FUNZIONI GENERATRICI di trasf. canoniche

Consideriamo la trasf.

$$\begin{cases} p = \sqrt{2\tilde{p}}\cos\tilde{q} \\ q = \sqrt{2\tilde{p}}\,\mathrm{sen}\,\tilde{q} \end{cases}$$

Vediamo che possiamo invertire la prima eq. per ricavare $\tilde{p}$ in funzione di p. Otteniamo

$$\tilde{p} = \frac{p^2}{2\cos^2\tilde{q}}$$

Se poi sostituiamo nella seconda eq. abbiamo

$$q = p\,\mathrm{tg}\,\tilde{q}$$

Quindi un modo equivalente di dare la trasf. è fornire le relazioni

$$\begin{cases} \tilde{p} = \dfrac{p^2}{2\cos^2\tilde{q}} \\ q = p\,\mathrm{tg}\,\tilde{q} \end{cases} \qquad \text{cioè} \qquad \begin{cases} \tilde{p} = \tilde{p}(p,\tilde{q}) \\ \tilde{q} = \tilde{q}(p,\tilde{q}) \end{cases}$$

In questo modo posso usare $(p,\tilde{q})$ come coord. indipendenti sullo spazio delle fasi (anche se non sono coord. canoniche, cioè le eq. del moto per i moti $(p(t),\tilde{q}(t))$ espressi in qte coord. non sono nella forma delle eq. di Hamilton).

Qta cosa la posso generalizzare a un caso generico e scegliendo come coord. indip. tra le diverse coppie

$(q,\tilde{q})$ $\quad(\tilde{p},q)$ $\quad(p,\tilde{q})$ $\quad(p,\tilde{p})$!

Facciamo la scelta $(\tilde{p}, q)$ come coordinate indipendenti e diamo la trasf. di coord. nella forma

$$\begin{cases} p_h = \varrho_h(\tilde{p}, q, t) \\ \tilde{q}_k = \tilde{\mu}_k(\tilde{p}, q, t) \end{cases} \quad \xrightarrow{} \quad p_h = u_h(\tilde{p}, \tilde{q}, t) \equiv \varrho_h(\tilde{p}, v(\tilde{p}, \tilde{q}, t), t)$$

$$\underset{\text{inversione}}{\rightsquigarrow} \quad q_k = v_k(\tilde{p}, \tilde{q}, t)$$

Possiamo pensare di definire le funzioni ϱ_h e $\tilde{\mu}_k$ a partire da una funzione $F_2(\tilde{p}, q, t)$ come

$$\varrho_h(\tilde{p}, q, t) \equiv \frac{\partial F_2}{\partial q_h}(\tilde{p}, q, t)$$

$$\mu_k(\tilde{p}, q, t) \equiv \frac{\partial F_2}{\partial \tilde{p}_k}(\tilde{p}, q, t)$$

Per invertire $\tilde{q} = \frac{\partial F_2}{\partial \tilde{p}}(\tilde{p}, q, t)$ nelle q, bisogna che $\det\left(\frac{\partial^2 F_2}{\partial \tilde{p}_h \partial q_k}\right) \neq 0$.

<u>Prop</u>. Quando una trasf. di coord. sullo sp. delle fasi è definita in maniera implicita da

$$(\circ) \quad \begin{cases} p_h = \dfrac{\partial F_2}{\partial q_h}(\tilde{p}, q, t) \\[2ex] \tilde{q}_k = \dfrac{\partial F_2}{\partial \tilde{p}_k}(\tilde{p}, q, t) \end{cases} \qquad \text{con } \det\left(\frac{\partial^2 F_2}{\partial \tilde{p} \partial q}\right) \neq 0$$

allora tale trasf. è CANONICA con $K = \tilde{H} + \frac{\partial F_2}{\partial t}$.

La funzione $F_2(\tilde{p}, q, t)$ è detta FUNZIONE GENERATRICE della trasf. canonica.

Dim. Riscriviamo (0) in termini delle funz. $u(\tilde{p},\tilde{q},t)$ e $v(\tilde{p},\tilde{q},t)$ che def. la trasf. canonica $p = u(\tilde{p},\tilde{q},t)$ e $q = v(\tilde{p},\tilde{q},t)$

$$(0)' \begin{cases} u_h(\tilde{p},\tilde{q},t) = \dfrac{\partial F_2}{\partial q_h}(\tilde{p}, v(\tilde{p},\tilde{q},t),t) \\ \tilde{q}_k = \dfrac{\partial F_2}{\partial \tilde{p}_k}(\tilde{p}, v(\tilde{p},\tilde{q},t),t) \end{cases}$$

Affinché sia canonica, dobbiamo dimostrare che le par. di Poisson fondam. sono preservate, cioè che

$$\{v_h, u_k\} = \delta_{hk} \qquad \{v_h, v_k\} = 0 \qquad \{u_h, u_k\} = 0$$

Per far ciò deriviamo prima le $(0)'$, ottenendo altre relazioni che le funz. u e v devono soddisfare:

$$(\mathrm{I}) \quad \frac{\partial u_h}{\partial \tilde{p}_\ell} = \frac{\partial^2 F_2}{\partial q_h \partial \tilde{p}_\ell} + \sum_m \frac{\partial^2 F_2}{\partial q_h \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_\ell}$$

$$(\mathrm{II}) \quad \frac{\partial u_h}{\partial \tilde{q}_\ell} = \sum_m \frac{\partial^2 F_2}{\partial q_h \partial q_m} \frac{\partial v_m}{\partial \tilde{q}_\ell}$$

$$\begin{cases} u_h(\tilde{p},\tilde{q},t) = \dfrac{\partial F_2}{\partial q_h}(\tilde{p}, v(\tilde{p},\tilde{q},t),t) \\ \tilde{q}_k = \dfrac{\partial F_2}{\partial \tilde{p}_k}(\tilde{p}, v(\tilde{p},\tilde{q},t),t) \end{cases}$$

$$(\mathrm{III}) \quad \delta_{k\ell} = \sum_m \frac{\partial F_2}{\partial \tilde{p}_k \partial q_m} \frac{\partial v_m}{\partial \tilde{q}_\ell}$$

$$(\mathrm{IV}) \quad 0 = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial \tilde{p}_\ell} + \sum_m \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_\ell}$$

Definiamo

$$A_{hm} \equiv \frac{\partial^2 F_2}{\partial q_h \partial q_m} \qquad B_{km} \equiv \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m} \qquad C_{k\ell} \equiv \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial \tilde{p}_\ell}$$

Allora:

$$(\mathrm{III}) \Rightarrow \quad \frac{\partial v_m}{\partial \tilde{q}_\ell} = (B^{-1})_{m\ell}$$

$$(\mathrm{IV}) \Rightarrow \quad \frac{\partial v_m}{\partial \tilde{p}_\ell} = -(B^{-1}C)_{m\ell}$$

$$(\mathrm{II}) \Rightarrow \quad \frac{\partial u_h}{\partial \tilde{q}_\ell} = (AB^{-1})_{h\ell}$$

$$(\mathrm{I}) \Rightarrow \quad \frac{\partial u_h}{\partial \tilde{p}_\ell} = (B^T)_{h\ell} - (AB^{-1}C)_{h\ell}$$

In particolare, da qste relaz possiamo ricavare lo Jacobiano della trasf. can. $J_{ij} = \frac{\partial w_i}{\partial \tilde{x}_j}$

$$J = \begin{pmatrix} B^T - AB^{-1}C & AB^{-1} \\ -B^{-1}C & B^{-1} \end{pmatrix}$$

Possiamo ora calcolare le par. di Poisson fondamentali:

$$\{v_m, u_h\} = \sum_\ell \left(\frac{\partial v_m}{\partial \tilde{q}_\ell} \frac{\partial u_h}{\partial \tilde{p}_\ell} - \frac{\partial v_m}{\partial \tilde{p}_\ell} \frac{\partial u_h}{\partial \tilde{q}_\ell} \right) =$$

$$= \Big[\underbrace{B^{-1}(B}_{=\mathbb{1}} - \cancel{\underset{=C}{C^T}} B^{-T} \underset{=\Lambda}{A^T}) + B^{-1}\cancel{C\,B^{-T}} A^T \Big]_{mh}$$

$$= \delta_{mh}$$ ← In particolare, la trasf. con. è UNIVALENTE

$$\{v_m, v_h\} = \left[-B^{-1}C^T B^{-T} + B^{-1} C B^{-T} \right]_{mh} = 0$$

$$\{u_m, u_h\} = \left[AB^{-1}(B - C^T B^{-T} A^T) - (B^T - AB^{-1}C) B^{-T} A^T \right] = 0$$

La nuova Hamiltoniana è $K(\tilde{p}, \tilde{q}, t) = \tilde{H}(\tilde{p}, \tilde{q}, t) + K_0(\tilde{p}, \tilde{q}, t)$.

Dove $K_0(\tilde{p}, \tilde{q}, t)$ è t.c. $\mathbb{E}\, \nabla_{\tilde{x}} K_0 = \frac{\partial \tilde{w}}{\partial t}(w(\tilde{x}, t), t)$ (*).

Dimostriamo che $K_0(\tilde{p}, \tilde{q}) = \frac{\partial F_2}{\partial t}(\tilde{p}, v(\tilde{p}, \tilde{q}, t))$ soddisfa (*)

- Calcoliamo il termine di destra in $(*)$:

$$w_i(\tilde{w}(x,t),t) = x_i \underset{\substack{\uparrow \\ \text{deriviamo} \\ \partial/\partial t}}{} \Rightarrow \sum_j \frac{\partial w_i}{\partial x_j} \frac{\partial \tilde{w}_j}{\partial t} + \frac{\partial w_i}{\partial t} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial w}{\partial t} = - J \frac{\partial \tilde{w}}{\partial t}$$

- Quindi dimostrare $(*)$ corrisponde a dimostrare che

$$\frac{\partial w}{\partial t} = - J E \nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} \qquad (*)'$$

- Deriviamo $(\circ)'$ per t e definiamo $a_k = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial t}$ $\quad b_h = \frac{\partial^2 F_2}{\partial q_h \partial t}$

$$\begin{cases} \frac{\partial u_h}{\partial t} = \frac{\partial^2 F_2}{\partial q_h \partial t} + \sum_m \frac{\partial^2 F_2}{\partial q_h \partial q_m} \frac{\partial v_m}{\partial t} \\ 0 = \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial t} + \sum_m \frac{\partial^2 F_2}{\partial \tilde{p}_k \partial q_m} \frac{\partial v_m}{\partial t} \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial t} = b - AB^{-1}a \\ \frac{\partial v}{\partial t} = -B^{-1}a \end{cases}$$

- $$\frac{\partial w}{\partial t} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} b - AB^{-1}a \\ -B^{-1}a \end{pmatrix}$$

- $$\frac{\partial}{\partial \tilde{p}_k}\left(\frac{\partial F_2}{\partial t}\right) = \frac{\partial^2 F_2}{\partial t \partial \tilde{p}_k} + \sum_m \frac{\partial^2 F_2}{\partial t \partial q_m} \frac{\partial v_m}{\partial \tilde{p}_k} = a_k - (B^{-1}C)^T b$$

$$\frac{\partial}{\partial \tilde{q}_h}\left(\frac{\partial F_2}{\partial t}\right) = \sum_m \frac{\partial^2 F_2}{\partial t \partial q_m} \frac{\partial v_m}{\partial \tilde{q}_h} = (B^{-T} b)_h$$

$$\Rightarrow \nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} = \begin{pmatrix} a - CB^{-T}b \\ B^{-T}b \end{pmatrix}$$

- $$-JE\nabla_{\tilde{x}} \frac{\partial F_2}{\partial t} = - \begin{pmatrix} B^T - AB^{-1}C & AB^{-1} \\ -B^{-1}C & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a - CB^{-T}b \\ B^{-T}b \end{pmatrix}$$

$$= \begin{pmatrix} -B^T + AB^{-1}C & -AB^{-1} \\ B^{-1}C & -B^{-1} \end{pmatrix} \begin{pmatrix} -B^{-T}b \\ a - CB^{-T}b \end{pmatrix}$$

$$= \begin{pmatrix} b - AB^{-1}CB^{-T}b - AB^{-1}a + AB^{-1}CB^{-T}b \\ -B^{-1}CB^{-T}b - B^{-1}a + B^{-1}CB^{-T}b \end{pmatrix} = \begin{pmatrix} b - AB^{-1}a \\ -B^{-1}a \end{pmatrix}$$

il che verifica $(*)'$.

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Funzioni tipo F_2 generano la maggior parte delle trasf. interessanti:

- Identità: $F_2(\tilde{p}, q, t) = \sum_k \tilde{p}_k q_k$: $\quad p_h = \frac{\partial F_2}{\partial q_h} = \tilde{p}_h \qquad \tilde{q}_h = \frac{\partial F_2}{\partial \tilde{p}_h} = q_h$

- Trasf. puntuali estese : $F_2 = \sum_k \tilde{p}_k \hat{v}_k(q,t)$:

$$p_h = \frac{\partial F_2}{\partial q_h} = \sum_k \tilde{p}_k \frac{\partial \hat{v}_k(q,t)}{\partial q_h} \qquad \tilde{q}_h = \frac{\partial F_2}{\partial \tilde{p}_h} = \hat{v}_h(q,t)$$

Con dim. analoga a F_2, si può dim. che trasf. canoniche in forma implicita si possono ottenere anche con funz. di altro tipo:

$$F_1(q, \tilde{q}, t) \to \begin{cases} p_h = \dfrac{\partial F_1}{\partial q_h} \\ \tilde{p}_k = -\dfrac{\partial F_1}{\partial \tilde{q}_k} \end{cases} \qquad K = \tilde{H} + \frac{\partial F_1}{\partial t}$$

$$F_3(p, \tilde{q}, t) \to \begin{cases} q_h = -\dfrac{\partial F_3}{\partial p_h} \\ \tilde{p}_k = -\dfrac{\partial F_3}{\partial \tilde{q}_k} \end{cases} \qquad F_4(p, \tilde{p}, t) \to \begin{cases} q_h = -\dfrac{\partial F_4}{\partial p_h} \\ \tilde{q}_k = \dfrac{\partial F_4}{\partial \tilde{p}_k} \end{cases}$$

$$K = \tilde{H} + \frac{\partial F_3}{\partial t} \qquad K = \tilde{H} + \frac{\partial F_4}{\partial t}$$

APPENDICE: Funzione $F_1(q,\tilde{q},t)$

Consideriamo una funzione $F_1(q,\tilde{q},t)$; essa definisce una trasf. di coord. in maniera implicita con

$$\begin{cases} p_h(q,\tilde{q},t) = \dfrac{\partial F_1}{\partial q_h}(q,\tilde{q},t) \\ \tilde{p}_k(q,\tilde{q},t) = -\dfrac{\partial F_1}{\partial \tilde{q}_k}(q,\tilde{q},t) \end{cases} \quad (\star) \quad \text{con} \quad \det\left(\frac{\partial^2 F_1}{\partial q_h \partial \tilde{q}_k}\right) \neq 0$$

Verifichiamo che una trasf. così definita è una trasf. CANONICA.

Riscriviamo $(\star)$ in termini delle coord. $\tilde{p},\tilde{q}$ e delle funzioni $p(\tilde{p},\tilde{q},t)$, $q(\tilde{p},\tilde{q},t)$:

$$\begin{cases} p_h(\tilde{p},\tilde{q},t) = \dfrac{\partial F_1}{\partial q_h}(q(\tilde{p},\tilde{q},t),\tilde{q},t) \\ \tilde{p}_k = -\dfrac{\partial F_1}{\partial \tilde{q}_k}(q(\tilde{p},\tilde{q},t),\tilde{q},t) \end{cases}$$

Derivando in $\tilde{p}$ e $\tilde{q}$, si ottengono le seguenti relaz. che le $p(\tilde{p},\tilde{q},t)$ e $q(\tilde{p},\tilde{p},t)$ soddisfano

$$\frac{\partial p_h}{\partial \tilde{p}_\ell} = \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m}\frac{\partial q_m}{\partial \tilde{p}_\ell} \qquad \delta_{k\ell} = -\sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m}\frac{\partial q_m}{\partial \tilde{p}_\ell}$$

$$\frac{\partial p_h}{\partial \tilde{q}_\ell} = \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m}\frac{\partial q_m}{\partial \tilde{q}_\ell} + \frac{\partial^2 F_1}{\partial q_h \partial \tilde{q}_\ell} \qquad 0 = \sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m}\frac{\partial q_m}{\partial \tilde{q}_\ell} + \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial \tilde{q}_\ell}$$

Definiamo

$$A_{hm} \equiv \frac{\partial^2 F_1}{\partial q_h \partial q_m} \qquad B_{km} \equiv \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m} \qquad C_{ke} \equiv \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial \tilde{q}_e}$$

Allora:

$$\frac{\partial q_m}{\partial \tilde{p}_e} = -(B^{-1})_{me} \qquad \frac{\partial q_m}{\partial \tilde{q}_e} = -(B^{-1}C)_{me}$$

$$\frac{\partial p_h}{\partial \tilde{p}_e} = -(AB^{-1})_{he} \qquad \frac{\partial p_h}{\partial \tilde{q}_e} = -(AB^{-1}C)_{he} + B^T_{he}$$

$$\rightarrow J = \begin{pmatrix} -AB^{-1} & -AB^{-1}C + B^T \\ -B^{-1} & -B^{-1}C \end{pmatrix}$$

Ora siamo pronti a dimostrare che le Parentesi di Poisson fondamentali sono PRESERVATE:

$$\{q_m, p_h\} = \sum_e \left(\frac{\partial q_m}{\partial \tilde{q}_e}\frac{\partial p_h}{\partial \tilde{p}_e} - \frac{\partial q_m}{\partial \tilde{p}_e}\frac{\partial p_h}{\partial \tilde{q}_e} \right) =$$

$$= \sum_e (B^{-1}C)_{me}(AB^{-1})_{he} - \sum_e (B^{-1})_{me}\left((AB^{-1}C)_{he} - B^T_{he}\right)$$

$$= \left(AB^{-1}\underbrace{C^T}_{=C}B^{-T} - AB^{-1}CB^{-T} + B^{-1}B\right)_{mh} = \delta_{mh}$$

$$\{q_m, q_h\} = \sum_e (B^{-1}C)_{me} B^{-1}_{he} - \sum_e (B^{-1}C)_{he} B^{-1}_{me} =$$

$$= (B^{-1}CB^{-T})_{mh} - (B^{-1}C^TB^{-T})_{mh} = 0$$

$$\{p_m, p_h\} = \sum_e \left[(AB^{-1}C)_{me} - B^T_{me}\right](AB^{-1})_{he} - \sum_e \left[(AB^{-1}C)_{he} - B^T_{he}\right](AB^{-1})_{me} =$$

$$= (AB^{-1}CB^{-T}A^T)_{mh} - (B^TB^{-T}A)_{mh} - (AB^{-1}C^TB^{-T}A^T)_{mh} + (AB^{-1}B)_{mh} = 0$$

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Deriviamo ora rispetto al tempo e definiamo $a_h = \frac{\partial^2 F_1}{\partial q_h \partial t}$

$b_k = \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial t}$

$$\begin{cases} \frac{\partial p_h}{\partial t} = \frac{\partial^2 F_1}{\partial q_h \partial t} + \sum_m \frac{\partial^2 F_1}{\partial q_h \partial q_m} \frac{\partial q_m}{\partial t} \\ 0 = \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial t} + \sum_m \frac{\partial^2 F_1}{\partial \tilde{q}_k \partial q_m} \frac{\partial q_m}{\partial t} \end{cases} \Longrightarrow \frac{\partial q_m}{\partial t} = -(B^{-1} b)_m$$

$$\Downarrow$$

$$\frac{\partial p_h}{\partial t} = a_h - (A B^{-1} b)_h$$

$$\frac{\partial w}{\partial t} = \begin{pmatrix} a - AB^{-1} b \\ -B^{-1} b \end{pmatrix} \qquad J = \begin{pmatrix} -AB^{-1} & -AB^{-1}C + B^T \\ -B^{-1} & -B^{-1}C \end{pmatrix} \leftarrow \frac{\partial w}{\partial \tilde{x}}$$

$$x = w(\tilde{w}(x,t),t) \rightarrow 0 = J \frac{\partial \tilde{w}}{\partial t} + \frac{\partial w}{\partial t} \Rightarrow \frac{\partial \tilde{w}}{\partial t} = -J^{-1} \frac{\partial w}{\partial t}$$

$$\frac{\partial F_1}{\partial t} = K_0 \rightsquigarrow E \nabla_{\tilde{x}} K_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 F_1}{\partial t \partial q} \frac{\partial q}{\partial \tilde{p}} \\ \frac{\partial^2 F_1}{\partial t \partial q} \frac{\partial q}{\partial \tilde{q}} + \frac{\partial^2 F_1}{\partial t \partial \tilde{q}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -B^{-T} a \\ -(C B^{-T}) a + b \end{pmatrix} = \begin{pmatrix} (C B^{-T}) a - b \\ -B^{-T} a \end{pmatrix} = \frac{\partial \tilde{w}}{\partial t}$$

$$\frac{\partial w}{\partial t} = -J \frac{\partial \tilde{w}}{\partial t} = \begin{pmatrix} AB^{-1} & AB^{-1}C - B^T \\ B^{-1} & B^{-1}C \end{pmatrix} \begin{pmatrix} (CB^{-T}) a - b \\ -B^{-T} a \end{pmatrix} =$$

$$= \begin{pmatrix} AB^{-1}CB^{-T} a - AB^{-1} b - AB^{-1}CB^{-T} a + a \\ B^{-1}CB^{-T} a - B^{-1} b - B^{-1}CB^{-T} a \end{pmatrix} =$$

$$= \begin{pmatrix} a - AB^{-1} b \\ -B^{-1} b \end{pmatrix} \quad //$$