Chapter 8
Dynamical Decoupling and Quantum Error Mitigation

The greatest problem in the development of quantum computers are the presence of errors and noises. QEC works in theory: the threshold theorem guarantees it. However, it requires a number of physical qubits ($\sim 10^3$ to $\sim 10^6$) that is often beyond what possible with the current technology. In this chapter, we introduce two possible routes to tackle the problem. These are the Dynamical Decoupling and the Quantum Error Mitigation.

8.1 Dynamical Decoupling

The dynamical decoupling (DD) approach leverage on averaging out the unwanted effects of the surrounding environment by applying a control on the system.

Two introduce the idea, we focus on the specific model of a qubit coupled to a thermal bath of harmonic oscillators, with $\hat{b}_k$ being the annihilation operator of the k oscillator. The total Hamiltonian reads

$$\hat{H}_0 = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{SB}} = \tfrac{1}{2}\hbar\omega_0\hat{\sigma}_z + \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k \hbar\hat{\sigma}_z (g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k), \tag{8.1}$$

where the first and second contributions are the free Hamiltonian of the qubit system and thermal bath respectively, while the last term describes their interaction being weighted by the constants g_k. Eventually, one focuses on the dynamics of the qubit alone. Thus, the state of interest is the reduced density matrix, which is obtained via

$$\hat{\rho}_{\mathrm{S}}(t) = \mathrm{Tr}^{(\mathrm{B})}\left[e^{-i\hat{H}_0 t/\hbar}\hat{\rho}_{\mathrm{T}}(0)e^{i\hat{H}_0 t/\hbar}\right], \tag{8.2}$$

where $\hat{\rho}_{\mathrm{T}}(0)$ is the total state at time $t = 0$. The latter can be decomposed on the computational basis $\{\,|0\rangle, |1\rangle\,\}$ as

$$\hat{\rho}_{\mathrm{S}}(t) = \sum_{i,j=0,1} \rho_{ij}(t) \left|i\right\rangle \left\langle j\right|. \tag{8.3}$$

Now, given the total Hamiltonian $\hat{H}_0$, one finds that the populations ρ_{ii} are conserved. Indeed, $\left[\hat{\sigma}_z, \hat{H}_0\right] = 0$ and thus the model describes a purely decohering mechanism, where no energy exchange between the system and the bath is present. Specifically, one can focus on the dynamics of the coherences $\rho_{01}(t)$ alone, and we do it in the interaction picture. Thus, we have that the total state is given by

$$\hat{\rho}_{\mathrm{T}}^{(\mathrm{I})}(t) = e^{i(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{B}})t/\hbar}\hat{\rho}_{\mathrm{T}}(t)e^{-i(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{B}})t/\hbar}, \tag{8.4}$$

with the effective Hamiltonian reading

$$\hat{H}_0^{(\mathrm{I})}(t) = \hbar\hat{\sigma}_z \sum_k \left(g_k \hat{b}_k^\dagger e^{i\omega_k t} + g_k^* \hat{b}_k e^{-i\omega_k t} \right). \tag{8.5}$$

Correspondingly, the unitary operator determining the time evolution in the interaction picture from time t_0 to t is

$$\hat{U}^{(\mathrm{I})}(t_0, t) = T \exp\left\{ -\frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}s \, \hat{H}_0^{(\mathrm{I})}(s) \right\}, \tag{8.6}$$

where T indicates the time-ordering operator. The corresponding Dyson expansion, which is effectively a Taylor expansion accounting also for the time-ordering, reads

$$\hat{U}^{(\mathrm{I})}(t_0, t) = \hat{\mathbb{1}} - \frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_1) - \frac{1}{\hbar^2} \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1) + \dots \tag{8.7}$$

Let us focus on the second order term, which can be rewritten in term of the integral from t_0 to t for both variables as

$$\int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1) = \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1) - \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_2}^{t} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1). \tag{8.8}$$

The last term corresponds to the integral over the area highlighted by the blue lines in Fig. 8.1: for each value of $t_2 \in [t_0, t]$, the t_1 integral runs from t_2 to t. Equivalently, this area can be described by the red lines: for each value of $t_1 \in [t_0, t]$, one performs the t_2 integral from t_0 to t_1.

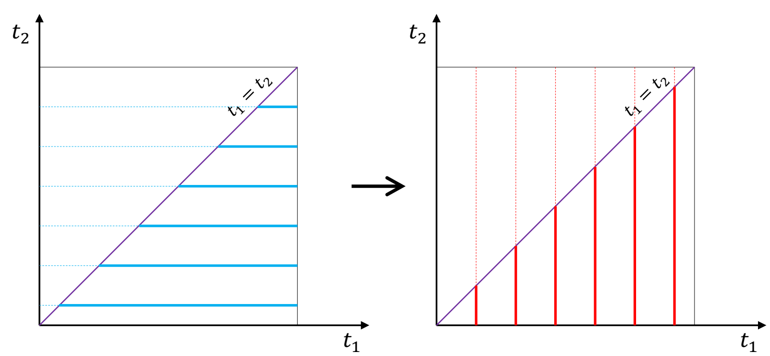

Fig. 8.1: Representation of different but equivalent ways of how to perform the integral in Eq. (8.9).

Mathematically, this implies that the following equality holds

$$\int_{t_0}^{t} \mathrm{d}t_2 \int_{t_2}^{t} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1) = \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1). \tag{8.9}$$

Moreover, we can recast the right-hand-side of the latter equation as

$$\begin{aligned} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1) &= \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left[\hat{H}_0^{(\mathrm{I})}(t_2), \hat{H}_0^{(\mathrm{I})}(t_1) \right] + \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, \hat{H}_0^{(\mathrm{I})}(t_1) \hat{H}_0^{(\mathrm{I})}(t_2), \\ &= \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left[\hat{H}_0^{(\mathrm{I})}(t_2), \hat{H}_0^{(\mathrm{I})}(t_1) \right] + \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, \hat{H}_0^{(\mathrm{I})}(t_2) \hat{H}_0^{(\mathrm{I})}(t_1), \end{aligned} \tag{8.10}$$

where we swapped the variables $t_1 \leftrightarrow t_2$ in the second term. Namely, such second term is identical to that in the left-hand-side of Eq. (8.8). Here, the non-equal time Hamiltonians do not commute, but give

$$\left[\hat{H}_0^{(\mathrm{I})}(t_2), \hat{H}_0^{(\mathrm{I})}(t_1)\right] = \hbar^2 \hat{\sigma}_z^2 \sum_{kk'} \left(g_k g_{k'}^* e^{i\omega_k t_2 - i\omega_{k'} t_1} \left[\hat{b}_k^\dagger, \hat{b}_{k'}\right] + g_k^* g_{k'} e^{-i\omega_k t_2 + i\omega_{k'} t_1} \left[\hat{b}_k, \hat{b}_{k'}^\dagger\right] \right), \\ = -2i\hbar^2 \sum_k |g_k|^2 \sin[\omega_k(t_2 - t_1)], \tag{8.11}$$

which is an imaginary number. Now, by merging the last four equations, we find

$$\int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1\, \hat{H}_0^{(\mathrm{I})}(t_2)\hat{H}_0^{(\mathrm{I})}(t_1) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t} \mathrm{d}t_1\, \hat{H}_0^{(\mathrm{I})}(t_2)\hat{H}_0^{(\mathrm{I})}(t_1) - \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left[\hat{H}_0^{(\mathrm{I})}(t_2), \hat{H}_0^{(\mathrm{I})}(t_1)\right], \\ = \frac{1}{2}\left(\int_{t_0}^{t} \mathrm{d}t_1\, \hat{H}_0^{(\mathrm{I})}(t_1)\right)^2 + i\hbar^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \sum_k |g_k|^2 \sin[\omega_k(t_2 - t_1)]. \tag{8.12}$$

This allows to recast the Dyson expansion as

$$\hat{U}^{(\mathrm{I})}(t_0, t) = \hat{\mathbb{1}} - \frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1\, \hat{H}_0^{(\mathrm{I})}(t_1) + \frac{1}{2}\left(-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1\, \hat{H}_0^{(\mathrm{I})}(t_1)\right)^2 - i\phi(t_0, t) + \ldots, \tag{8.13}$$

where

$$\phi(t_0, t) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \sum_k |g_k|^2 \sin[\omega_k(t_2 - t_1)]. \tag{8.14}$$

By summing all the terms, one gets

$$\hat{U}^{(\mathrm{I})}(t_0, t) = e^{-i\varphi(t_0,t)} \exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}s\, \hat{H}_0^{(\mathrm{I})}(s)\right\}, \tag{8.15}$$

where one has an extra global phase, which is hiowever unimportant since

$$\hat{U}^{(\mathrm{I})}(t_0, t)\hat{\rho}\left[\hat{U}^{(\mathrm{I})}(t_0, t)\right]^\dagger = \exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}s\, \hat{H}_0^{(\mathrm{I})}(s)\right\}\hat{\rho}\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}s\, \hat{H}_0^{(\mathrm{I})}(s)\right\}, \tag{8.16}$$

and there is not time-ordering operator.

Specifically, one can perform explicitly the time integral in the exponential of $\hat{U}^{(\mathrm{I})}(t_0, t)$, which reads

$$-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}s\, \hat{H}_0^{(\mathrm{I})}(s) = \frac{1}{2}\hat{\sigma}_z \sum_k \left(\hat{b}_k^\dagger e^{i\omega_k t_0}\xi_k(t - t_0) - \hat{b}_k e^{-i\omega_k t_0}\xi_k^*(t - t_0)\right), \tag{8.17}$$

where

$$\xi_k(t - t_0) = \frac{2g_k}{\omega_k}(1 - e^{i\omega_k(t - t_0)}). \tag{8.18}$$

Now, the quantity of interest is the coherence, which in the interaction picture is given by

$$\rho_{01}^{(\mathrm{I})}(t) = \langle 0|\hat{\rho}_{\mathrm{S}}^{(\mathrm{I})}(t)|1\rangle = \langle 0|\, \mathrm{Tr}^{(\mathrm{B})}\left[\hat{U}^{(\mathrm{I})}(t_0, t)\hat{\rho}_{\mathrm{T}}(t_0)\left(\hat{U}^{(\mathrm{I})}(t_0, t)\right)^\dagger\right]|1\rangle, \tag{8.19}$$

and it can be computed analytically under the following assumptions:

1) The total initial state is separable, namely

$$\hat{\rho}_{\mathrm{T}}(t_0) = \hat{\rho}_{\mathrm{S}}(t_0) \otimes \hat{\rho}_{\mathrm{B}}(t_0); \tag{8.20}$$

2) The initial state of the bath is a thermal state of the form

$$\hat{\rho}_{\mathrm{B}}(t_0) = \prod_k (1 - e^{-\beta\hbar\omega_k}) \exp\left(-\beta\hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right), \tag{8.21}$$

with $\beta = (k_{\mathrm{B}}T)^{-1}$ being the inverse temperature.

By going back to the Schrödinger picture, Eq. (8.19) reads

$$\begin{aligned}\rho_{01}(t) &= e^{-i\omega_0(t-t_0)}\rho_{01}^{(\mathrm{I})}(t),\\ &= e^{-i\omega_0(t-t_0)} \langle 0| \operatorname{Tr}^{(\mathrm{B})}\left[\exp\left[\frac{1}{2}\hat{\sigma}_z \sum_k \left(\hat{b}_k^\dagger e^{i\omega_k t_0}\xi_k(t-t_0) - \hat{b}_k e^{-i\omega_k t_0}\xi_k^*(t-t_0)\right)\right]\hat{\rho}_{\mathrm{T}}(t_0)\right.\\ &\left.\times \exp\left[-\frac{1}{2}\hat{\sigma}_z \sum_k \left(\hat{b}_k^\dagger e^{i\omega_k t_0}\xi_k(t-t_0) - \hat{b}_k e^{-i\omega_k t_0}\xi_k^*(t-t_0)\right)\right]\right]|1\rangle.\end{aligned} \tag{8.22}$$

By applying with $\hat{\sigma}_z$ on the $\langle 0|$ and $|1\rangle$ states, we get

$$\rho_{01}(t) = e^{-i\omega_0(t-t_0)} \operatorname{Tr}^{(\mathrm{B})}\left[\exp\left[\sum_k \left(c_k \hat{b}_k^\dagger - c_k^* \hat{b}_k\right)\right]\hat{\rho}_{\mathrm{B}}(t_0)\right]\rho_{01}(t_0), \tag{8.23}$$

where we exploited the cyclicity of the partial trace with respect to the bath operators, the assumption of separability of the initial state, the definition of the initial coherence $\rho_{01}(t_0) = \langle 0|\hat{\rho}_{\mathrm{S}}(t_0)|1\rangle$, and defined

$$c_k = c_k(t_0, t) = e^{i\omega_k t_0}\xi_k(t - t_0). \tag{8.24}$$

In Eq. (8.23), one can recognise the displacement operator. Namely, the latter equation can be recasted as

$$\rho_{01}(t) = e^{-i\omega_0(t-t_0)} \operatorname{Tr}^{(\mathrm{B})}\left[\prod_k \hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}}(t_0)\right]\rho_{01}(t_0), \tag{8.25}$$

where

$$\hat{D}(\beta) = \exp(\beta\hat{b}_k^\dagger - \beta^*\hat{b}) \tag{8.26}$$

is the displacement operator.

Recall 8.1 (Coherent states)
Given the ground state $|0\rangle$, one can construct a coherent state via the application of a displacement operator to $|0\rangle$. Namely,

$$\hat{D}(\beta)\,|0\rangle = |\beta\rangle\,, \tag{8.27}$$

where $\beta \in \mathbb{C}$. The coherent states form an overcomplete basis of the Hilbert space $\mathbb{H}$, for which one has

$$\hat{\mathbb{1}} = \int \frac{\mathrm{d}^2 z}{\pi}\,|z\rangle\langle z|\,, \tag{8.28}$$

where $\mathrm{d}^2 z = \mathrm{d}(\Re z)\mathrm{d}(\Im z)$. The combination of two displacement operators is governed by

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \beta\alpha^*)}\hat{D}(\alpha+\beta). \tag{8.29}$$

Finally, one can express the coherent states in terms of the Fock basis $\{\ |n\rangle\ \}$, where

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}e^{-|z|^2/2}, \tag{8.30}$$

determines the weights between the two basis.

Let us then consider the partial trace over the k-th mode of the displacement operator and the thermal state, which explicitly reads

$$\begin{aligned} \mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}_k}(t_0)\right] &= (1-e^{-\beta\hbar\omega_k})\,\mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)e^{-\beta\hbar\omega_k\hat{b}_k^\dagger\hat{b}_k}\right], \\ &= (1-e^{-\beta\hbar\omega_k})\int\frac{\mathrm{d}^2z}{\pi}\,\langle z|e^{-\beta\hbar\omega_k\hat{b}_k^\dagger\hat{b}_k}\hat{D}_k(c_k)|z\rangle\,, \end{aligned} \tag{8.31}$$

where we exploited the coherent basis to compute the partial trace and exploit its cyclicity. By applying the composition of displacement opertors (namely, Eq. (8.29) merged with Eq. (8.27)), and introducing an identity in the Fock basis, i.e. $\hat{\mathbb{1}} = \sum_{n=0}^{+\infty}|n\rangle\langle n|$, we get

$$\begin{aligned} \mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}_k}(t_0)\right] &= (1-e^{-\beta\hbar\omega_k})\int\frac{\mathrm{d}^2z}{\pi}\sum_n\langle z|n\rangle\,e^{-\beta\hbar\omega_k n}e^{\frac{1}{2}(c_kz^*-zc_k^*)}\,\langle n|z+c_k\rangle\,, \\ &= (1-e^{-\beta\hbar\omega_k})\int\frac{\mathrm{d}^2z}{\pi}\sum_n e^{-\beta\hbar\omega_k n}e^{\frac{1}{2}(c_kz^*-zc_k^*)}\frac{(z^*)^n}{\sqrt{n!}}e^{-|z|^2/2}\frac{(z+c_k)^n}{\sqrt{n!}}e^{-|z+c_k|^2/2}, \end{aligned} \tag{8.32}$$

where we applied Eq. (8.30). By putting together the exponentials, we obtain

$$\mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}_k}(t_0)\right] = (1-e^{-\beta\hbar\omega_k})\int\frac{\mathrm{d}^2z}{\pi}e^{-|z|^2}e^{-|c_k|^2/2}e^{-zc_k^*}S(z,c_k) \tag{8.33}$$

where

$$\begin{aligned} S(z,c_k) &= \sum_n e^{-\beta\hbar\omega_k n}\frac{(z^*)^n}{\sqrt{n!}}\frac{(z+c_k)^n}{\sqrt{n!}}, \\ &= \sum_n\frac{1}{n!}\left[e^{-\beta\hbar\omega_k}(|z|^2+c_kz^*)\right]^n, \\ &= \exp\left[e^{-\beta\hbar\omega_k}(|z|^2+c_kz^*)\right]. \end{aligned} \tag{8.34}$$

Thus,

$$\mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}_k}(t_0)\right] = (1-e^{-\beta\hbar\omega_k})\int\frac{\mathrm{d}^2z}{\pi}e^{-|z|^2\alpha_k}e^{-|c_k|^2/2}e^{-zc_k^*}\exp[c_kz^*e^{-\beta\hbar\omega_k}], \tag{8.35}$$

where

$$\alpha_k = 1-e^{-\beta\hbar\omega_k}, \tag{8.36}$$

is a positive quantity. Then, the Gaussian integral in Eq. (8.35) can be safely implemented and gives

$$\begin{aligned} \mathrm{Tr}^{(\mathrm{B})}\left[\hat{D}_k(c_k)\hat{\rho}_{\mathrm{B}_k}(t_0)\right] &= (1-e^{-\beta\hbar\omega_k})\frac{1}{\alpha_k}\exp\left[-\frac{|c_k|^2}{\alpha_k}e^{-\beta\hbar\omega_k}\right]e^{-|c_k|^2/2}, \\ &= \exp\left[-|c_k|^2\left(\frac{1}{2}+\frac{e^{-\beta\hbar\omega_k}}{(1-e^{-\beta\hbar\omega_k})}\right)\right], \\ &= \exp\left[-\frac{|c_k|^2}{2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right)\right]. \end{aligned} \tag{8.37}$$

Finally, by merging together the latter equation with Eq. (8.25), we obtain

$$\rho_{01}(t) = e^{-i\omega_0(t-t_0)}e^{-\Gamma(t_0,t)}\rho_{01}(t_0), \tag{8.38}$$

where we defined

$$
\begin{aligned}
\Gamma(t_0,t) &= \sum_k \frac{|c_k^2|}{2} \coth\left(\frac{\beta\hbar\omega_k}{2}\right), \\
&= \sum_k \frac{|e^{i\omega_k t_0}\xi_k(t-t_0)|^2}{2} \coth\left(\frac{\beta\hbar\omega_k}{2}\right), \\
&= \sum_k \frac{|\frac{2g_k}{\omega_k}(1-e^{i\omega_k(t-t_0)})|^2}{2} \coth\left(\frac{\beta\hbar\omega_k}{2}\right), \\
&= \sum_k \frac{4|g_k|^2}{\omega_k^2}\left(1-\cos[\omega_k(t-t_0)]\right) \coth\left(\frac{\beta\hbar\omega_k}{2}\right)
\end{aligned}
\tag{8.39}
$$

where we simply substituted the definitions of c_k and $\xi_k(t-t_0)$. Now, by introducing the spectral density $I(\omega)$ as

$$
I(\omega) = \sum_k \delta(\omega-\omega_k)|g_k|^2, \tag{8.40}
$$

which determines the strength of the coupling between the system and each bath's modes, we can rewrite $\Gamma(t_0,t)$ as

$$
\begin{aligned}
\Gamma(t_0,t) &= 4\int_0^{+\infty} \mathrm{d}\omega\, I(\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right) \frac{(1-\cos[\omega(t-t_0)])}{\omega^2}, \\
&= 4\int_0^{+\infty} \mathrm{d}\omega\, I(\omega)\left(2\bar{n}(\omega,T)+1\right) \frac{(1-\cos[\omega(t-t_0)])}{\omega^2},
\end{aligned}
\tag{8.41}
$$

where $\bar{n}(\omega,T)$ is the mean number of excitations of the mode ω at the temperature $T=(k_{\mathrm{B}}\beta)^{-1}$. Notably, $\Gamma(t_0,t)$ is positive. This can be seen explicitly from the first line of Eq. (8.39). This means, that — as expected — the interaction with the environment reduces the coherences [cf. Eq. (8.38)].