ROTATION: at which scales the ambient rotation becomes an important factor in controlling the fluid motions?

STRATIFICATION: at which conditions the stratification effects have a dynamic role on fluid motions?

ROTATION + STRATIFICATION

ROTATION: at which scales the ambient rotation Ω becomes an important factor in controlling the fluid motions?

=> Determine which are the physical phenomena affected by Ω

- def. ambient rotation rate **Ω** (daily rotation + orbital revolution)

$$\Omega = \frac{2\pi\ radians}{time\ of\ 1\ rotation} = \frac{2\pi}{24\ hr} + \frac{2\pi}{1\ yr} = (7.27 + 0.02) \cdot 10^{-5} s^{-1} = \frac{2\pi}{T_R}$$

$$\Rightarrow \Omega = \frac{2\pi}{T_R} = 7.29 \cdot 10^{-5} s^{-1}$$

- def. motion time scale T (linked with local time variation ∂_t)

$$\omega = \frac{time\ of\ 1\ rotation}{motion\ time\ scale} = \frac{2\pi/_{\Omega}}{T} = \frac{T_R}{T}$$

ROTATION: at which scales the ambient rotation Ω becomes an important factor in controlling the fluid motions?

=> Determine which are the physical phenomena affected by Ω

- IF $\omega \lesssim 1 \Rightarrow T \gtrsim \frac{2\pi}{\Omega} = T_R \approx 24hr \Rightarrow$ the fluid will feel the effect of rotation
- now let's consider velocity and length scales of motion: U and L (linked with advection time scale $u\partial_x$)
- IF a parcel travelling at velocity U covers the distance L in a time L/U $\gtrsim T_R$ then its trajectory will be affected by rotation:

$$\varepsilon = \frac{time\ of\ 1\ rotation}{time\ taken\ to\ cover\ L\ at\ U} = \frac{2\pi/\Omega}{L/U} = \frac{2\pi U}{L\Omega}$$

- IF $\varepsilon \lesssim 1 \Rightarrow L/U \gtrsim 2\pi/\Omega = T_R \Rightarrow$ rotation will affect parcel's trajectory

ROTATION: at which scales the ambient rotation Ω becomes an important factor in controlling the fluid motions?

=> Determine which are the physical phenomena affected by Ω

• IF $\omega \lesssim 1 \Rightarrow T \gtrsim \frac{2\pi}{\Omega} = T_R \approx 24hr \Rightarrow$ the fluid will feel the effect of rotation

• now let's consider velocity and length scales of motion: U and L (linked with advection time scale $u\partial_x$)

• IF a parcel travelling at velocity U covers the distance L in a time L/U $\gtrsim T_R$ then its trajectory will be affected by rotation:

$$\varepsilon = \frac{time\ of\ 1\ rotation}{time\ taken\ to\ cover\ L\ at\ U} = \frac{2\pi/_{\Omega}}{L/_{U}} = \frac{2\pi U}{L\Omega}$$

• IF $\varepsilon \lesssim 1 \Rightarrow {}^{L}/_{U} \gtrsim {}^{2\pi}/_{\Omega} = T_R \Rightarrow$ rotation will affect parcel's trajectory

Table 1.1 LENGTH AND VELOCITY SCALES OF MOTIONS IN WHICH ROTATION EFFECTS ARE IMPORTANT

$L = 1$ m	$U \leq 0.012$ mm/s
$L = 10$ m	$U \leq 0.12$ mm/s
$L = 100$ m	$U \leq 1.2$ mm/s
$L = 1$ km	$U \leq 1.2$ cm/s
$L = 10$ km	$U \leq 12$ cm/s
$L = 100$ km	$U \leq 1.2$ m/s
$L = 1000$ km	$U \leq 12$ m/s
$L =$ Earth radius $= 6371$ km	$U \leq 74$ m/s

given L, rotation is important if $U \leq L/T_R$ with $T_R = 24$ hours

most engineering applications (water flows, pipe flows, turbines, airfoils…) do not satisfy the inequality and in these cases rotation can be neglected

STRATIFICATION: at which conditions the stratification effects have a dynamic role on fluid motions?

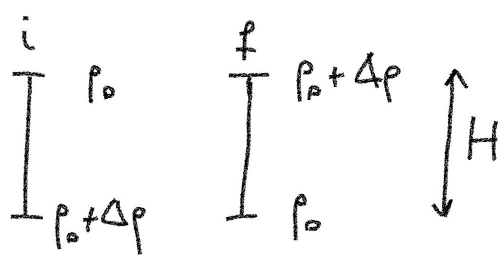
=> Determine at which conditions the stratification effects have a dynamic role on fluid motions

- GFD flows consist of fluid masses at different density which under gravitational action tend to arrange in vertical stacks => state of minimal energy

- BUT motions continuously perturb the equilibrium, tending to raise dense fluid and lower light fluid (potential vs kinetic energy): $\Delta P\ increase \Leftrightarrow \Delta K\ decrease$

- dynamical role of stratification can be evaluated by comparison of P and K

- ρ_0 = average density
- $\Delta\rho$ = density variations
- H = $\Delta\rho$ height scale



$$\Delta P = P_f - P_i = (\rho_0 + \Delta\rho)gH + \rho_0 \mathrm{g} \cdot 0 - \big((\rho_0 + \Delta\rho)g \cdot 0 + \rho_0 \mathrm{gH}\big) = \Delta\rho g H$$

$$K = \frac{1}{2}\rho_0 U^2$$

STRATIFICATION: at which conditions the stratification effects have a dynamic role on fluid motions?

=> Determine at which conditions the stratification effects have a dynamic role on fluid motions

$$\sigma = \frac{kinetic\ energy\ available}{potential\ energy\ available} = \frac{\frac{1}{2}\rho_0 U^2}{\Delta\rho g H}$$

- IF $\sigma \sim 1$ a ΔP increase to perturb the equilibrium will consume a ΔK of the same order modifying the flow field substantially
- IF $\sigma \ll 1$ ΔK is too low to perturb the stratification that largely affects the flow
- IF $\sigma \gg 1$ ΔP occurs at very little ΔK variation so stratification does not affect the flow
- $\Rightarrow$ Stratification effects are important when $\sigma \lesssim 1$

STRATIFICATION: at which conditions the stratification effects have a dynamic role on fluid motions?

=> Determine at which conditions the stratification effects have a dynamic role on fluid motions

$$\sigma = \frac{kinetic\ energy\ available}{potential\ energy\ available} = \frac{\frac{1}{2}\rho_0 U^2}{\Delta\rho g H}$$

- IF $\sigma \sim 1$ a ΔP increase to perturb the equilibrium will consume a ΔK of the same order modifying the flow field substantially
- IF $\sigma \ll 1$ ΔK is too low to perturb the stratification that largely affects the flow
- IF $\sigma \gg 1$ ΔP occurs at very little ΔK variation so stratification does not affect the flow
- $\Rightarrow$ Stratification effects are important when $\sigma \lesssim 1$

ROTATION + STRATIFICATION

⇒ Determine what happens when both R + S affect the flow

$\varepsilon \sim 1 \Rightarrow L \sim \frac{U}{\Omega} \ldots \sigma \sim 1 \Rightarrow \; U \sim \sqrt{\frac{\Delta\rho}{\rho_0} gH} \; \Rightarrow \mathrm{L} \sim \frac{1}{\Omega} \sqrt{\frac{\Delta\rho}{\rho_0} gH}$ fundamental length scale

- In a given fluid of mean density ρ_0 and density variation $\Delta\rho$ occupying a height H in a planet rotating at Ω with a gravitational acceleration g, the scale L represents the preferential lenght over which motions take place

ROTATION + STRATIFICATION

⇒ **Determine what happens when both R + S affect the flow**

$$\varepsilon \sim 1 \Rightarrow L \sim \frac{U}{\Omega} \; \ldots \; \sigma \sim 1 \Rightarrow \; U \sim \sqrt{\frac{\Delta\rho}{\rho_0} gH} \; \Rightarrow \boxed{\mathrm{L} \sim \frac{1}{\Omega}\sqrt{\frac{\Delta\rho}{\rho_0} gH} \text{ fundamental length scale}}$$

- In a given fluid of mean density ρ_0 and density variation $\Delta\rho$ occupying a height H in a planet rotating at Ω with a gravitational acceleration g, the scale L represents the preferential lenght over which motions take place

$\Omega = 7.29 \cdot 10^{-5}\ s^{-1}$

$g = 9.81\ m\,s^{-2}$

ATM. $\rho_0 = 1.2\ Kg/m^3$, $\Delta\rho = 0.03\ Kg/m^3$, $H = 5000\ m$ } $L \sim 500\ Km$, $U \sim 30\ m/s$

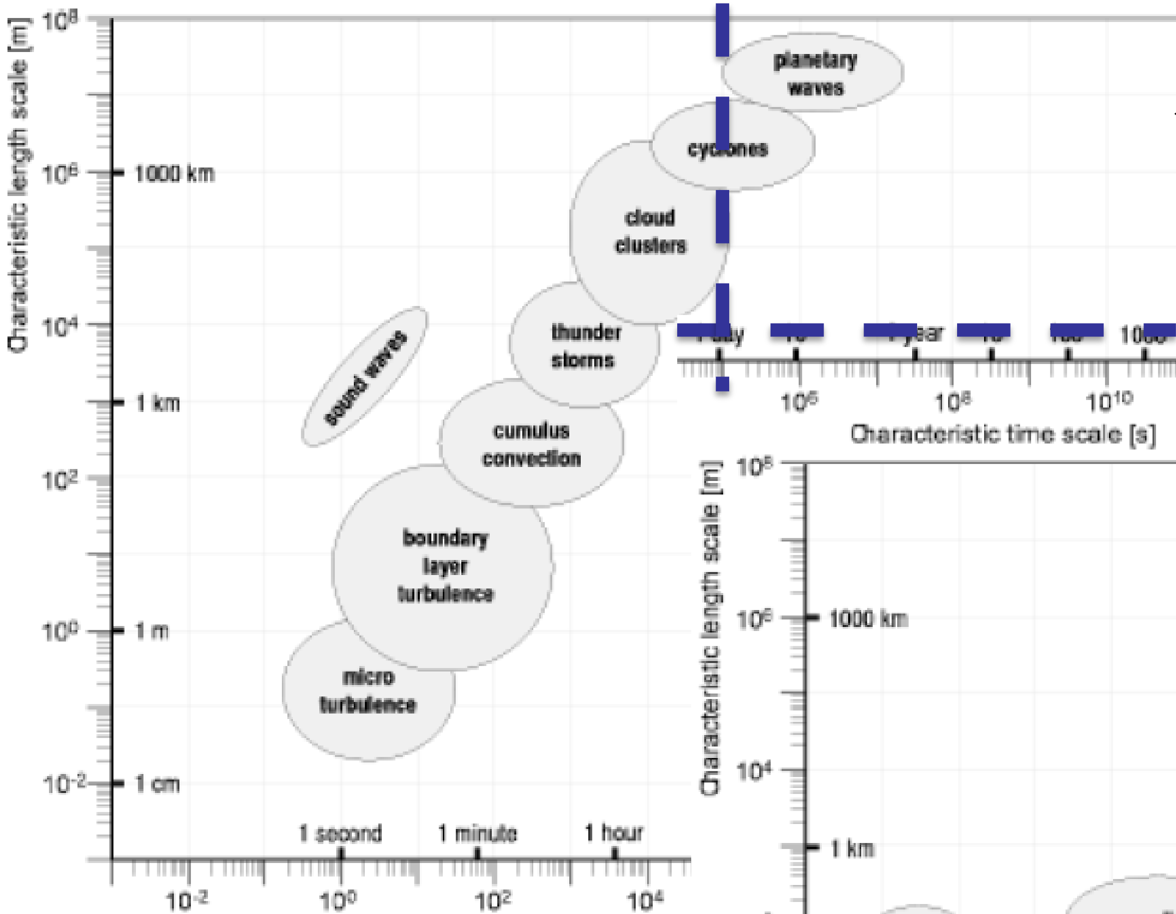
OC. $\rho_0 = 1028\ kg/m^3$, $\Delta\rho = 2\ kg/m^3$, $H = 1000\ m$ } $L \sim 60\ km$, $U \sim 4\ m/s$

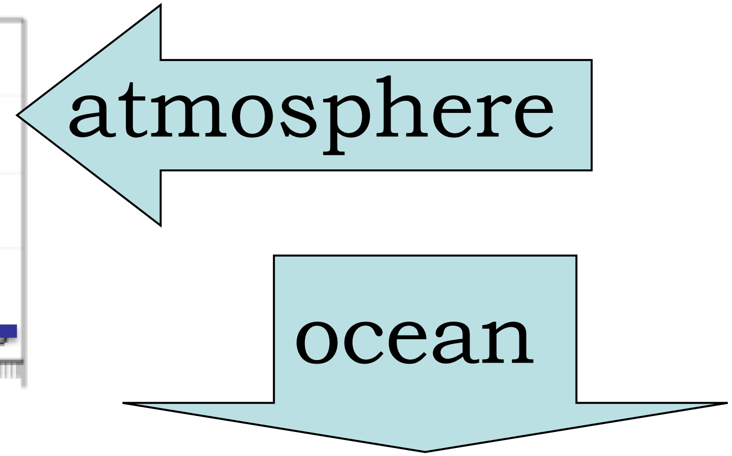
- These are «natural scales»: for **ATM**, range of [L,U] is typical for wind in weather patterns in lower troposphere, for **OC**, range of [L,U] is typical for currents in upper layers, with $T \sim T_R$
- => MESOSCALE ATM ≠ OC => implications for numerical models

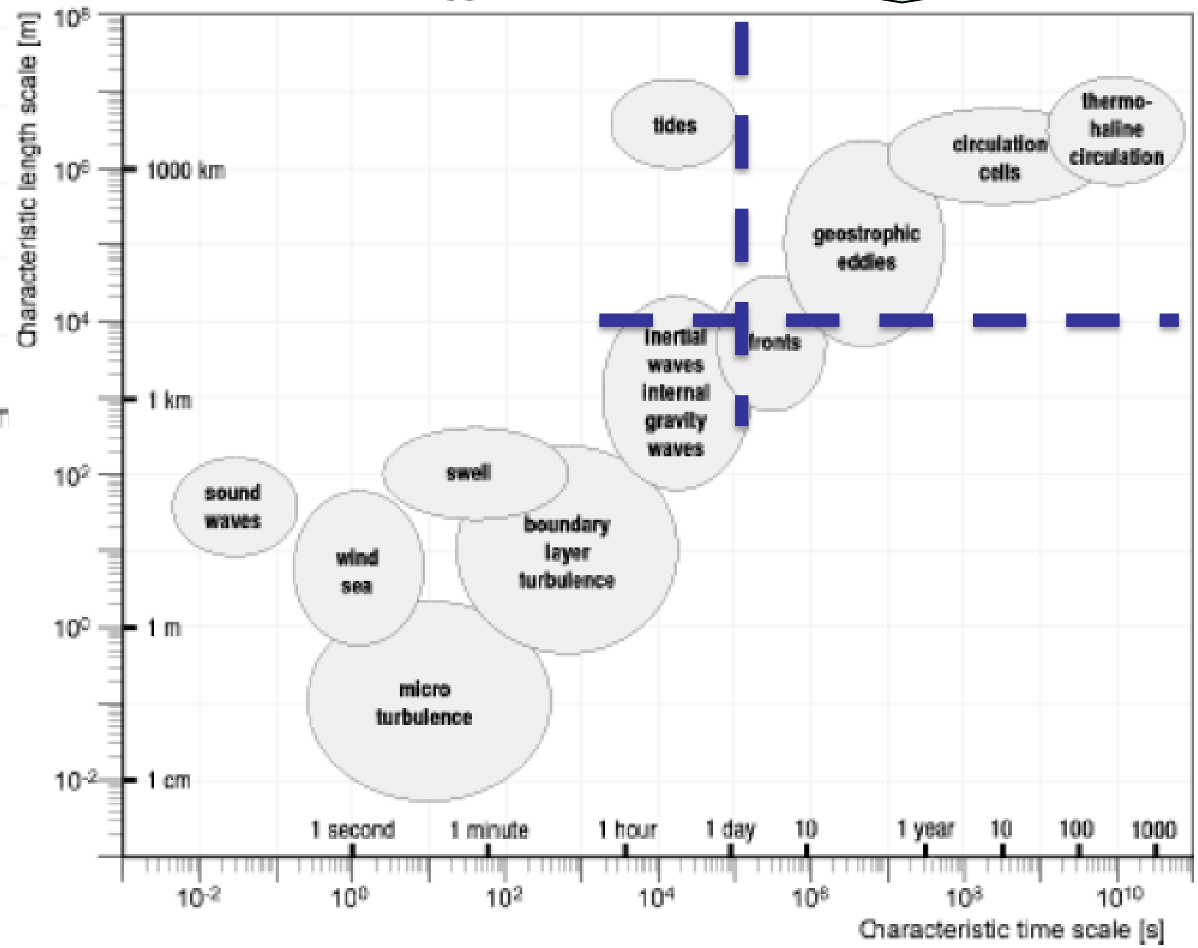
Table 1.2 LENGTH, VELOCITY AND TIME SCALES IN THE EARTH'S ATMOSPHERE AND OCEANS

Phenomenon	Length Scale L	Velocity Scale U	Time Scale T
Atmosphere:			
Microturbulence	10–100 cm	5–50 cm/s	few seconds
Thunderstorms	few km	1–10 m/s	few hours
Sea breeze	5–50 km	1–10 m/s	6 hours
Tornado	10–500 m	30–100 m/s	10–60 minutes
Hurricane	300–500 km	30–60 m/s	Days to weeks
Mountain waves	10–100 km	1–20 m/s	Days
Weather patterns	100–5000 km	1–50 m/s	Days to weeks
Prevailing winds	Global	5–50 m/s	Seasons to years
Climatic variations	Global	1–50 m/s	Decades and beyond
Ocean:			
Microturbulence	1–100 cm	1–10 cm/s	10–100 s
Internal waves	1–20 km	0.05–0.5 m/s	Minutes to hours
Tides	Basin scale	1–100 m/s	Hours
Coastal upwelling	1–10 km	0.1–1 m/s	Several days
Fronts	1–20 km	0.5–5 m/s	Few days
Eddies	5–100 km	0.1–1 m/s	Days to weeks
Major currents	50–500 km	0.5–2 m/s	Weeks to seasons
Large-scale gyres	Basin scale	0.01–0.1 m/s	Decades and beyond

GFD







Various types of processes and structures in the atmosphere and oceans, ranked according to their respective L and T. (courtesy of Hans von Storch; GFD2)

ATMOSPHERE vs OCEAN

- Many ocean processes are caused by lateral boundaries, that do not exist in atmosphere
- Atmospheric motions are strongly dependent on moisture content, that does not exist in ocean (but ocean has salinity)
- Atm is thermodynamically driven: the Solar radiation is the main source of energy
- Oc is forced by different mechanisms: tides, wind stress, surface fluxes

ATMOSPHERE vs OCEAN

A note on wind/current direction:

- *In meteorology, winds «come from» a direction (UPSTREAM), e.g. easterly wind = from the East*
- *In oceanography, currents «go toward» a direction (DOWNSTREAM), e.g. eastward current = to the East*

THE CORIOLIS FORCE

$\Rightarrow$ **Navier-Stokes equations for FD are written for an inertial system of reference (I)**

$$\left.\frac{d\bar{u}}{dt}\right|_I = -\frac{1}{\rho}\nabla p + \nabla\phi + \frac{1}{\rho}\bar{F}$$

parcel acceleration in I = pressure grad. + body forces + surface forces

- The N.-S. eqs must be re-written in a rotating frame of reference (**R**), since we observe motion on Earth
- HOW observers in **R** and **I** would describe the motion of a vector $\vec{a}$

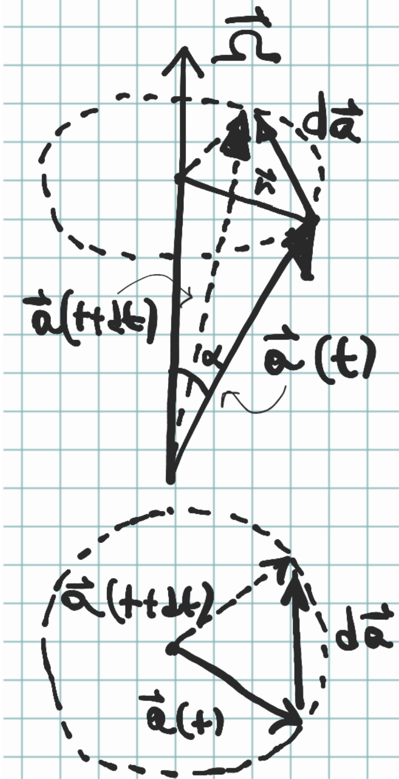
THE CORIOLIS FORCE

⇒ **Navier-Stokes equations for FD are written for an inertial system of reference (I)**

$$\left.\frac{d\bar{u}}{dt}\right|_I = -\frac{1}{\rho}\nabla p + \nabla\phi + \frac{1}{\rho}\bar{F}$$

parcel acceleration in I = pressure grad. + body forces + surface forces

- The N.-S. eqs must be re-written in a rotating frame of reference (**R**), since we observe motion on Earth
- HOW observers in **R** and **I** would describe the motion of a vector $\vec{a}$



1) IF $\vec{a}(t)$ is FIXED

$$\left.\frac{d\vec{a}}{dt}\right|_R = 0$$

⇒ obs in R does not seen a variation

2) in I, $\vec{a}$ rotates with $\vec{\Omega}$:

$$|d\vec{a}| = v\,dt = \Omega r\,dt = \Omega a \sin\alpha\, dt$$

$$\Rightarrow |d\vec{a}| = |\vec{\Omega}\times\vec{a}|\,dt$$

$$\Rightarrow \left.\frac{d\vec{a}}{dt}\right|_I = \vec{\Omega}\times\vec{a} \quad : \perp\vec{\Omega}, \perp\vec{a}$$

THE CORIOLIS FORCE

3) NOW, $\left.\frac{d\vec{a}}{dt}\right|_R \neq 0$

$$\Rightarrow \left.\frac{d\vec{a}}{dt}\right|_I = \left.\frac{d\vec{a}}{dt}\right|_R + \vec{\Omega}\times\vec{a}$$

4) $\vec{a} \rightsquigarrow \vec{u}_I$: parcel velocity in I

$$\Rightarrow \left.\frac{d\vec{u}_I}{dt}\right|_I = \left.\frac{d\vec{u}_I}{dt}\right|_R + \vec{\Omega}\times\vec{u}_I \qquad (*)$$

... but we are interested in $\left.\frac{d\vec{u}_R}{dt}\right|_R$!

(since we are obs. in R)

since $\vec{u}_I = \left.\frac{d\vec{r}}{dt}\right|_I$ and $\vec{u}_R = \left.\frac{d\vec{r}}{dt}\right|_R$

$$\Rightarrow \vec{u}_I = \vec{u}_R + \vec{\Omega}\times\vec{r}$$

$$\Rightarrow \left.\frac{d\vec{u}_I}{dt}\right|_R = \left.\frac{d\vec{u}_R}{dt}\right|_R + \left.\frac{d}{dt}(\vec{\Omega}\times\vec{r})\right|_R$$

THE CORIOLIS FORCE

3) NOW, $\left.\frac{d\vec{a}}{dt}\right|_R \neq 0$

$$\Rightarrow \left.\frac{d\vec{a}}{dt}\right|_I = \left.\frac{d\vec{a}}{dt}\right|_R + \vec{\Omega}\times\vec{a}$$

4) $\vec{a} \rightsquigarrow \vec{u}_I$: parcel velocity in I

$$\Rightarrow \left.\frac{d\vec{u}_I}{dt}\right|_I = \left.\frac{d\vec{u}_I}{dt}\right|_R + \vec{\Omega}\times\vec{u}_I \qquad (\star)$$

... but we are interested in $\left.\frac{d\vec{u}_R}{dt}\right|_R$! (since we are obs. in R)

since $\vec{u}_I = \left.\frac{d\vec{r}}{dt}\right|_I$ and $\vec{u}_R = \left.\frac{d\vec{r}}{dt}\right|_R$

$$\Rightarrow \vec{u}_I = \vec{u}_R + \vec{\Omega}\times\vec{r}$$

$$\Rightarrow \left.\frac{d\vec{u}_I}{dt}\right|_R = \left.\frac{d\vec{u}_R}{dt}\right|_R + \left.\frac{d}{dt}(\vec{\Omega}\times\vec{r})\right|_R$$

$$\Rightarrow \left.\frac{d\vec{u}_I}{dt}\right|_R = \left.\frac{d\vec{u}_R}{dt}\right|_R + \underbrace{\left.\frac{d\vec{\Omega}}{dt}\right|_R}_{=0\ \ \vec{\Omega}=\text{cst.}}\times\vec{r} + \vec{\Omega}\times\underbrace{\left.\frac{d\vec{r}}{dt}\right|_R}_{\vec{u}_R}$$

$$= \left.\frac{d\vec{u}_R}{dt}\right|_R + \vec{\Omega}\times\vec{u}_R$$

in R!

in I!

$$\left.\frac{d\vec{u}_I}{dt}\right|_I \overset{(\star)}{=} \underbrace{\left.\frac{d\vec{u}_R}{dt}\right|_R + \vec{\Omega}\times\vec{u}_R}_{\left.\frac{d\vec{u}_I}{dt}\right|_R} + \underbrace{\vec{\Omega}\times(\vec{u}_R + \vec{\Omega}\times\vec{r})}_{\vec{\Omega}\times\vec{u}_I}$$

$$= \left.\frac{d\vec{u}_R}{dt}\right|_R + 2\vec{\Omega}\times\vec{u}_R + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$2\vec{\Omega}\times\vec{u}_R$: CORIOLIS FORCE

$\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$: CENTRIFUGAL FORCE ~ body force

THE CORIOLIS FORCE

⇒ **Navier-Stokes equations for FD are written for inertial system of reference (I)**

$$\left.\frac{d\bar{u}}{dt}\right|_I = -\frac{1}{\rho}\nabla p + \nabla\phi + \frac{1}{\rho}\bar{F}$$

$$\frac{d\bar{u}}{dt} + 2\bar{\Omega}x\bar{u} = -\frac{1}{\rho}\nabla p + \nabla\hat{\phi} + \frac{1}{\rho}\bar{F}$$

…including the centrifugal force in $\nabla\phi$ and neglecting |R

The centrifugal force acts as an outward pull whereas the Coriolis force depends on velocity

$$\Rightarrow \left.\frac{d\bar{u}_I}{dt}\right|_R = \left.\frac{d\bar{u}_R}{dt}\right|_R + \underbrace{\left.\frac{d\bar{\Omega}}{dt}\right|_R}_{=0\ \ \bar{\Omega}=\text{cst.}} \times \vec{r} + \vec{\Omega} \times \underbrace{\left.\frac{d\vec{r}}{dt}\right|_R}_{\vec{u}_R}$$

in R!

$$= \left.\frac{d\vec{u}_R}{dt}\right|_R + \bar{\Omega} \times \vec{u}_R$$

in I!

$$\left.\frac{d\bar{u}_I}{dt}\right|_I \overset{(*)}{=} \underbrace{\left.\frac{d\bar{u}_R}{dt}\right|_R + \bar{\Omega} \times \bar{u}_R}_{\left.\frac{d\bar{u}_I}{dt}\right|_R} + \underbrace{\bar{\Omega} \times (\bar{u}_R + \bar{\Omega} \times \bar{r})}_{\vec{\Omega} \times \bar{u}_I}$$

$$= \left.\frac{d\bar{u}_R}{dt}\right|_R + \underbrace{2\bar{\Omega} \times \bar{u}_R}_{\text{CORIOLIS FORCE}} + \underbrace{\bar{\Omega} \times (\bar{\Omega} \times \bar{r})}_{\text{CENTRIFUGAL FORCE}}$$

CENTRIFUGAL FORCE ~ body force

THE CORIOLIS FORCE

⇒ **Coriolis Force depends on $\bar{u}$ and $\overline{\Omega}$: it is $\neq 0$ only for $\bar{u} \neq 0$**

⇒ **Centrifugal Force depends on $\bar{r}$ and $\overline{\Omega}$: even at rest, parcels have an outward pull, balanced by gravity force**

THE CORIOLIS FORCE

⇒ Coriolis Force depends on $\bar{u}$ and $\bar{\Omega}$: it is $\neq 0$ only for $\bar{u} \neq 0$

⇒ Centrifugal Force depends on $\bar{r}$ and $\bar{\Omega}$: even at rest, parcels have an outward pull, balanced by gravity force

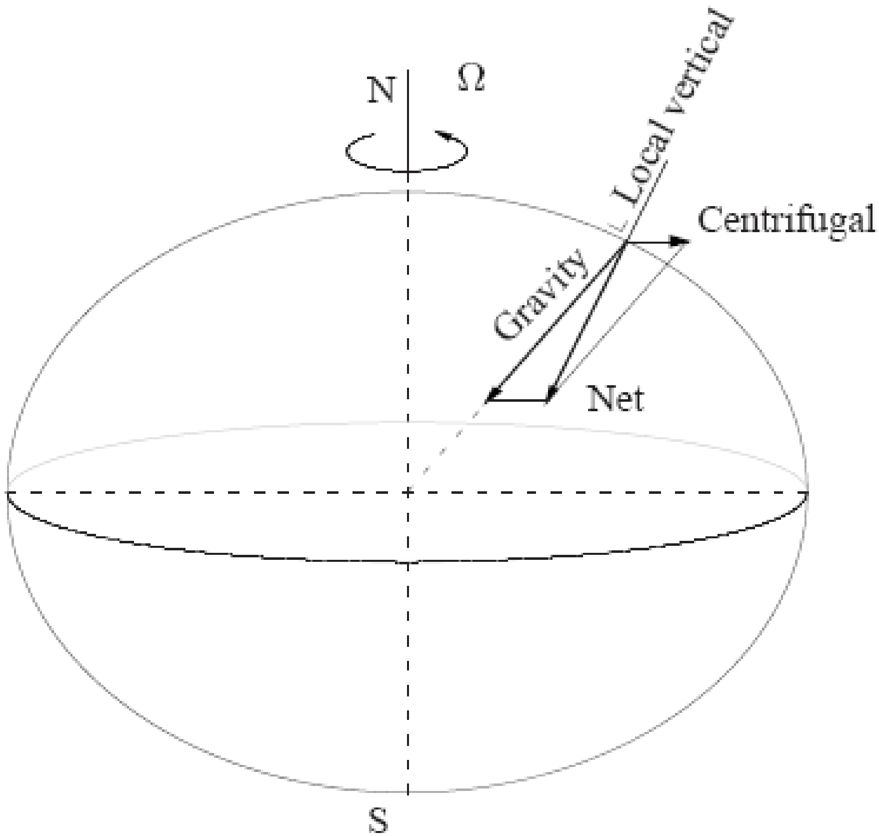
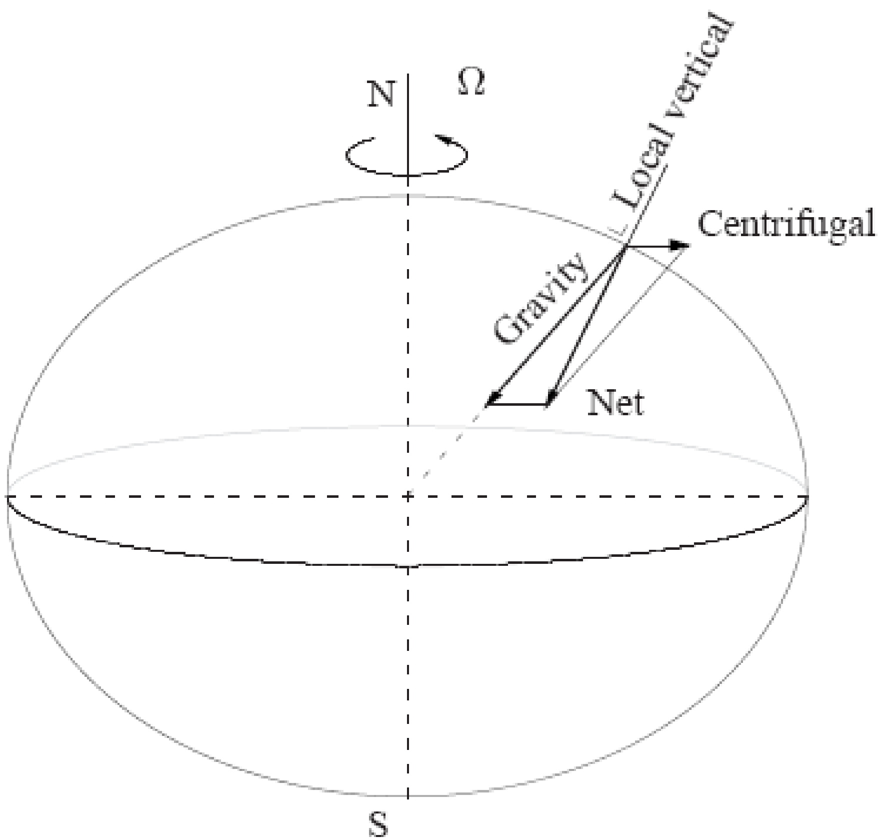


Figure 2-2 How the flattening of the rotating earth (grossly exaggerated in this drawing) causes the gravitational and centrifugal forces to combine into a net force aligned with the local vertical, so that equilibrium is reached.

THE CORIOLIS FORCE

⇒ **Coriolis Force depends on $\bar{u}$ and $\bar{\Omega}$: it is $\neq 0$ only for $\bar{u} \neq 0$**

⇒ **Centrifugal Force depends on $\bar{r}$ and $\bar{\Omega}$: even at rest, parcels have an outward pull, balanced by gravity force**



NOTE: in absence of rotation, gravitational forces keep the matter together to form a spherical body – the outward pull of the centrifugal force produces a FLATTENING, which is necessary to keep the planet in equilibrium given its rotation rate

$$\frac{R_{EQUATORIAL}}{R_{POLAR}} = \frac{6378\ km}{6357\ km} \approx 1.003$$

$$\frac{centrifugal}{gravity} = \frac{\Omega^2 R cos\varphi}{GM/R^2} \sim 1\%$$

⇒ ***We call gravitational force = NET resultant of gravity and centrifugal***

MOTION OF A FREE PARTICLE ON A ROTATING PLANE

⇒ **Effect of Coriolis force on a particle not subjected to any external force, in I: $\bar{F} = 0 \implies \frac{d\bar{u}}{dt} = 0$ (uniform motion) but in R: $\frac{d\bar{u}}{dt} + 2\bar{\Omega} x \bar{u} = 0$**

$$\frac{du}{dt} - 2\Omega v = 0 \qquad \Rightarrow \qquad u = V \sin(ft + \phi)$$

$$\frac{dv}{dt} + 2\Omega u = 0 \qquad \qquad v = V \cos(ft + \phi)$$

$f := 2\Omega$ CORIOLIS PARAMETER

V, ϕ constants of integration

$\Rightarrow \sqrt{u^2 + v^2} = V$: particle speed is constant determined by IC $(t=0)$

BUT $u = u(t)$, $v = v(t)$ $\Rightarrow$ $\exists$ change in direction : let's find the trajectory

$$\dot{x} = u, \quad \dot{y} = v \quad \Rightarrow \quad x = x_0 - \frac{V}{f}\cos(ft + \phi), \quad x_0 = x(t=0)$$

$$y = y_0 + \frac{V}{f}\sin(ft + \phi), \quad y_0 = y(t=0)$$

MOTION OF A FREE PARTICLE ON A ROTATING PLANE

⇒ **Effect of Coriolis force on a particle not subjected to any external force, in I:** $\bar{F} = 0 \implies \frac{d\bar{u}}{dt} = 0$ **(uniform motion) but in R:** $\frac{d\bar{u}}{dt} + 2\bar{\Omega}x\bar{u} = 0$

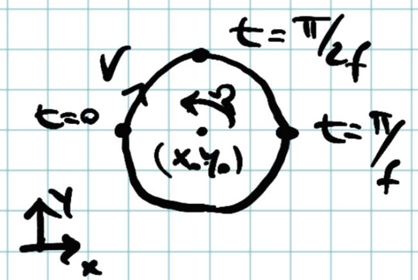
$$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = (V/f)^2$$

CIRCLE
$C(x_0, y_0)$
$R = V/|f|$

$t = 0 \quad x = x_0 - V/f \quad y = y_0$

$t = \frac{\pi}{2f} \quad x = x_0 \quad y = y_0 + V/f$

$t = \frac{\pi}{f} \quad x = x_0 + V/f \quad y = y_0$

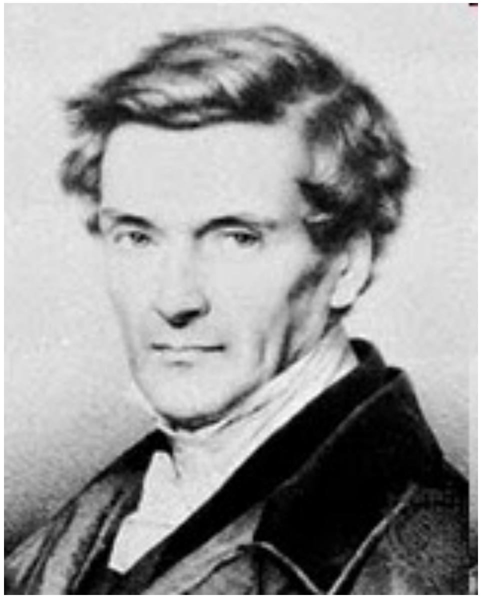


def. INERTIAL OSCILLATIONS

$T = \frac{2\pi}{f} = \frac{\pi}{\Omega}$ period of I.O. (INERTIAL PERIOD)

$T_R = \frac{2\pi}{\Omega} = 2T$ period of rotation

def. INERTIAL RADIUS $R = V/|f|$



Gaspard Gustave de Coriolis

(21 May 1792 – 19 September 1843)
French mathematician, mechanical engineer and scientist

http://en.wikipedia.org/wiki/Gaspard-Gustave_Coriolis

TIP: Look for example videos on Youtube:

Coriolis force 1

Coriolis force 2

Coriolis force 3

Foucault Pendulum

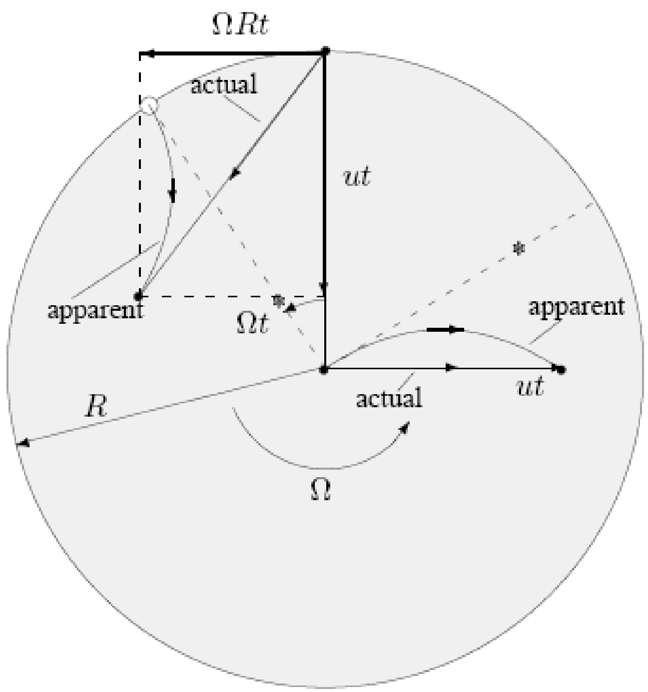
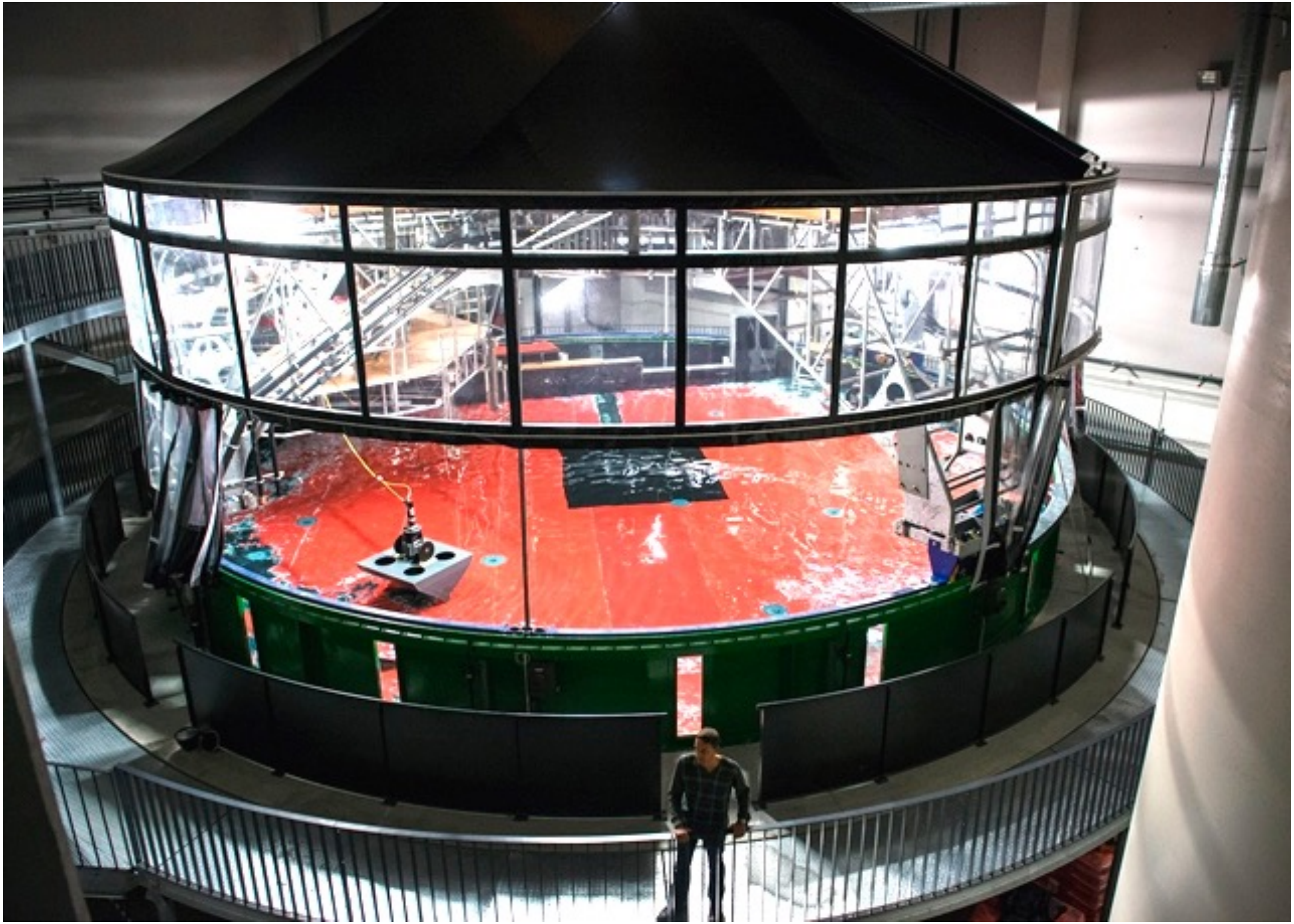


Figure 2-5 Geometrical interpretation of the apparent veering of a particle trajectory viewed in a rotating framework. The veering is to the right when the ambient rotation is counterclockwise, as shown here for two particular trajectories, one originating from the rim, the other from the axis of rotation.

The Coriolis rotating platform at LEGI (Grenoble): diameter = 13 m



http://www.legi.grenoble-inp.fr/web/spip.php?article757

MOTION ON A 3d ROTATING EARTH

⇒ **Let's consider Earth as a perfect sphere rotating around N-S axis**

⇒ **Local Cartesian frame of reference = x:E, y:N, z:up**

2.5. ROTATING PLANET

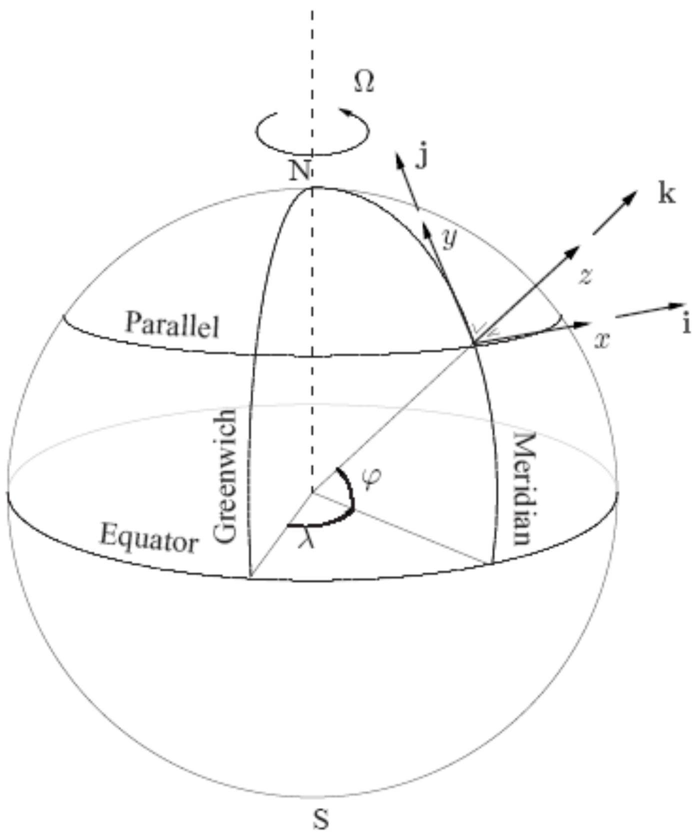


Figure 2-9 Definition of a local Cartesian framework of reference on a spherical earth. The coordinate x is directed eastward, y northward, and z upward.

$R = 6370$ km

$H_{atm} = 10$ km

$H_{oce} = 5$ km

=> H/R is order 1/1000

…like 1 mm over 1 m !

…less than an apple's peel

MOTION ON A 3d ROTATING EARTH

⇒ **Let's consider Earth as a perfect sphere rotating around N-S axis**

⇒ **Local Cartesian frame of reference = x:E, y:N, z:up**

⇒ Earth's rotation vector : $\overline{\Omega} = \Omega cos\varphi\hat{j} + \Omega sin\varphi\hat{k}$

$$\frac{d\bar{u}}{dt} + 2\overline{\Omega} x \bar{u} = \begin{cases} x: \dfrac{du}{dt} + 2\Omega cos\varphi w - 2\Omega sin\varphi v \\ y: \dfrac{dv}{dt} + 2\Omega sin\varphi u \\ z: \dfrac{dw}{dt} - 2\Omega cos\varphi u \end{cases}$$

- CORIOLIS PARAMETER: $f = 2\Omega sin\varphi$
- RECIPROCAL CORIOLIS PARAMETER: $f^* = 2\Omega cos\varphi$

$$\frac{du}{dt} - 2\Omega v = 0$$
$$\frac{dv}{dt} + 2\Omega u = 0$$

⇒

$$u = V \sin(ft + \phi)$$
$$v = V \cos(ft + \phi)$$

INERTIAL PERIOD is now:

$$T = \frac{2\pi}{f} = \frac{\pi}{\Omega sin\varphi}$$

12 hr [poles] $\leq T \leq \infty$ [eq]

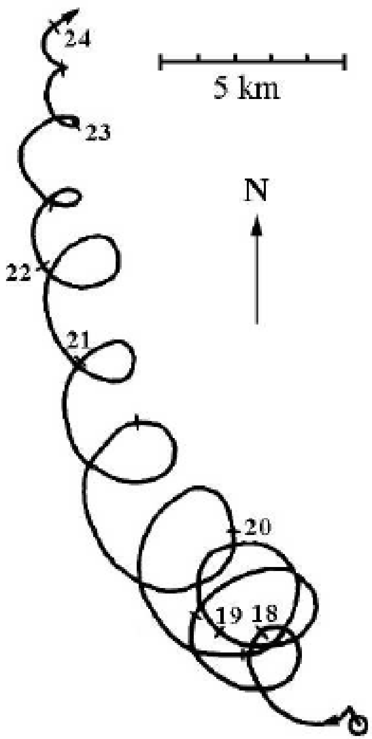


Figure 2-10 Evidence of inertial oscillations in the Baltic Sea, as reported by Gustafson and Kullenberg (1936). The plot is a progressive–vector diagram constructed by the successive addition of velocity measurements at a fixed location. For weak or uniform velocities, such a curve approximates the trajectory that a particle starting at the point of observation would have followed during the period of observation. Numbers indicate days of the month. Note the persistent veering to the right, at a period of about 14 hours, which is the value of $2\pi/f$ at that latitude (57.8°N). [From Gustafson and Kullenberg, 1936, as adapted by Gill, 1982]

THE GOVERNING EQUATIONS

⇒ **Equations governing the movement of a stratified fluid in a rotating environment**

1. MOMENTUM EQUATIONS:

- **Coming from N.-S. eqs. in R:** $\rho\left(\frac{d\bar{u}}{dt} + 2\bar{\Omega} x \bar{u}\right) = -\nabla p + \nabla \cdot \tau_{ij} - \rho g \hat{k}$
- where $\tau_{ij} = -\frac{2}{3}\mu\delta_{ij}\nabla \cdot \bar{u} + 2\mu S_{ij}$ with $S_{ij} = (\partial_i u_j + \partial_j u_i)$

$$x:\quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z}$$

$$y:\quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z}$$

$$z:\quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z} - \rho g$$

- where $f = 2\Omega sin\varphi$ and $f_* = 2\Omega cos\varphi$ and ρg is grav. force + centrif. force

THE GOVERNING EQUATIONS

⇒ **The acceleration in fluids is measured NOT as $\frac{\Delta v}{\Delta t}$ at a fixed position but as $\frac{\Delta v}{\Delta t}$ of a parcel as it moves with the flow**

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{u}\cdot\nabla = \text{LOCAL TIME RATE OF CHANGE} + \text{ADVECTIVE TERMS DUE TO FLOW MOTION}$$

NOTE: in the curvilinear coordination system <u>curvature terms</u> enter in equations:

$$x: \quad -\frac{uv\tan\varphi}{r} + \frac{uw}{r}$$

$$y: \quad \frac{u^2\tan\varphi}{r} + \frac{vw}{r}$$

$$z: \quad -\frac{u^2+v^2}{r}$$

Since GFD restricts to motions on LARGE SCALES $L \sim 1000\ km$ but $L \ll r$ where r is the Earth radius ⇒ in 1st approx. we can neglect the curvature terms ⇒ similar to consider a portion of sphere as a plane

✓ **WE HAVE 5 UNKNOWNS u, v, w, ρ, p IN 3 EQUATIONS**

THE GOVERNING EQUATIONS

2. MASS CONSERVATION EQUATION:

- **The imbalance between convergence and divergence in xyz must be translated into a local compression or expansion of fluid**

⇒ IF sink / spring of mass exists in a given volume, mass is transported to / from the external of the volume

THE GOVERNING EQUATIONS

2. MASS CONSERVATION EQUATION:

- **The imbalance between convergence and divergence in xyz must be translated into a local compression or expansion of fluid**

⇒ IF sink / spring of mass exists in a given volume, mass is transported to / from the external of the volume

WE KNOW THAT "$\partial_t C = -\nabla (\text{FLUX of } C)$"

$$\Rightarrow \partial_t \rho = -\nabla (\rho \bar{u}) \quad \text{when } C = \rho = \frac{m}{V}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \bar{u} \cdot \nabla \rho}_{} + \rho \nabla \cdot \bar{u} = 0$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \bar{u} = 0 \Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \bar{u}$$

$\nabla \cdot \bar{u} > 0$: DIVERGENCE of flow ⇒ DECREASE of mass

$\nabla \cdot \bar{u} < 0$: CONVERGENCE of flow ⇒ INCREASE of mass

THE GOVERNING EQUATIONS

3. ENERGY EQUATION:

- **INTERNAL ENERGY OF A FLUID PARCEL**
- **The energy budget can be simplified for most GFD applications where ΔT and $\Delta S \ll 1$ (GFD flows do no have internal heat sources)**
- **<u>1st Law of Thermodynamics</u>**: internal energy gained by a parcel ΔU is equal to the heat received Q minus the mechanical work performed $\Delta W \Rightarrow$ $\Delta U = Q - \Delta W$ (written per units of mass/time: $[\Delta U] = Watt/kg$)
- **Internal energy**: $\Delta U = \frac{de}{dt}$ where $e = C_V T$ internal energy per mass
  - C_V <u>heat capacity</u> at constant volume: $\begin{cases} AIR\text{: } C_V = 718 \frac{J}{kg \cdot K} \\ SEAWATER\text{: } C_V = 3990 \frac{J}{kg \cdot K} \end{cases}$
- **Rate of heat gained per unit mass**: Q result of heat diffusion: $\rho Q = k \nabla^2 T$ following Fourier's law of diffusion and k <u>thermal conductivity</u> of the fluid $[k] = Watt/m \cdot K$ (local heat flux density $\bar{q} = -k \nabla T$ and $\rho Q = -\nabla \cdot \bar{q}$)
- **Mechanical work**: $\Delta W = p \frac{dv}{dt}$ where $v = {}^1/_\rho$ specific volume

THE GOVERNING EQUATIONS

3. ENERGY EQUATION:

- $\Delta U = Q - \Delta W \;\Rightarrow\; \frac{de}{dt} = \frac{k}{\rho}\nabla^2 T - p\frac{dv}{dt}$
- $\frac{dv}{dt} = \frac{d\left({}^{1}\!/\!{}_{\rho}\right)}{dt} = -\frac{1}{\rho^2}\frac{d\rho}{dt} \Rightarrow \rho C_V\frac{dT}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = k\nabla^2 T$
- from continuity eq.: $-\frac{1}{\rho}\frac{d\rho}{dt} = \nabla\cdot\bar{u} \Rightarrow \boxed{\rho C_V\frac{dT}{dt} + p\nabla\cdot\bar{u} = k\nabla^2 T}$

✓ **WE HAVE NOW 5+1 UNKNOWNS u, v, w, ρ, p, T IN 5 EQUATIONS**

✓ **So we need another equation…**

THE GOVERNING EQUATIONS

4. EQUATION OF STATE:

- **Every fluid has density as function of pressure and temperature:** $\boldsymbol{\rho = \rho(p,T) \rightarrow AIR \neq WATER}$

- **DRY AIR in atmosphere behaves as an ideal gas:** $\boldsymbol{\rho = \frac{P}{RT}}$

$$R = 287\,\frac{m^2}{s^2K} = C_P - C_V \; with \; \begin{cases} C_P = 1005 \; {m^2}/{s^2K} \\ C_V = 718 \; {m^2}/{s^2K} \end{cases}$$

C_P , C_V heat capacity at constant pressure / volume

- **MOIST AIR:** $\boldsymbol{\rho = \frac{P}{RT(1+0.608q)}}$ where $\boldsymbol{q}$ is **humidity**

✓ **WE HAVE NOW 5+2 UNKNOWNS u, v, w, ρ, p, T, q IN 6 EQUATIONS**

✓ **So we need another equation for humidity:** $\boldsymbol{\frac{dq}{dt} = k_q \nabla^2 q}$ **diffusion**

THE GOVERNING EQUATIONS

4. EQUATION OF STATE:

- **WATER ~ incompressible:** $\frac{\partial \rho}{\partial p} = 0 \rightarrow$ **SEAWATER:** $\rho = \rho(T, S)$
- In 1st approx. linear: $\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)]$
  - $\rho_0 = 1028\, kg/m^3 \ldots T_0 = 10°C \ldots S_0 = 35\, psu$ reference values
  - $\alpha = 1.7\times10^{-4}\, K^{-1} \rightarrow$ coefficient of thermal expansion
  - $\beta = 7.6\times10^{-4} \rightarrow$ coefficient of saline contraction
- seawater has **salinity**: grams of salt over 1 kg of water

✓ **WE HAVE NOW 5+2 UNKNOWNS u, v, w, ρ, p, T, S IN 6 EQUATIONS**

✓ **So we need another equation:** $\frac{dS}{dt} = k_S \nabla^2 S$ **diffusion**

- **Seawater parcels conserve their salt content and we can neglect salt sources**

SEAWATER

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{3.1}$$

$$x: \quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z} \tag{3.2a}$$

$$y: \quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z} \tag{3.2b}$$

$$z: \quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}, \tag{3.2c}$$

$$\rho = \rho_0\,[1 - \alpha(T - T_0) + \beta(S - S_0)], \tag{3.4}$$

$$\rho C_v \frac{dT}{dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k_T\,\nabla^2 T. \tag{3.10}$$

$$\frac{dS}{dt} = \kappa_S\,\nabla^2 S, \tag{3.14}$$

AIR

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \quad (3.1)$$

$$x: \quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z} \quad (3.2a)$$

$$y: \quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z} \quad (3.2b)$$

$$z: \quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}, \quad (3.2c)$$

$$\rho = \frac{p}{RT}, \quad (3.5)$$

$$\rho C_v \frac{dT}{dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k_T \nabla^2 T. \quad (3.10)$$

$$\frac{dq}{dt} = \kappa_q \nabla^2 q, \quad (3.15)$$

$$\boxed{\theta = T\left(\frac{\rho_0}{\rho}\right)^{R/C_v}}$$

$$\rho C_v \frac{d\theta}{dt} = k_T \frac{\theta}{T} \nabla^2 T.$$

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on analysis of the phenomena and the processes we want to study…

- **Approx based on considerations on density**
- **Approx based on Newtonian fluid hypothesis**
- **Approx based on hydrostatic balance**
- **Approx based on effects of turbulence mixing**
- **Approx based on orders of magnitude (scale analysis)**
- **Ignoring frictional forces**
- **Considering fast rotating flows**
- **Considering homogenous fluids**
- **…**

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on analysis of the phenomena and the processes we want to study…

- **Approx based on considerations on density**
  - **Variations of density in ocean and atmosphere are order of few % w.r.t. background density**
  - **Ocean:** reference density 1028 kg/m^3 → density variations are below 3-4 kg/m^3 (less than 1%), in estuaries (where salinity increase from zero to 34) reaching 2%
  - **Atmosphere:** reference density 1.2 kg/m^3 → in the troposphere density variations are around 0.05 kg/m^3 (around 5%), and these are responsible of pressure gradients and therefore wind patterns
  - **In most cases, we can consider mean density as a reference value and rewrite density field as a constant value plus a variation, dependent on space and time**
  - **=> BOUSSINESQ APPROXIMATION !**

Joseph Valentin Boussinesq

(13 March 1842 – 19 February 1929)
French mathematician and physicist

http://en.wikipedia.org/wiki/Joseph_Valentin_Boussinesq

GFD: **Density differences are sufficiently small to be neglected, except when multiplied by gravity acceleration**

$$\rho = \rho_0 + \rho'(x,y,z,t) \quad \text{where} \quad \rho' << \rho_0$$

[…] Through the Boussinesq approx* the mass conservation becomes volume conservation: $\nabla \cdot \bar{u} = 0$ *which can be expected since, with uniform density, volume is a good proxy for mass!*

*also, in GFD flows variations of density are smaller than variations of velocity

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on the analysis of the phenomena and the processes we want to study…

- **Approx based on considerations on stress tensor**
  - **NEWTONIAN FLUID hypothesis: stresses are proportional to velocity gradients =>** $\tau_{ij} = \mu(\partial_i u_j + \partial_j u_i)$ **where**
    - μ **is coefficient of dynamic viscosity**
    - $\nu = \mu/\rho_0$ **is coefficient of kinematic viscosity**
  - **Viscous terms in momentum equations become [...]** $\nabla^2 u, \nabla^2 v, \nabla^2 w$

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on the analysis of the phenomena and the processes we want to study…

- **Approx based on hydrostatic balance**
  - **in the z-momentum equation density at LHS $\rho = \rho_0 + \rho' \rightarrow \rho_0$ but at RHS $-\rho g$ accounts for the weight of fluid, causing:**
    - **increase of pressure with depth in the ocean and**
    - **decrease of pressure with height in the atmosphere**
  - **following Boussinesq approx. also pressure may be separated in 2 terms hydrostatic pressure + dynamic pressure: $p = p_0(z) + p'(x, y, z, t)$**
  - **hydrostatic balance: $\frac{dp_0}{dz} = -\rho_0 g$ expresses the variation of pressure in the vertical due only to fluid weight**
  - **Integrating from a reference height (or depth) […] :**

$$p_0(z) = P_o - \rho_0 g z$$

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on the analysis of the phenomena and the processes we want to study…

- **Following the Boussinesq approx. and the mass conservation:**
  - **Energy equation with $\nabla \cdot \bar{u} = 0$ becomes a diffusion equation for temperature:** $\rho_0 C_V \frac{dT}{dt} = k \nabla^2 T$
    - $\boldsymbol{k_T} = \frac{\boldsymbol{k}}{\rho_0 C_V}$ **is heat diffusivity → $[\boldsymbol{k_T}] = m^2/s$**
    - $\boldsymbol{k}$ **is thermal conductivity: 0.02 / 0.60 $Watt/m \cdot K$ in air/water**
  - $\frac{dT}{dt} = k_T \nabla^2 T$ **is isomorphic to salt/humidity equation […]**
  - **GFD motions are mostly turbulent: turbulence rules diffusion on large scales where efficient diffusion is accomplished by eddies or vortexes which mix properties (temperature, salt, humidity) at equal rates => EDDY DIFFUSIVITY $\boldsymbol{K \sim 10^{-2}\, m^2/s}$ ($\boldsymbol{K_{T,E}}$) used for T and S:** $\frac{d\rho'}{dt} = K \nabla^2 \rho'$

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on the analysis of the phenomena and the processes we want to study…

- **<u>Approx based on orders of magnitude (scale analysis)</u>**
  - **SCALE = a dimensional constant of dimensions identical to that of the variable and having a numerical value representative of the values of the variable**

THE GOVERNING EQUATIONS

Governing Eqs cannot be solved as they are: GFD adopts different simplifying approximations, based on the analysis of the phenomena and the processes we want to study…

- **<u>Approx based on orders of magnitude (scale analysis)</u>**
  - **SCALE = a dimensional constant of dimensions identical to that of the variable and having a numerical value representative of the values of the variable**

Table 4.1 TYPICAL SCALES OF ATMOSPHERIC AND OCEANIC FLOWS

Variable	Scale	Unit	Atmospheric value	Oceanic value
x, y	L	m	$100 \text{ km} = 10^5 \text{ m}$	$10 \text{ km} = 10^4 \text{ m}$
z	H	m	$1 \text{ km} = 10^3 \text{ m}$	$100 \text{ m} = 10^2 \text{ m}$
t	T	s	$\geq \frac{1}{2} \text{ day} \simeq 4 \times 10^4 \text{ s}$	$\geq 1 \text{ day} \simeq 9 \times 10^4 \text{ s}$
u, v	U	m/s	10 m/s	0.1 m/s
w	W	m/s		
p	P	$\text{kg}/(\text{m}\cdot\text{s}^2)$	variable	
ρ	$\Delta\rho$	kg/m^3	**1% of ρ_0**	**0.1% of ρ_0**

THE GOVERNING EQUATIONS

- Approx based on orders of magnitude (scale analysis)
  - SCALE = a dimensional constant of dimensions identical to that of the variable and having a numerical value representative of the values of the variable

Table 4.1 TYPICAL SCALES OF ATMOSPHERIC AND OCEANIC FLOWS

Variable	Scale	Unit	Atmospheric value	Oceanic value
x, y	L	m	100 km = 10^5 m	10 km = 10^4 m
z	H	m	1 km = 10^3 m	100 m = 10^2 m
t	T	s	$\geq \frac{1}{2}$ day $\simeq 4 \times 10^4$ s	≥ 1 day $\simeq 9 \times 10^4$ s
u, v	U	m/s	10 m/s	0.1 m/s
w	W	m/s		
p	P	kg/(m·s^2)	variable	
ρ	$\Delta\rho$	kg/m^3	1% of ρ_0	0.1% of ρ_0

GFD large-scale motions:

$L \gg H$ and $T \geq 1/\Omega$ [and $\Omega \geq U/L$]

THE GOVERNING EQUATIONS

- **Approx based on orders of magnitude (scale analysis)**
  - **Considerations on continuity equation…**
  - **… […]**
  - **Considerations on x,y - momentum equations…**
  - **… […]**
  - **Considerations on z - momentum equations…**
  - **… […]**
  - **Considerations on energy equation…**
  - **… […]**

THE GOVERNING EQUATIONS

- **<u>Approx based on orders of magnitude (scale analysis)</u>**
  - **Considerations on continuity equation…**
  - **… […]**
  - **Considerations on x,y - momentum equations…**
  - **… […]**
  - **Considerations on z - momentum equations…**
  - **… […]**
  - **Considerations on energy equation…**
  - **… […]**
  - **Large-scale geophysical flows tend to be fully hydrostatic even in presence of substantial motions**

Primitive Equations of GFD

x − momentum: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right) \quad (4.21a)$$

y − momentum: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right) \quad (4.21b)$$

z − momentum: $$0 = -\frac{\partial p}{\partial z} - \rho g \quad (4.21c)$$

continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4.21d)$$

energy: $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right) \quad (4.21e)$$

5 equations for 5 variables u, v, w, ρ, p

$f = 2\Omega \sin\varphi$ and ρ_0, g constant, $\nu_E(z)$, $\kappa_E(z)$

THE PRIMITIVE EQUATIONS

- **Scale analysis on x,y momentum equations:**

$$\frac{U}{T}, \frac{U^2}{L}, \frac{U^2}{L}, \frac{WU}{H}, \Omega U, \frac{P}{\rho_0 L}, \frac{\nu U}{H^2}$$

$\Rightarrow$ **Rotation (Coriolis) term ΩU is fundamental to measure the importance of the terms relative to it:**

$$\frac{1}{\Omega T}, \frac{U}{\Omega L}, \frac{U}{\Omega L}, \frac{WU}{H} \cdot \frac{1}{\Omega U} \cdot \frac{L}{L}, 1, \frac{P}{\rho_0 L \Omega U}, \frac{\nu}{\Omega H^2}$$

- def. Temporal Rossby number: $Ro_T = \frac{1}{\Omega T} = \frac{local\ velocity\ variation}{Coriolis\ term}$ ... (ω)
- def. Rossby number: $Ro = \frac{U}{\Omega L} = \frac{advection}{Coriolis\ term}$ ... (ε)
- $\frac{WU}{H} \cdot \frac{1}{\Omega U} \cdot \frac{L}{L} = \frac{WL}{UH} \cdot \frac{U}{\Omega L} = \frac{WL}{UH} \cdot Ro = \frac{vertical\ convergence/divergence}{horizontal\ convergence/divergence} \cdot Ro$
- def. Ekman number: $Ek = \frac{\nu}{\Omega H^2} = \frac{friction\ (z)}{Coriolis\ term}$

$\Rightarrow$ **GFD has** $Ro_T \lesssim 1 \ldots Ro \lesssim 1 \ldots Ek \ll 1\ far\ from\ B.L.$

- def. Reynolds number: $Re = \frac{UL}{\nu} = \frac{U}{\Omega L} \cdot \frac{\Omega H^2}{\nu} \cdot \frac{L^2}{H^2} = \frac{Ro}{Ek} \cdot \frac{L^2}{H^2} = \frac{advection}{friction\ (xy)} \gg 1$

THE PRIMITIVE EQUATIONS

- **Fluid turbulence at sub-geophysical scales (small eddies) can act as dissipative mechanism: molecular viscosity ν can be substituted by a much larger EDDY VISCOSITY ν_T or ν_E**
  - **For water: $\nu \sim 10^{-6} m^2/s$ and $\nu_T \sim 10^{-2} m^2/s$**
- **Even with eddy viscosity, Ekman number remains small ($Ek \sim 10^{-2}$) but friction becomes essential near boundary layers ($Ek \sim 1$)**

- $\frac{P}{\rho_0 L\Omega U} = \frac{pressure\ gradient}{Coriolis\ term} \Rightarrow if\ terms\ are\ comparable$: $P \sim \rho_0 L\Omega U$
- **from z-momentum:** $0 = -\frac{\partial p}{\partial z} - \rho g \Rightarrow P \sim \Delta\rho g H$

$$\frac{\Delta\rho g H}{\rho_0 L\Omega U} \cdot \frac{U}{U} = \frac{\Delta\rho g H}{\rho_0 U^2} \cdot \frac{U}{L\Omega} = Ri \cdot Ro$$

- def. Richardson number: $Ri = \frac{\Delta\rho g H}{\rho_0 U^2} = \frac{potential\ energy}{kinetic\ energy}$ ... $(1/\sigma)$

✓ *Exercise: find scales for $\rho_0 L\Omega U$, $\Delta\rho g H$, $\rho_0 g H$ in Ocean and Atmosphere using Table 4.1*