Esempio

Consideriamo la seguente trasf. di coord. nello sp. delle fasi $(n=1)$

$$\begin{cases} p = \sqrt{2\tilde{p}}\cos\tilde{q} \\ q = \sqrt{2\tilde{p}}\,\mathrm{sen}\,\tilde{q} \end{cases}$$

Verifichiamo la CANONICITÀ.

1) PARENTESI DI POISSON (fondamentali)

$$\{p,q\} = \{\sqrt{2\tilde{p}}\,\mathrm{sen}\tilde{q},\ \sqrt{2\tilde{p}}\cos\tilde{q}\} =$$

$$= 2\left[\frac{\partial}{\partial\tilde{q}}(\sqrt{\tilde{p}}\,\mathrm{sen}\tilde{q})\frac{\partial}{\partial\tilde{p}}(\sqrt{\tilde{p}}\cos\tilde{q}) - \frac{\partial}{\partial\tilde{p}}(\sqrt{\tilde{p}}\,\mathrm{sen}\tilde{q})\frac{\partial}{\partial\tilde{q}}(\sqrt{\tilde{p}}\cos\tilde{q})\right]$$

$$= \cancel{2}\left[\cancel{\sqrt{\tilde{p}}}\cos\tilde{q}\,\frac{1}{\cancel{2}\cancel{\sqrt{\tilde{p}}}}\cos\tilde{q} - \cancel{\sqrt{\tilde{p}}}(-\mathrm{sen}\tilde{q})\frac{1}{\cancel{2}\cancel{\sqrt{\tilde{p}}}}\mathrm{sen}\tilde{q}\right] =$$

$$= \cos^2\tilde{q} + \mathrm{sen}^2\tilde{q} = 1$$

$\{q,q\} = 0 \qquad \{p,p\} = 0$ //

2) CONDIZIONE DI LIE

$$p\,dq = \sqrt{2\tilde{p}}\cos\tilde{q}\; d(\sqrt{2\tilde{p}}\,\mathrm{sen}\tilde{q}) = 2\sqrt{\tilde{p}}\cos\tilde{q}\left(\frac{1}{2\sqrt{\tilde{p}}}\mathrm{sen}\tilde{q}\,d\tilde{p} + \sqrt{\tilde{p}}\cos\tilde{q}\,d\tilde{q}\right) =$$

$$= \cos\tilde{q}\,\mathrm{sen}\tilde{q}\,d\tilde{p} + 2\tilde{p}\cos^2\tilde{q}\,d\tilde{q} \qquad (\circ)$$

→ trasf. soddisfa cond. di Lie se $\exists F$ t.c.

$$\underbrace{p\,dq}_{\text{dato da } (\circ)} = dF + \tilde{p}\,d\tilde{q}$$

⇝ F dev'essere tale che

$$\frac{\partial F}{\partial\tilde{p}}d\tilde{p} + \frac{\partial F}{\partial\tilde{q}}d\tilde{q} = \cos\tilde{q}\,\mathrm{sen}\tilde{q}\,d\tilde{p} + \tilde{p}(2\cos^2\tilde{q}-1)\,d\tilde{q}$$

cioè:

$$\frac{\partial F}{\partial \tilde{p}} = \cos\tilde{q}\,\operatorname{sen}\tilde{q} \longrightarrow F = \tilde{p}\cos\tilde{q}\,\operatorname{sen}\tilde{q} + f(\tilde{q})$$

inseriamo qui sotto:

$$\frac{\partial F}{\partial \tilde{q}} = 2\tilde{p}\cos^2\tilde{q} - \tilde{p} \rightleftharpoons \tilde{p}(2\cos^2\tilde{q}-1) + f'(\tilde{q})$$

$$f'(\tilde{q}) = 0 \Rightarrow f(\tilde{q}) = \text{cost.}$$

$\Rightarrow \exists\, F = \tilde{p}\cos\tilde{q}\,\operatorname{sen}\tilde{q} + \text{cost.}$ t.c. cond. di Lie è soddisfatta. //

3) JACOBIANO SIMPLETTICO

$$J_{ij} = \frac{\partial w_i}{\partial \tilde{x}_j} = \begin{pmatrix} \frac{\partial p}{\partial \tilde{p}} & \frac{\partial p}{\partial \tilde{q}} \\ \frac{\partial q}{\partial \tilde{p}} & \frac{\partial q}{\partial \tilde{q}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\tilde{p}}}\cos\tilde{q} & -\sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} \\ \frac{1}{\sqrt{2\tilde{p}}}\operatorname{sen}\tilde{q} & \sqrt{2\tilde{p}}\cos\tilde{q} \end{pmatrix}$$

$$J^T E J = \begin{pmatrix} \frac{1}{\sqrt{2\tilde{p}}}\cos\tilde{q} & \frac{1}{\sqrt{2\tilde{p}}}\operatorname{sen}\tilde{q} \\ -\sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} & \sqrt{2\tilde{p}}\cos\tilde{q} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2\tilde{p}}}\cos\tilde{q} & -\sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} \\ \frac{1}{\sqrt{2\tilde{p}}}\operatorname{sen}\tilde{q} & \sqrt{2\tilde{p}}\cos\tilde{q} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2\tilde{p}}}\operatorname{sen}\tilde{q} & -\frac{1}{\sqrt{2\tilde{p}}}\cos\tilde{q} \\ \sqrt{2\tilde{p}}\cos\tilde{q} & \sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2\tilde{p}}}\cos\tilde{q} & -\sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} \\ \frac{1}{\sqrt{2\tilde{p}}}\operatorname{sen}\tilde{q} & \sqrt{2\tilde{p}}\cos\tilde{q} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = E \quad //$$

4) ESISTENZA DI UNA FUNZIONE GENERATRICE, per es. di tipo $F_3(p,\tilde{q})$

$$\begin{cases} p = \sqrt{2\tilde{p}}\cos\tilde{q} \\ q = \sqrt{2\tilde{p}}\,\operatorname{sen}\tilde{q} \end{cases} \longrightarrow \begin{cases} \tilde{p} = \dfrac{p^2}{2\cos^2\tilde{q}} & \longleftarrow \; = -\dfrac{\partial F_3}{\partial \tilde{q}} \\ q = p\,\operatorname{tg}\tilde{q} & \longleftarrow \; = -\dfrac{\partial F_3}{\partial p} \end{cases}$$

Quindi F_3 deve essere t.c.

$$p \, tg \tilde{q} = -\frac{\partial F_3}{\partial p} \rightarrow F_3 = -\frac{p^2}{2} tg \tilde{q} + f(\tilde{q})$$

Con tale F_3, otteniamo

$$-\frac{\partial F_3}{\partial \tilde{q}} = \frac{p^2}{2} \frac{1}{\cos^2 \tilde{q}} - f'(\tilde{q}) \xrightarrow[\text{prima eq.}]{\text{per la}} f'(\tilde{q}) = 0 .$$

//

Consideriamo l'oscillatore armonico con $m = 1/\omega$

$$H(p,q) = \frac{\omega}{2}(p^2 + q^2) \rightarrow \text{eq. di Ham.} \begin{cases} \dot{p} = -\omega q \\ \dot{q} = \omega p \end{cases} \qquad (\sim)$$

Verifichiamo che per questa H, $\exists K(\tilde{p}, \tilde{q})$ t.c. le eq. del moto si possono ancora scrivere in forma di eq. di Hamilton.

Data $p = p(\tilde{p}(t), \tilde{q}(t))$ e $q = q(\tilde{p}(t), \tilde{q}(t))$

$$\begin{cases} \dot{p} = \frac{d}{dt}\left(\sqrt{2\tilde{p}(t)} \cos \tilde{q}(t)\right) = \frac{1}{\sqrt{2\tilde{p}}} \cos\tilde{q} \, \dot{\tilde{p}} - \sqrt{2\tilde{p}} \, \text{sen}\tilde{q} \, \dot{\tilde{q}} \\ \dot{q} = \frac{d}{dt}\left(\sqrt{2\tilde{p}(t)} \, \text{sen} \, \tilde{q}(t)\right) = \frac{1}{\sqrt{2\tilde{p}}} \text{sen}\tilde{q} \, \dot{\tilde{p}} + \sqrt{2\tilde{p}} \cos\tilde{q} \, \dot{\tilde{q}} \end{cases}$$

Scriviamo le $\dot{\tilde{p}}$ e $\dot{\tilde{q}}$ in funzione di $\dot{p}$ e $\dot{q}$ (sist. lineare):

$$\dot{\tilde{p}} = \sqrt{2\tilde{p}} \left(\cos\tilde{q} \, \dot{p} + \text{sen}\tilde{q} \, \dot{q}\right) = q\dot{p} + p\dot{q}$$

$$\dot{\tilde{q}} = \frac{1}{\sqrt{2\tilde{p}}} \left(-\text{sen}\tilde{q} \, \dot{p} + \cos\tilde{q} \, \dot{q}\right) = \frac{1}{2\tilde{p}} (p\dot{q} - q\dot{p})$$

Usando le eq. di Ham. $(\sim)$

$$\begin{cases} \dot{\tilde{p}} = 0 \\ \dot{\tilde{q}} = \omega \end{cases} \rightarrow \exists K = \omega \tilde{p} \text{ che genera qste eq. di Hamilton}$$

La nuova $K(\tilde{p},\tilde{q})$ è proprio $= H\left(p(\tilde{p},\tilde{q}), q(\tilde{p},\tilde{q}) \right)$.

Le nuove eq. del moto sono semplicissime, con soluz.:

$$\tilde{p}(t) = \tilde{p}^{\circ}$$
$$\tilde{q}(t) = \omega t + \tilde{q}^{\circ}$$

Le soluz. nelle vecchie coord. sono allora

$$p(t) = p(\tilde{p}(t), \tilde{q}(t)) = \sqrt{2\tilde{p}^{\circ}}\ \mathrm{sen}(\omega t + \tilde{q}^{\circ})$$

$$q(t) = q(\tilde{p}(t), \tilde{q}(t)) = \sqrt{2\tilde{p}^{\circ}} \cos(\omega t + \tilde{q}^{\circ})$$

che sono le note soluzioni dell'oscillatore armonico.