Suppose now that we want to perturb the system so the induce a spin-flip transition. Physically, since the interaction Hamiltonian $\hat{H}_{\rm SB}$ is proportional to $\hat{\sigma}_z$, then opposite contributions arise when the system is in $|0\rangle$ and $|1\rangle$. Thus, by making the system change fast between $|0\rangle$ and $|1\rangle$, one can average out the contributions from $\hat{H}_{\rm SB}$, effectively decoupling the system from the environment.

Specifically, we will consider a modified Hamiltonian reading

$$\hat{H}_0 \to \hat{H}(t) = \hat{H}_0 + \hat{H}_{\rm P}(t), \tag{8.42}$$

where the Hamiltonian perturbation $\hat{H}_{\rm P}(t)$ can be implemented via a monocromatic alternating magnetic field applied at the resonance. Its explicit form we consider is

$$\begin{aligned} \hat{H}_{\rm P}(t) &= \sum_{n=1}^{n_{\rm P}} V^{(n)}(t) \left\{ \hat{\sigma}_x \cos[\omega_0(t - t_{\rm P}^{(n)})] + \hat{\sigma}_y \sin[\omega_0(t - t_{\rm P}^{(n)})] \right\}, \\ &= \sum_{n=1}^{n_{\rm P}} V^{(n)}(t) \left(\hat{\sigma}_+ e^{i\omega_0(t - t_{\rm P}^{(n)})} + \hat{\sigma}_- e^{-i\omega_0(t - t_{\rm P}^{(n)})} \right), \end{aligned} \tag{8.43}$$

with $n_{\rm P}$ being the number of pulses, $t_{\rm P}^{(n)}$ is the time at which the pulse is switched on every Δt, namely

$$t_{\rm P}^{(n)} = t_0 + n\Delta t, \quad \text{with} \quad n \in \{1, \dots, n_{\rm P}\}. \tag{8.44}$$

Finally, the switch of the impulse is determined by $V^{(n)}(t)$, which is defined as

$$V^{(n)}(t) = \begin{cases} V, & \text{for } t \in [t_{\rm P}^{(n)}, t_{\rm P}^{(n)} + \tau_{\rm P}], \\ 0, & \text{otherwise}, \end{cases} \tag{8.45}$$

where $\tau_{\rm P}$ is the duration time of the pulses.

The exact dynamics with respect to the modified Hamiltonian $\hat{H}(t)$ cannot be solved. However, we can assume that during the pulses the contribution of $\hat{H}_{\rm SB}$ is negligible and we completely neglect it. Then, the dynamics becomes piecewise, alternating $\hat{H}_{\rm SB}$ to $\hat{H}_{\rm P}$.

As for the unperturbed case, we tackle the problem in the interaction picture. Namely, the effective Hamiltonian becomes

$$\hat{H}^{(\rm I)}(t) = \hat{H}_0^{(\rm I)}(t) + \hat{H}_{\rm P}^{(\rm I)}(t), \tag{8.46}$$

where $\hat{H}_0^{(\rm I)}(t)$ is shown in (8.5) and

$$\begin{aligned} \hat{H}_{\rm P}^{(\rm I)}(t) &= \exp\left[\frac{i}{\hbar}\left(\hat{H}_{\rm S} + \hat{H}_{\rm B}\right)\right] \hat{H}_{\rm P}(t) \exp\left[-\frac{i}{\hbar}\left(\hat{H}_{\rm S} + \hat{H}_{\rm B}\right)\right], \\ &= e^{i\omega_0\hat{\sigma}_z t/2} \sum_{n=1}^{n_{\rm P}} V^{(n)}(t) \left(\hat{\sigma}_+ e^{i\omega_0(t - t_{\rm P}^{(n)})} + \hat{\sigma}_- e^{-i\omega_0(t - t_{\rm P}^{(n)})} \right) e^{-i\omega_0\hat{\sigma}_z t/2}. \end{aligned} \tag{8.47}$$

However, one has that

$$\begin{aligned} e^{i\omega_0\hat{\sigma}_z t/2}\hat{\sigma}_- e^{-i\omega_0\hat{\sigma}_z t/2} &= e^{i\omega_0\hat{\sigma}_z t/2} |0\rangle \langle 1| e^{-i\omega_0\hat{\sigma}_z t/2}, \\ &= e^{i\omega_0 t} |0\rangle \langle 1|, \\ &= e^{i\omega_0 t}\hat{\sigma}_-, \end{aligned} \tag{8.48}$$

and similarly

$$e^{i\omega_0\hat{\sigma}_z t/2}\hat{\sigma}_+ e^{-i\omega_0\hat{\sigma}_z t/2} = e^{-i\omega_0 t}\hat{\sigma}_+. \tag{8.49}$$

Then, we obtain

$$\begin{aligned}\hat{H}_{\mathrm{P}}^{(\mathrm{I})}(t) &= \sum_{n=1}^{n_{\mathrm{P}}} V^{(n)}(t)\left(\hat{\sigma}_{+}e^{-i\omega_0 t_{\mathrm{P}}^{(n)}} + \hat{\sigma}_{-}e^{i\omega_0 t_{\mathrm{P}}^{(n)}}\right), \\ &= \sum_{n=1}^{n_{\mathrm{P}}} V^{(n)}(t) e^{i\omega_0 \hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2},\end{aligned} \tag{8.50}$$

where we exploited that $\hat{\sigma}_+ + \hat{\sigma}_- = \hat{\sigma}_x$. Notably, the only time dependence is in $V^{(n)}(t)$, but it is only formal as one can see from Eq. (8.45). Then, when considering the corresponding unitary, we have

$$\begin{aligned}\hat{\mathcal{V}}_n^{(\mathrm{I})}(\tau_{\mathrm{P}}) &= \exp\left(-\frac{i}{\hbar}\int_{t_{\mathrm{P}}^{(n)}}^{t_{\mathrm{P}}^{(n)}+\tau_{\mathrm{P}}} \mathrm{d}s\, \hat{H}_{\mathrm{P}}^{(\mathrm{I})}(s)\right), \\ &= \exp\left(-\frac{i}{\hbar} V e^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\tau_{\mathrm{P}}\right).\end{aligned} \tag{8.51}$$

By Taylor expanding

$$\begin{aligned}\hat{\mathcal{V}}_n^{(\mathrm{I})}(\tau_{\mathrm{P}}) &= \sum_k \frac{1}{k!}\left(-\frac{i}{\hbar} V e^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\tau_{\mathrm{P}}\right)^k, \\ &= e^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\sum_k \frac{1}{k!}\left(-\frac{i}{\hbar}V\hat{\sigma}_x\tau_{\mathrm{P}}\right)^k e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}, \\ &= e^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2} e^{-\frac{i}{\hbar}V\hat{\sigma}_x\tau_{\mathrm{P}}} e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}.\end{aligned} \tag{8.52}$$

We finally fix V and τ_{P} so to have an actual bit-flip. This is provided by setting

$$\frac{V\tau_{\mathrm{P}}}{\hbar} = \frac{\pi}{2}, \tag{8.53}$$

which gives

$$e^{-\frac{i}{\hbar}V\hat{\sigma}_x\tau_{\mathrm{P}}} = e^{-i\frac{\pi}{2}\hat{\sigma}_x} = -i\hat{\sigma}_x. \tag{8.54}$$

Notably, we can consider the limit of the time pulses that go to zero, i.e. $\tau_{\mathrm{P}} \to 0$, as long as $V \to \infty$ and Eq. (8.53) holds. Since from here V does not appear explicitly, this will only simplify the calculations.

Then, we have that

$$\hat{\mathcal{V}}_n^{(\mathrm{I})}(\tau_{\mathrm{P}}) = \hat{\mathcal{V}}_n^{(\mathrm{I})} = -ie^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}/2}. \tag{8.55}$$

By considering that the following relation holds

$$e^{-i\omega_0\hat{\sigma}_z t/2} = \cos(\omega_0 t/2)\hat{\mathbb{1}} - i\sin(\omega_0 t/2)\hat{\sigma}_z, \tag{8.56}$$

and the anticommutation relation $\{\hat{\sigma}_x, \hat{\sigma}_z\} = 0$, we have that

$$\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t/2} = e^{i\omega_0\hat{\sigma}_z t/2}\hat{\sigma}_x. \tag{8.57}$$

It follows that one can write the operator $\hat{\mathcal{V}}_n^{(\mathrm{I})}$ in two equivalent ways:

$$\hat{\mathcal{V}}_n^{(\mathrm{I})} = -ie^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}}\hat{\sigma}_x = -i\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(n)}}. \tag{8.58}$$

Let us now consider the time evolution of the first entire cycle of spin-flips: this is from time t_0 through time $t_{\mathrm{P}}^{(1)}$ when the spin flips the first time, to time $t_{\mathrm{P}}^{(2)}$ when the spin flips back to the original spin state. In particular, we define this latter time as $t_1 = t_0 + 2\Delta t$. The unitary dynamics from t_0 to t_1 is given by

$$\begin{aligned}\hat{\mathcal{U}}_{\mathrm{P}}^{(\mathrm{I})}(t_0,t_1) &= \hat{\mathcal{V}}_2^{(\mathrm{I})}\hat{U}^{(\mathrm{I})}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\hat{\mathcal{V}}_1^{(\mathrm{I})}\hat{U}^{(\mathrm{I})}(t_0,t_{\mathrm{P}}^{(1)}),\\ &= \left[\hat{\mathcal{V}}_2^{(\mathrm{I})}\hat{\mathcal{V}}_1^{(\mathrm{I})}\right]\left[\left(\hat{\mathcal{V}}_1^{(\mathrm{I})}\right)^{-1}\hat{U}^{(\mathrm{I})}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\hat{\mathcal{V}}_1^{(\mathrm{I})}\right]\hat{U}^{(\mathrm{I})}(t_0,t_{\mathrm{P}}^{(1)}),\end{aligned} \tag{8.59}$$

where

$$\hat{U}^{(\mathrm{I})}(t_\alpha,t_\beta) = \exp\left[\frac{1}{2}\hat{\sigma}_z\sum_k\left(\hat{b}_k^\dagger e^{i\omega_k t_\alpha}\xi_k(t_\beta-t_\alpha)-\hat{b}_k e^{-i\omega_k t_\alpha}\xi_k^*(t_\beta-t_\alpha)\right)\right]. \tag{8.60}$$

The first square parenthesis in the last line of Eq. (8.59) is given by

$$\begin{aligned}\hat{\mathcal{V}}_2^{(\mathrm{I})}\hat{\mathcal{V}}_1^{(\mathrm{I})} &= \left(-ie^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(2)}}\hat{\sigma}_x\right)\left(-i\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(1)}}\right),\\ &= -e^{i\omega_0\hat{\sigma}_z(t_{\mathrm{P}}^{(2)}-t_{\mathrm{P}}^{(1)})}.\end{aligned} \tag{8.61}$$

Similarly, the second square parenthesis in the last line of Eq. (8.59) can be rewritten as

$$\left(\hat{\mathcal{V}}_1^{(\mathrm{I})}\right)^{-1}\hat{U}^{(\mathrm{I})}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\hat{\mathcal{V}}_1^{(\mathrm{I})} = e^{i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(1)}}\hat{\sigma}_x\exp\left[\frac{1}{2}\hat{\sigma}_z\hat{B}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\right]\hat{\sigma}_x e^{-i\omega_0\hat{\sigma}_z t_{\mathrm{P}}^{(1)}} \tag{8.62}$$

where $B(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)}) = \sum_k\left(\hat{b}_k^\dagger e^{i\omega_k t_{\mathrm{P}}^{(1)}}\xi_k(\Delta t)-\hat{b}_k e^{-i\omega_k t_{\mathrm{P}}^{(1)}}\xi_k^*(\Delta t)\right)$. Here, we Taylor expand the central exponential, which gives

$$\exp\left[\frac{1}{2}\hat{\sigma}_z\hat{B}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\right] = \sum_l\frac{1}{l!}\left(\frac{1}{2}\hat{\sigma}_z\hat{B}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\right)^l. \tag{8.63}$$

We divide the contributions to the sum in those with even and odd values of l. For even values, we have $\hat{\sigma}_z^l = \hat{\sigma}_z^{2l'}$, where $l=2l'$; then $\hat{\sigma}_z^l = \hat{\mathbb{1}} = (-\hat{\sigma}_z)^l$. For odd values of l, we have $\hat{\sigma}_z^l = \hat{\sigma}_z^{2l'+1}$, where $l=2l'+1$; then $\hat{\sigma}_z^l = \hat{\sigma}_z$. But more specifically, we also have that $\hat{\sigma}_x\hat{\sigma}_z^l\hat{\sigma}_x = \hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x = -\hat{\sigma}_z = -\hat{\sigma}_z^l$. Thus, it follows that

$$\left(\hat{\mathcal{V}}_1^{(\mathrm{I})}\right)^{-1}\hat{U}^{(\mathrm{I})}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\hat{\mathcal{V}}_1^{(\mathrm{I})} = \exp\left[-\frac{1}{2}\hat{\sigma}_z\hat{B}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\right]. \tag{8.64}$$

Thus, we have that Eq. (8.59) reads

$$\hat{\mathcal{U}}_{\mathrm{P}}^{(\mathrm{I})}(t_0,t_1) = -e^{i\omega_0\hat{\sigma}_z(t_{\mathrm{P}}^{(2)}-t_{\mathrm{P}}^{(1)})}\exp\left[-\frac{1}{2}\hat{\sigma}_z\hat{B}(t_{\mathrm{P}}^{(1)},t_{\mathrm{P}}^{(2)})\right]\exp\left[\frac{1}{2}\hat{\sigma}_z\hat{B}(t_0,t_{\mathrm{P}}^{(1)})\right], \tag{8.65}$$

and can be recasted as

$$\hat{\mathcal{U}}_{\mathrm{P}}^{(\mathrm{I})}(t_0,t_1) = \exp\left[i\omega_0\hat{\sigma}_z(t_{\mathrm{P}}^{(2)}-t_{\mathrm{P}}^{(1)})+\frac{1}{2}\hat{\sigma}_z\sum_k\left(\hat{b}_k^\dagger e^{i\omega_k t_0}\eta_k(\Delta t)-\hat{b}_k e^{-i\omega_k t_0}\eta_k^*(\Delta t)\right)\right], \tag{8.66}$$

where we neglected the overall unimportant phase and we defined

$$\eta_k(\Delta t) = \xi(\Delta t)\left(1-e^{i\omega_k\Delta t}\right). \tag{8.67}$$

Now, the full evolution from time t_0 to time t_N after N entire cycles of spin-flip is simply given by

$$\prod_{n=1}^N\hat{\mathcal{U}}_{\mathrm{P}}^{(\mathrm{I})}(t_{n-1},t_n) = \exp\left[i\omega_0\hat{\sigma}_z(t_N-t_0)+\frac{1}{2}\hat{\sigma}_z\sum_k\left(\hat{b}_k^\dagger\sum_n e^{i\omega_k t_0}\eta_k(N,\Delta t)-\hat{b}_k\sum_n e^{-i\omega_k t_0}\eta_k^*(N,\Delta t)\right)\right], \tag{8.68}$$

where we introduced

$$\begin{aligned}\eta_k(N,\Delta t) &= e^{-i\omega_k t_0}\sum_n e^{i\omega_k t_{n-1}}\eta_k(\Delta t),\\ &= \eta_k(\Delta t)\sum_{n=1}^{N} e^{2i\omega_k \Delta t(n-1)}.\end{aligned} \tag{8.69}$$

Such an evolution is to be compared to that with no pulses on the same time period. This is given by Eq. (8.60) where one substitutes $t_\alpha \to t_0$ and $t_\beta \to t_N$. Then, since $t_N - t_0 = 2N\Delta t$, we have

$$\hat{U}^{(\mathrm{I})}(t_0,t_N) = \exp\left[\frac{1}{2}\hat{\sigma}_z\sum_k\left(\hat{b}_k^\dagger e^{i\omega_k t_0}\xi_k(2N\Delta t) - \hat{b}_k e^{-i\omega_k t_0}\xi_k^*(2N\Delta t)\right)\right]. \tag{8.70}$$

Notably, the expressions in Eq. (8.68) and Eq. (8.70) have a similar structure, with the important difference being the factor $\eta_k(N,\Delta t)$ substituted with $\xi_k(2N\Delta t)$. Thus, the decohering factor $\Gamma(t_0,t_N)$ will take a suitably modified expression as that in Eq. (8.39), namely

$$\Gamma(t_0,t_N) = \sum_k \frac{|e^{i\omega_k t_0}\eta_k(N,\Delta t)|^2}{2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right). \tag{8.71}$$

We now compare the difference between these two factors:

$$\eta_k(N,\Delta t) - \xi_k(2N\Delta t) = \eta_k(\Delta t)\sum_{n=1}^{N} e^{2i\omega_k\Delta t(n-1)} - \xi_k(2\Delta t)\sum_{n=1}^{N} e^{2i\omega_k\Delta t(n-1)}, \tag{8.72}$$

where we exploited the composition of the ξ_k terms. Then, by considering that

$$\begin{aligned}\xi_k(\Delta t)(1+e^{i\omega_k\Delta t}) &= \frac{2g_k}{\omega_k}(1-e^{i\omega_k\Delta t})(1+e^{i\omega_k\Delta t}),\\ &= \frac{2g_k}{\omega_k}(1-e^{2i\omega_k\Delta t}),\\ &= \xi_k(2\Delta t),\end{aligned} \tag{8.73}$$

and the definition of $\eta_k(\Delta t)$ in Eq. (8.67), we obtain

$$\eta_k(N,\Delta t) - \xi_k(2N\Delta t) = -2\xi_k(\Delta t)e^{i\omega_k\Delta t}\sum_{n=1} e^{2i\omega_k\Delta t(n-1)}. \tag{8.74}$$

Equivalently, we have

$$\eta_k(N,\Delta t) = \xi_k(2N\Delta t)\left(1 - f_k(N,\Delta t)\right), \tag{8.75}$$

where

$$f_k(N,\Delta t) = 2\frac{\xi_k(\Delta t)}{\xi_k(2N\Delta t)}e^{i\omega_k\Delta t}\sum_{n=1} e^{2i\omega_k\Delta t(n-1)}. \tag{8.76}$$

By exploiting the geometric series and the definition of ξ_k, we get

$$\begin{aligned}f_k(N,\Delta t) &= 2\frac{(1-e^{i\omega_k\Delta t})}{(1-e^{2i\omega_k N\Delta t})}e^{i\omega_k\Delta t}\frac{(1-e^{2i\omega_k N\Delta t})}{(1-e^{2i\omega_k\Delta t})},\\ &= 2\frac{(1-e^{i\omega_k\Delta t})}{(1-e^{2i\omega_k\Delta t})}e^{i\omega_k\Delta t}.\end{aligned} \tag{8.77}$$

Finally, by taking the limit of dense pulses, i.e. $\Delta t \to 0$, we obtain

$$\lim_{\Delta t\to 0} f_k(N,\Delta t) = 1, \tag{8.78}$$

which means that under the same limit we have

$$\lim_{\Delta t \to 0} \eta_k(N, \Delta t) = 0. \tag{8.79}$$

As a consequence, the decoherence factor vanishes: $\Gamma(t_0, t_N) \to 0$. Namely, the decohering effect of the environment on the system is cancelled. Effectively, one has a (dynamical) decoupling of the system from its environment.