	$(\exists x)\sim Fx$	(k)	6, QN
8.	$\sim Fa$	(k)	7, EI *a*
9.	$\Box Fb$	(n)	2, RUI *b* (*b* is new)
10.	Fb	(n)	9, □T
11.	Fb	(k)	5, 9, □R
	↑		

- In this tree we are unable to use UI with ***a*** at line 9 to get □F**a**   (n),

  because **a** does **not** occur in some formula in world n, but in some formula in world k, so a **new** one (b) in the path must be used.

  - This means that a does not exist in world n, but it does exist in world k.

Converse BF

- Is also CBF false in S5**R**QT=?

- CBF: $\Box(x)Fx \supset (x)\Box Fx$ (n) is:

“If in all worlds accessible to n, everything is F, then everything that exists in n is F in every world accessible to n.”

CBF is true in S5**R**QT= (RUI)

1.	$\sim(\Box(x)Fx \supset (x)\Box Fx)$	(n)	NTF
2.	$\Box(x)Fx$	(n)	1
3.	$\sim(x)\Box Fx$	(n)	1
4.	$(\exists x)\sim\Box Fx$	(n)	3, QN
5.	$\sim\Box Fa$	(n)	4, EI a
6.	$\Diamond\sim Fa$	(n)	5, MN
7.	nAk		6, $\Diamond$R
8.	$\sim Fa$	(k)	6, $\Diamond$R
9.	$(x)Fx$	(k)	2, 7, $\Box$R
10.	Fa	(k)	9, RUI a (a occurs in k at line 8)
	$\times$		

- So the CBF is valid also in S5**R**QT=.

- How to make it false?

- To make CBF false we need the following:

To make the **antecedent true**:

everything in all worlds is F.

To make the **consequent false:**

we need something in n that does not exist in another world, and if it does not exist there, statements about it are false.

So, an indexed formula F(a) (n) is **false** in n if:

1. **a** exists in n but **a** does not have the property F

OR

2. **a** does not exist in n.

- To be precise we need this:

1. Variable domains

2. A formula F(a) is necessary only if a exists in all worlds, and a is F in all worlds.

→ Whether Fa is false or gappy in a world is not so crucial, as long as **□Fa is false when a does not exist in at least one world**.

Rules for a system non validating CBF

(HRUI)	$(\forall\eta)\alpha$	$(\omega)\ \backslash\kappa$
	$\vdots$	
	β	(ω)

where β is the result of replacing every free occurrence of η in α with an individual constant, say κ, provided that, in this path of the tree, κ has not been used for the EI *of a formula in a world other than ω and either κ occurs in a formula in ω or, if not, κ is* NEW *to this path of the tree*

1.	$\sim(\Box(x)Fx \supset (x)\Box Fx)$	(n)	NTF
2.	$\Box(x)Fx$	(n)	1
3.	$\sim(x)\Box Fx$	(n)	1
4.	$(x)Fx$	(n)	2, $\Box$T
5.	$(\exists x)\sim\Box Fx$	(n)	3, QN
6.	$\sim\Box Fa$	(n)	5, EI a
7.	$\Diamond\sim Fa$	(n)	6, MN
8.	nAk		7, $\Diamond$R
9.	$\sim Fa$	(k)	7, $\Diamond$R
10.	$(x)Fx$	(k)	2, 8, $\Box$R
11.	Fb	(k)	10, HRUI b (b is new)
12.	Fa	(n)	4, HRUI a
	$\uparrow$		

- According to HRUI, it is not possible to use **a** for instantiation of the formula in k at line 10, because **a** was used for the EI of a formula in n at line 5.

- So from the open tree we can have a counterexample to CBF.

- Using HRUI (instead of RUI),

  we have that neither CBF nor BF are valid.

  → But this does not tell us which rule is correct:

  UI, RUI or HRUI?