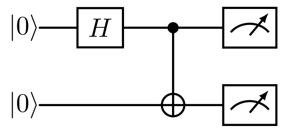
7.4 Fault-tolerant computation

Let us suppose we want to implement the following logical circuit



(7.135)

so that if an error occurs it can be successfully corrected via a QEC code. The error can essentially occur in any of the components of the circuit. Namely,

- in the state preparation,
- in the logic quantum gates,
- in the measurement,
- in the simple transition of the quantum information along the quantum wires.

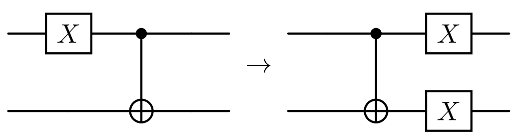
To combat the effect of the noise, one encodes the logical qubits in blocks of physical ones, using QEC codes. However, one needs also to replace the logical operations with encoded gates. Performing QEC periodically on the encoded states prevents the accumulation of errors in the state. However, it is not sufficient to prevent the build-up of errors, even if QEC is applied after each encoded gate.

There are two main reasons:

1) The encoded gates can cause the propagation of errors.
   Let us consider a specific example, where two qubits are connected by a CNOT gate and a X error occurs on the control qubit before the CNOT. Then, the error propagates also to the target qubit. This can be easily computed by considering that the CNOT is implemented by the unitary operator $\hat{U}_{\text{CNOT}}$. Then, one has

$$\hat{U}_{\text{CNOT}}\hat{\sigma}_x^{(1)} = \hat{U}_{\text{CNOT}}\hat{\sigma}_x^{(1)}\hat{U}_{\text{CNOT}}^{\dagger}\hat{U}_{\text{CNOT}} = \hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}\hat{U}_{\text{CNOT}}, \tag{7.136}$$

   which can be graphically represented with



(7.137)

   Then, one needs to design the encoded gates carefully, so that errors do not propagate on the entire block, but are limited to some physical qubits. In such a case the QEC code can remove these errors.
   Performing encoded gates in such a way is a fault-tollerant (FT) procedure.
2) Also QEC can introduce errors.
   An example is that graphically represented in Fig. 7.15, where an error is not correctly recovered.

To showcase the FT procedure, we introduce the Steane code.

7.4.1 Stean code or 7-qubit code

The Steane code is a stabiliser code employing 7 data qubit and 6 syndrome qubits for each logical qubit. The total Hilbert space $\mathbb{H}'$ has 128 dimensions that are divided in 64 subspaces of two dimensions. The graphical representation is given in Fig. 7.17. The generators of the stabilisers are

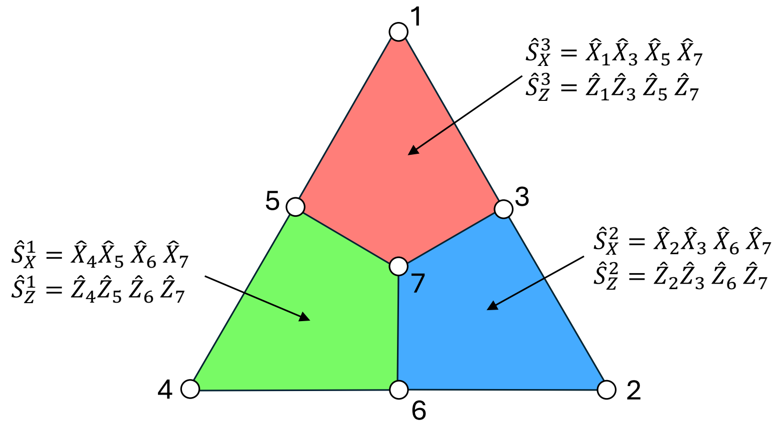


Fig. 7.17: Graphical representation of the Steane code with the data qubit represented as open circles and the corresponding stabilisers.

$$\{\,\hat{S}_k\,\}_{k=1}^{6} = \begin{Bmatrix} \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{X} & \hat{X} & \hat{X} & \hat{X}, \\ \hat{\mathbb{1}} & \hat{X} & \hat{X} & \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{X} & \hat{X}, \\ \hat{X} & \hat{\mathbb{1}} & \hat{X} & \hat{\mathbb{1}} & \hat{X} & \hat{\mathbb{1}} & \hat{X}, \\ \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{Z} & \hat{Z} & \hat{Z} & \hat{Z}, \\ \hat{\mathbb{1}} & \hat{Z} & \hat{Z} & \hat{\mathbb{1}} & \hat{\mathbb{1}} & \hat{Z} & \hat{Z}, \\ \hat{Z} & \hat{\mathbb{1}} & \hat{Z} & \hat{\mathbb{1}} & \hat{Z} & \hat{\mathbb{1}} & \hat{Z} \end{Bmatrix} = \begin{Bmatrix} \hat{X}_4\hat{X}_5\hat{X}_6\hat{X}_7, \\ \hat{X}_2\hat{X}_3\hat{X}_6\hat{X}_7, \\ \hat{X}_1\hat{X}_3\hat{X}_5\hat{X}_7, \\ \hat{Z}_4\hat{Z}_5\hat{Z}_6\hat{Z}_7, \\ \hat{Z}_2\hat{Z}_3\hat{Z}_6\hat{Z}_7, \\ \hat{Z}_1\hat{Z}_3\hat{Z}_5\hat{Z}_7, \end{Bmatrix}. \tag{7.138}$$

With the Steane code, any single-qubit error can be correctly recovered as it send the code space $\mathbb{H}_{\mathrm{C}}$ in one of the other subspaces $\mathbb{H}_i$. Specifically, the code space $\mathbb{H}_{\mathrm{C}}$ is given by the span of two logical states $\{\,|\,0_{\mathrm{L}}\rangle,\ |\,1_{\mathrm{L}}\rangle\,\}$. These are encoded as

$$\begin{aligned} |0_{\mathrm{L}}\rangle =& \frac{1}{\sqrt{8}}\left[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\right], \\ |1_{\mathrm{L}}\rangle =& \frac{1}{\sqrt{8}}\left[|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\right]. \end{aligned} \tag{7.139}$$

The logical operations are given by the normalisers. In the Pauli group, we have the logical X-gate and the logical Z-gate. These are constructed by applying the corresponding single qubit operators to each physical qubit. Namely,

$$\begin{aligned} \hat{X}_{\mathrm{L}} =& \hat{X}_1\hat{X}_2\hat{X}_3\hat{X}_4\hat{X}_5\hat{X}_6\hat{X}_7, \\ \hat{Z}_{\mathrm{L}} =& \hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\hat{Z}_5\hat{Z}_6\hat{Z}_7. \end{aligned} \tag{7.140}$$

These operations are allowed, as there are still two free degrees of freedom. There are also other logical operations that one can construct. These are normalisers that do not appartain to the Pauli group. An example is the Hadamard gate, which can be also implemented as the application of single qubit Hadamards:

$$\hat{H}_{\mathrm{L}} = \hat{H}_1\hat{H}_2\hat{H}_3\hat{H}_4\hat{H}_5\hat{H}_6\hat{H}_7. \tag{7.141}$$

At this point, one would naively extrapolate that any logical operation can be constructed via the application of the corresponding gate on the single data qubits where the logical state is encoded. However, this is not the case. An example is the phase gate $\hat{S}$, whose representation in the computation basis reads

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{7.142}$$

The corresponding logical gate is constructed as

$$\hat{S}_{\mathrm{L}} = \hat{S}_1^\dagger \hat{S}_2^\dagger \hat{S}_3^\dagger \hat{S}_4^\dagger \hat{S}_5^\dagger \hat{S}_6^\dagger \hat{S}_7^\dagger. \tag{7.143}$$

Indeed, when applied to the logical computational basis we find

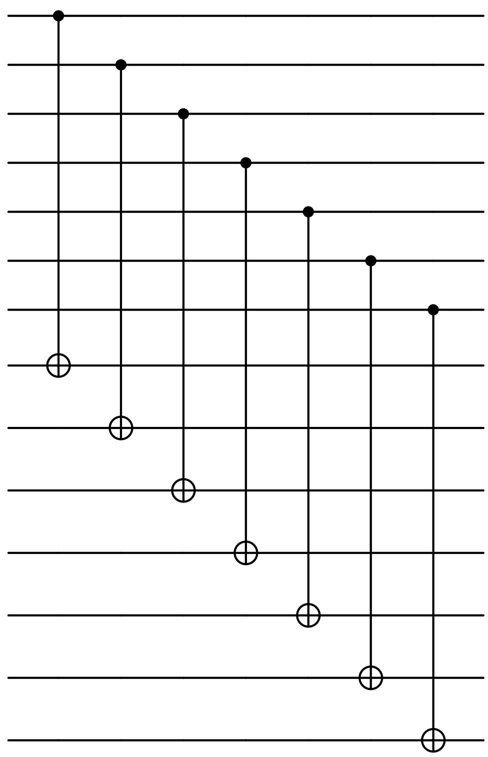
$$\begin{aligned} \hat{S}_{\mathrm{L}} \left|0_{\mathrm{L}}\right\rangle &= \left|0_{\mathrm{L}}\right\rangle, \\ \hat{S}_{\mathrm{L}} \left|1_{\mathrm{L}}\right\rangle &= i \left|1_{\mathrm{L}}\right\rangle. \end{aligned} \tag{7.144}$$

Conversely, if one would have defined a $\hat{S}'_{\mathrm{L}}$ as just simply applying the S-gate to each physical qubit, one would have obtained

$$\begin{aligned} \hat{S}'_{\mathrm{L}} \left|0_{\mathrm{L}}\right\rangle &= \left|0_{\mathrm{L}}\right\rangle, \\ \hat{S}'_{\mathrm{L}} \left|1_{\mathrm{L}}\right\rangle &= -i \left|1_{\mathrm{L}}\right\rangle. \end{aligned} \tag{7.145}$$

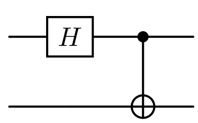
Thus, the construction of logical gates needs to be done with care and it depends on the QEC code that one is employing.

Let us consider the case of a CNOT gate. Here, we have two logical qubits, each corresponding to 7 data and 6 syndrome qubits. One can see that the logical CNOT gate can be implemented pairwise as described by the following ciurcuit



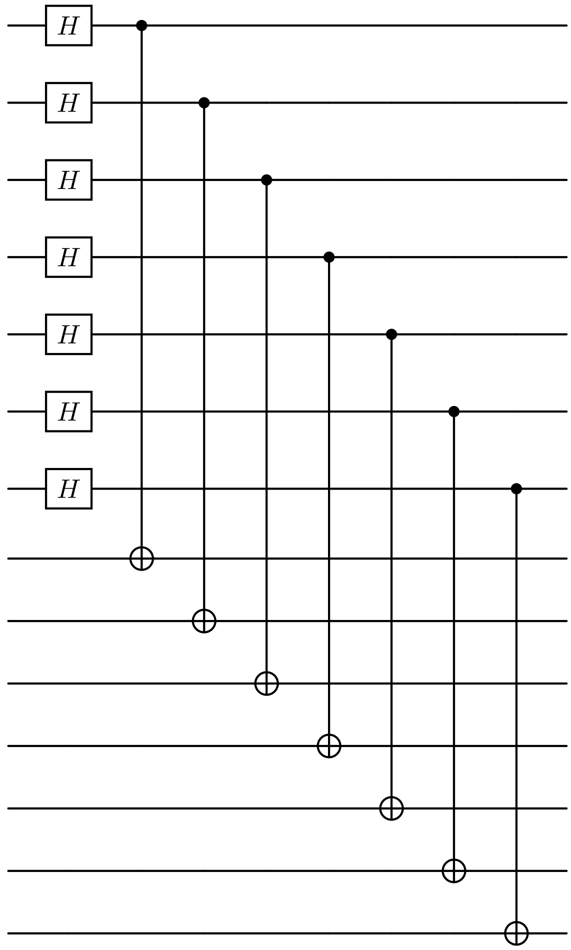
(7.146)

It follows that one can implement the logical circuit



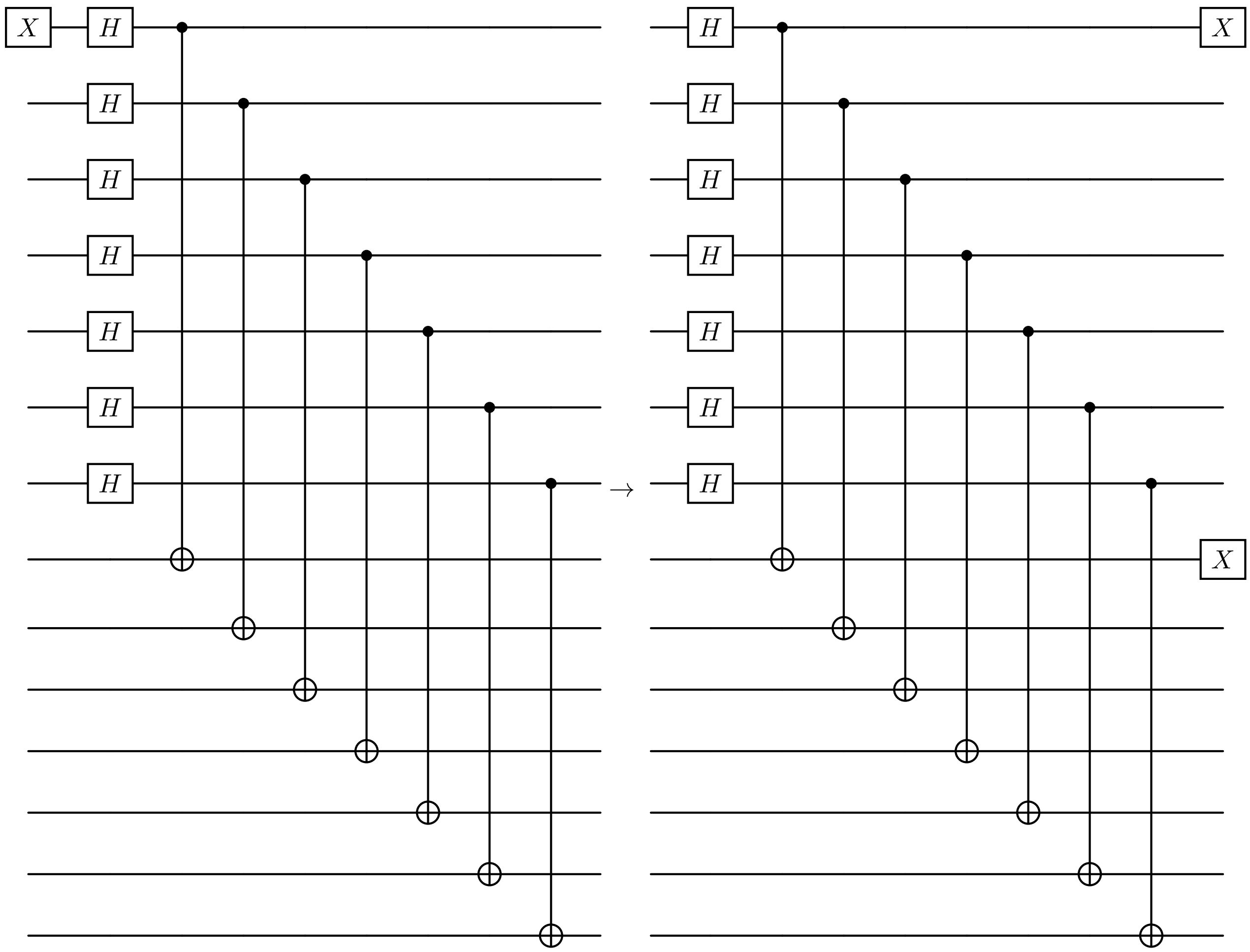
(7.147)

as



(7.148)

This way of implementing logical gates is called transversal construction. This is a really easy and straightforward construction, but most importantly allows to confine errors, as they are unable to propagate. Let us take as an example the circuit in Eq. (7.149) and suppose we have initially a X error on the first qubit of the first block. In such a case, the error propagates accordingly to



(7.149)

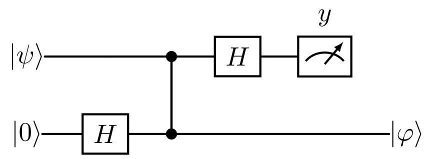
Thus, the error has been copied to the first qubit of the second block, but all the other qubits are not affected. The Steane code can correct for such an error. This holds true for any transverse construction.

Till now, we have covered the case for the normalisers $\{\hat{X}_{\mathrm{L}}, \hat{Z}_{\mathrm{L}}, \hat{H}_{\mathrm{L}}, \hat{S}_{\mathrm{L}}, \mathrm{CNOT}_{\mathrm{L}}\}$, which is the so-called Clifford group. Notably, the Gottesman-Knill theorem indicates that operations performed using only elements of this group can be simulated classically. Thus, one cannot have a real quantum advantage over a classical computer. Moreover, the Clifford group is not universal, meaning that the composition of elements of this group is not sufficient to implement any arbitrary gate. This is essentially the argument of the Solovay-Kitaev theorem. One needs to extend the Clifford group with the addition of at least an extra gate not appartaining to the group. This can be the T-gate or the Toffoli gate.

Unfortunately, one does not know how to implement through unitary operations the T-gate in a transversal way using the Steane code. This is a code-related problem. One could consider a different QEC code and implement the T-gate, but they will have problems with another gate, e.g. the Hadarmard gate. The Eastin-Knill theorem indicates that it is not possible to construct a fault-tolerant universal set with unitary operations.

Notably, the latter theorem applies only to unitary operations. One can still have the entire universal set by substituting the problematic gate with an effective implementation. Below, we focus on the derivation of an effective T-gate for the Steane code.

Consider the following circuit



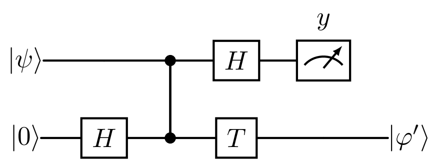
(7.150)

where one obtains that the state $|\varphi\rangle$ is given by

$$|\varphi\rangle = \hat{X}^y \hat{H} |\psi\rangle , \tag{7.151}$$

with $y = 0, 1$. Notably, the three operations performed here, the two Hadamards and the control-Z gates, are all fault-tolerant.

Suppose we slightly modify the circuit by adding a T-gate on the second qubit as follows



(7.152)

where the representation of the T-gate on the computational basis reads

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix} \tag{7.153}$$

and the final state of the second qubit now is

$$|\varphi'\rangle = \hat{T}\hat{X}^y \hat{H} |\psi\rangle . \tag{7.154}$$

Since, a part from a total phase, the T-gate can be written as

$$\hat{T} = e^{i\frac{\pi}{8}\hat{Z}}, \tag{7.155}$$

one obtains

$$|\varphi'\rangle = \hat{X}^y \hat{T}^{(1-2y)} \hat{H} |\psi\rangle . \tag{7.156}$$

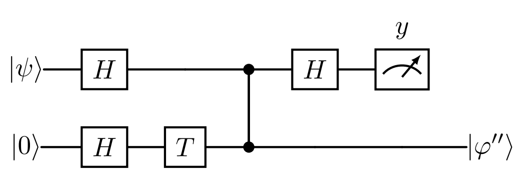
But, $\hat{T}^{(1-2y)}$ can be expressed as

$$\hat{T}^{(1-2y)} = \hat{T}e^{-i\frac{\pi y}{4}\hat{Z}} = e^{-i\frac{\pi y}{4}} \hat{S}^y \hat{T}, \tag{7.157}$$

where we can safely omit the phase in the last expression. Thus, we get

$$|\varphi'\rangle = \hat{X}^y \hat{S}^y \hat{T}\hat{H} |\psi\rangle = (\hat{X}\hat{S})^y \hat{T}\hat{H} |\psi\rangle . \tag{7.158}$$

Here, the last equality holds since the value of y can be only 0 or 1. Moreover, we also find that the T-gate commutes with the control-Z, since it commutes with $\hat{Z}$. Thus, by adding an extra Hadamard gate on the first qubit, we obtain

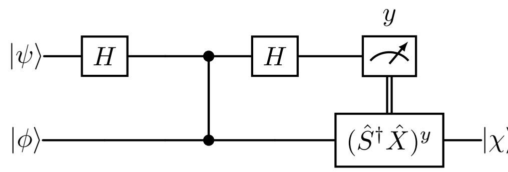


(7.159)

where

$$|\varphi''\rangle = (\hat{X}\hat{S})^y \hat{T} |\psi\rangle . \tag{7.160}$$

Specifically, the latter implies the following one



(7.161)

where

$$\begin{aligned} |\chi\rangle &= \hat{T} |\psi\rangle , \\ |\phi\rangle &= \hat{T}\hat{H} |0\rangle . \end{aligned} \tag{7.162}$$

This means that, if one is able to prepare the second qubit in the state $|\phi\rangle$, then they also can apply the T-gate to an arbitrary state $|\psi\rangle$ that was initially embedded in the first qubit. This is an effective application of the T-gate. Clearly, it requires the preparation of $|\phi\rangle$. Specifically, this is given by

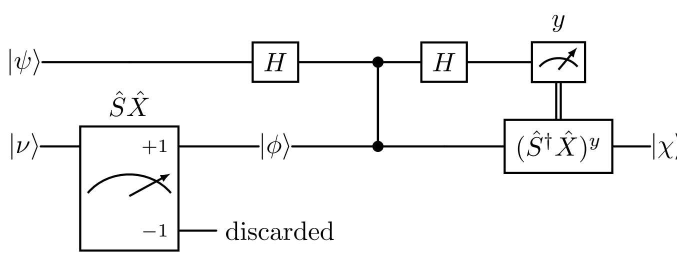
$$|\phi\rangle = \hat{T}\hat{H} |0\rangle = e^{-i\frac{\pi}{8}\hat{Z}} |+\rangle . \tag{7.163}$$

Now, the state $|+\rangle$ is eigeinstate of $\hat{X}$ with eigeinvalue $(+1)$, and in the same way $|\phi\rangle$ is eigeinstate of the operator

$$e^{-i\frac{\pi}{8}\hat{Z}} \hat{X} e^{i\frac{\pi}{8}\hat{Z}} = e^{-i\frac{\pi}{4}\hat{Z}} \hat{X} = \hat{S}\hat{X}, \tag{7.164}$$

with the eigeinvalue $(+1)$. Here, the last two equalities follow from the commutaion of $\hat{T}$ and $\hat{X}$ and the definition of the S-gate. Thus, $|\phi\rangle$ is eigeinstate of $\hat{S}\hat{X}$ with eigeinvalue $(+1)$. Similarly, the ortogonal state $|\phi_\perp\rangle$ is associated to the eigenvalue (-1). Now, this implies that given an arbitrary state $|\nu\rangle$ by measuring the operator $\hat{S}\hat{X}$, the state will collapse in $|\phi\rangle$ or $|\phi_\perp\rangle$ with corresponding outcomes respectively given by $(+1)$ and (-1).

The complete circuit thus becomes



(7.165)

and effectively applies the T-gate to an arbitrary state $|\psi\rangle$. If the measurement is also FT (this can be proven, but it will not be tackled here), then one has that the entire circuit is FT.