GEOSTROPHIC FLOWS

- **After inertial oscillations, homogeneous geostrophic flows are the second simple case where NSEq. can be solved, and can describe natural GFD phenomena**
- **Hypoteses:**
  - **Coriolis term dominates others (= rapidly rotating flows):** $Ro_T \ll 1 \; and \; Ro \ll 1$
  - **Homogeneous fluids:** $\rho_0 = cost \; and \; \rho' = 0$
  - **Ignore frictional effect (= far from B.L.):** $Ek \ll 1$
- **Primitive equations:**

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
+fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- **∂_z of x,y-momentum eq. [...] :** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$

=> Taylor-Proudman theorem: *"horizontal velocity field has no vertical shear and all the particles on the same vertical move in concert"*

GEOSTROPHIC FLOWS

- **Solving the x,y-momentum eqs.:**

$$u = \frac{-1}{\rho_0 f}\frac{\partial p}{\partial y}, \qquad v = \frac{+1}{\rho_0 f}\frac{\partial p}{\partial x}$$

- $(\boldsymbol{u}, \boldsymbol{v}) = \overline{\boldsymbol{u}} \perp \boldsymbol{\nabla p}$ **the flow is across-gradient (or isobaric):**
  - **NO pressure work is performed either on the fluid or by the fluid: Once initiated the flow can persist without a continuous energy source** $\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla p} = \mathbf{0}$

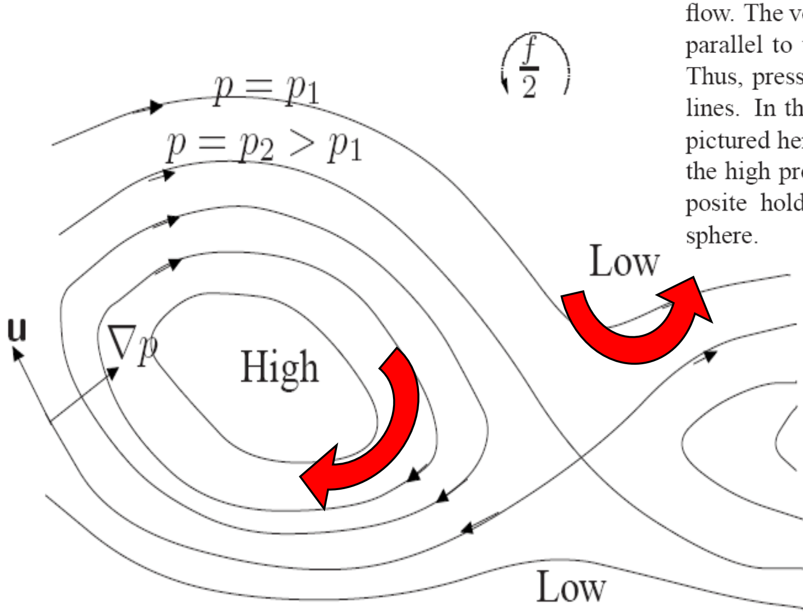
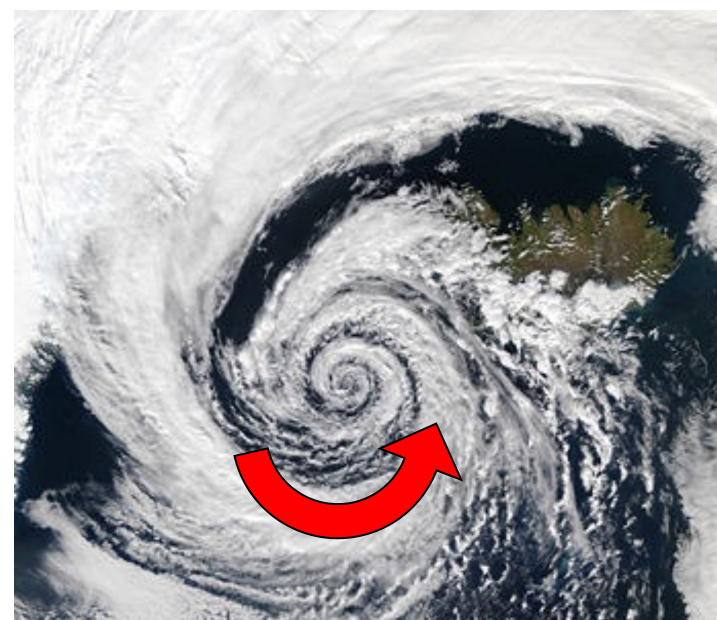


Figure 7-1 Example of geostrophic flow. The velocity vector is everywhere parallel to the lines of equal pressure. Thus, pressure contours act as streamlines. In the Northern Hemisphere (as pictured here), the fluid circulates with the high pressure on its right. The opposite holds for the Southern Hemisphere.

$f > 0$



http://it.wikipedia.org/wiki/Immagine:Low_pressure_system_over_Iceland.jpg

GEOSTROPHIC FLOWS

- **GEOSTROPHY comes from $\gamma\eta = Earth$ and $\sigma\tau\rho o\varphi\eta = turning$**
- **Balance between Coriolis force and pressure gradient**
- **All geostrophic flows are isobaric:**
  - **Northern hemisphere $f > 0$: currents flows with H on their right**
  - **Southern hemisphere $f < 0$: currents flows with H on their left**
- **IF the flow extends over a meridional span not too wide $L_y \ll L_x$: $\frac{\partial f}{\partial y} \rightarrow 0 \Rightarrow f = cost \Rightarrow f - PLANE$ and the horizontal divergence is**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right) = 0$$

- **Geostrophic flows are naturally non-divergent on the f-plane**

$\frac{\partial w}{\partial z} = 0 \Rightarrow w = cost$ and, if the fluid is bounded in the vertical by a flat surface: $w = 0$

- **Geostrophic flows are 2-dimensional**

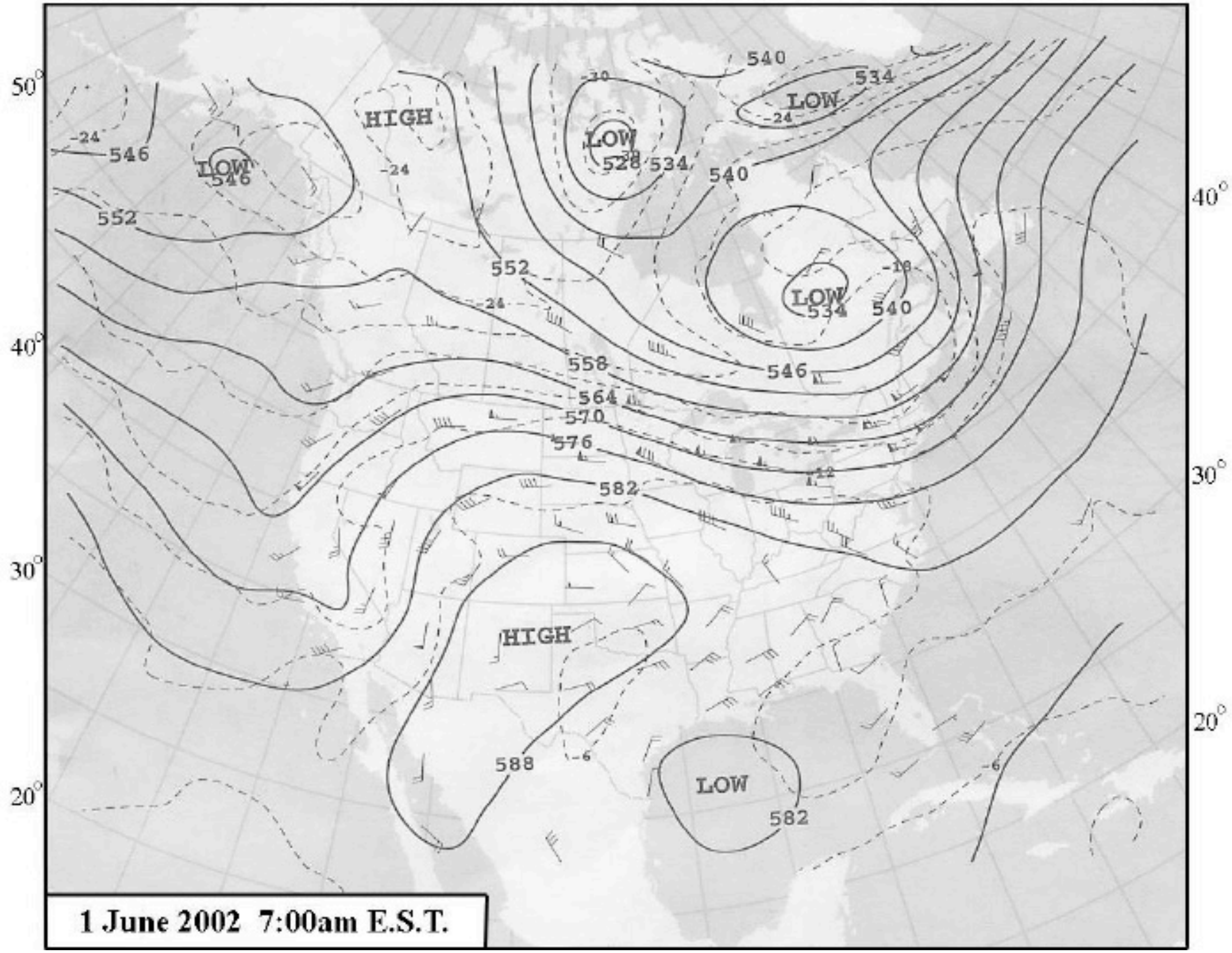


Figure 7-2 A meteorological example showing the high degree of parallelism between wind velocities and pressure contours (isobars), indicative of geostrophic balance. The solid lines are actually height contours of a given pressure (500 mb in this case) and not pressure at a given height. However, because atmospheric pressure variations are large in the vertical and weak in the horizontal, the two sets of contours are nearly identical by virtue of the hydrostatic balance. According to meteorological convention, wind vectors are depicted by arrows with flags and barbs: on each tail, a flag indicates a speed of 50 knots, a barb 10 knots and a half-barb 5 knots (1 knot = 1 nautical mile per hour = 0.5144 m/s). The wind is directed toward the bare end of the arrow, because meteorologists emphasize where the wind comes from, not where it is blowing. The dashed lines are isotherms. (*Chart by the National Weather Service, Department of Commerce, Washington, D.C.*)

A simplified schematic (top) of the AMOC. Warm water flows north in the upper ocean (red), gives up heat to the atmosphere (atmospheric flow gaining heat represented by changing color of broad arrows), sinks, and returns as a deep cold flow (blue).

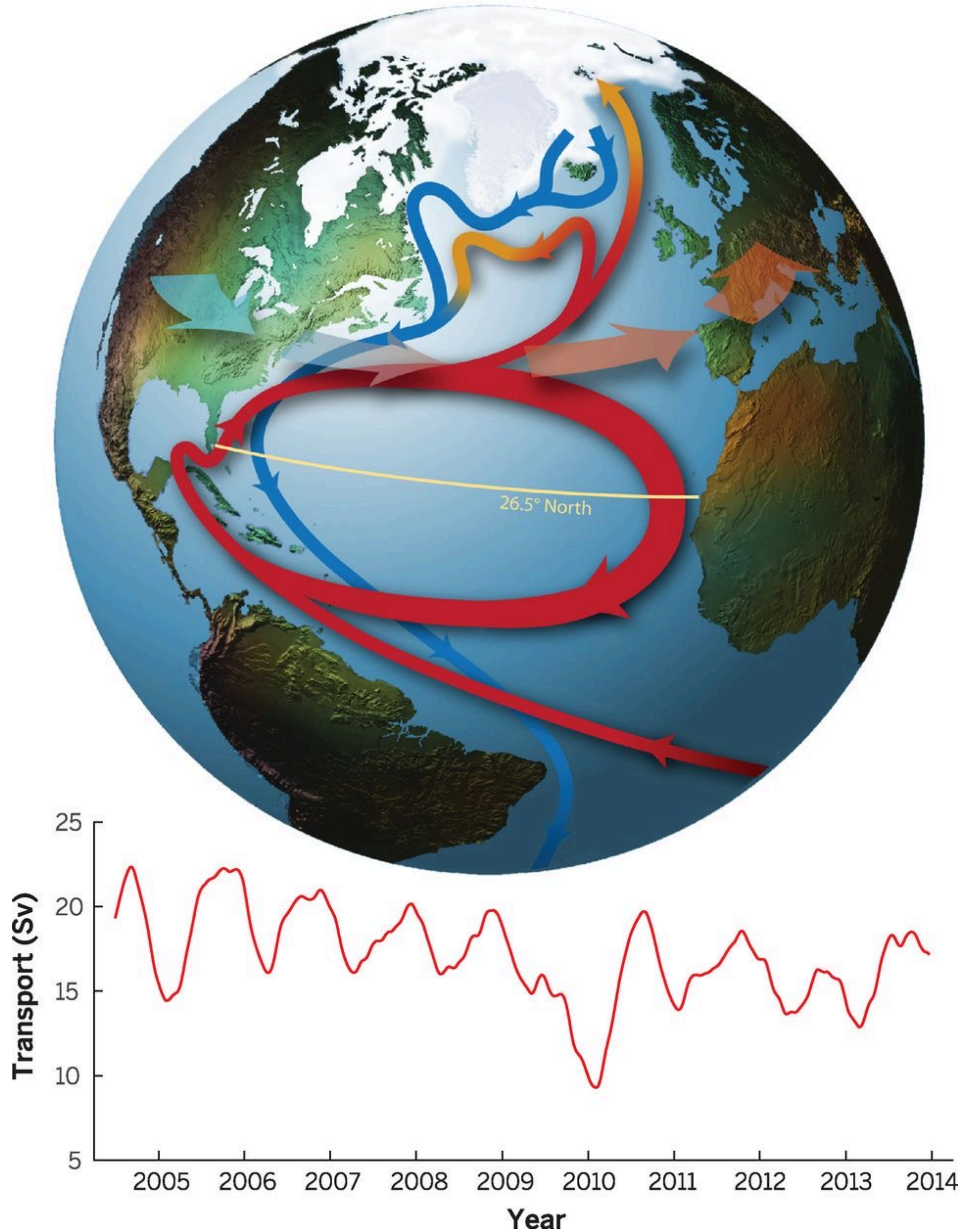
Latitude of the 26.5 N AMOC observations is indicated. The actual flow is considerably more complex.

(**Bottom**) The 10-year (April 2004 to March 2014) time series of the AMOC strength at 26.5 N in Sverdrups (1 Sv = 10^6 m^3 s^{-1}). This is the 180-day filtered version of the time series. Visible are the low AMOC event in 2009–2010 and the overall decline in AMOC strength over the 10-year period.

http://www.sciencemag.org/content/348/6241/1255575

See recent update on AMOC OMI @ CMS

and Lobelle et al. (2020), GRL
(statistical significance and projections; is slowing due to natural variability or CC?)



Published by AAAS

M. A. Srokosz, and H. L. Bryden Science 2015;348:1255575

GEOSTROPHIC FLOWS OVER IRREGULAR BOTTOM

- **Same framework, but with no-flat bottom:**

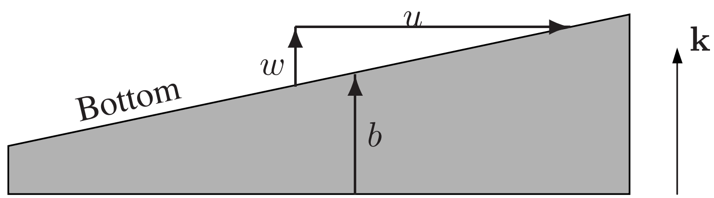


Figure 7-3 Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.

- Bottom elevation (bathymetry / topography): $b = b(x, y)$

- $w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \bar{u} \cdot \nabla z = 0 + u\,\frac{\partial b}{\partial x} + \mathrm{v}\,\frac{\partial b}{\partial y} + w\,\frac{\partial b}{\partial z} = u\,\frac{\partial b}{\partial x} + \mathrm{v}\,\frac{\partial b}{\partial y}$

- But on the f-plane $w = \text{cost}$ and since $w(z = H) = 0 \Rightarrow w = 0\ \forall z$

- $\Rightarrow \bar{\boldsymbol{u}} \cdot \nabla \boldsymbol{b} = \mathbf{0}$ **flow is directed to zones of equal depth: FREE geostrophic flows can occur only along closed isobaths**

- **ISOBARS = ISOBATHS**

- **If bumps or dips exist, the fluid can only go around them: due to vertical rigidity, fluid particles at all levels must likewise go around: TAYLOR COLUMNS are permanent tubes of fluid above bumps or dips**

GEOSTROPHIC FLOWS OVER IRREGULAR BOTTOM

- **Same framework, but with no-flat bottom [...]:**

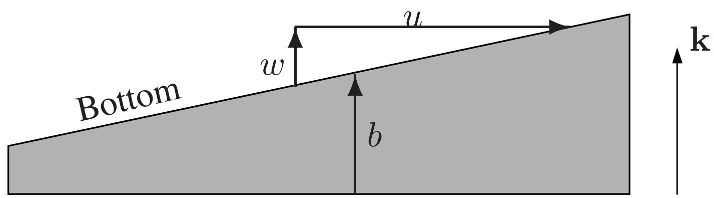


Figure 7-3 Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.

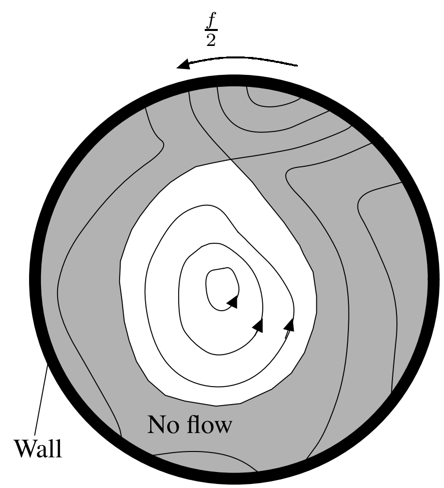


Figure 7-4 Geostrophic flow in a closed domain and over irregular topography. Solid lines are isobaths (contours of equal depth). Flow is permitted only along closed isobaths.

- **ISOBARS = ISOBATHS**
- **If bumps or dips exist, the fluid can only go around them: due to vertical rigidity, fluid particles at all levels must likewise go around: TAYLOR COLUMNS are permanent tubes of fluid above bumps or dips**

BAROTROPIC FLOWS

- **Generalization to non-geostrophic flows: "second level" of flows**
- **Hypoteses:**
  - Coriolis term DOES NOT dominate others: $Ro_T \sim 1 \text{ and } Ro \sim 1$
  - Homogeneous fluids: $\rho_0 = cost \text{ and } \rho' = 0$
  - Ignore frictional effect (SLIP is allowed [...]): $Ek \ll 1$
- **Primitive equations:**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- **IF T.-P. theorem still holds** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ **initially, it will hold also at all future time**
- **Advection, Coriolis and Pressure terms remain z-independent**

BAROTROPIC FLOWS

- **Generalization to non-geostrophic flows: "second level" of flows**
- **Hypoteses:**
  - **Coriolis term DOES NOT dominate others:** $Ro_T \sim 1$ *and* $Ro \sim 1$
  - **Homogeneous fluids:** $\rho_0 = cost$ *and* $\rho' = 0$
  - **Ignore frictional effect (SLIP is allowed [...]):** $Ek \ll 1$
- **Primitive equations:**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- **IF T.-P. theorem still holds** $\boxed{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0}$ **initially, it will hold also at all future time**
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BAROTROPIC FLOWS

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- **Hypoteses:**
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\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- **IF T.-P. theorem still holds** $\boxed{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0}$ **initially, it will hold also at all future time**
- **Advection, Coriolis and Pressure terms remain z-independent**

BAROTROPIC FLOWS

- **Although the flow has NO VERTICAL SHEAR, this remains the only similarity with geostrophic flows: Barotropic flows are not required to be aligned with isobars, neither be non-divergent on the horizontal plane, so they can develop a vertical velocity $w \neq 0$**
- **Integrating continuity eq. over the entire fluid depth [...]:**

$$\int_b^{b+h} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz = 0$$

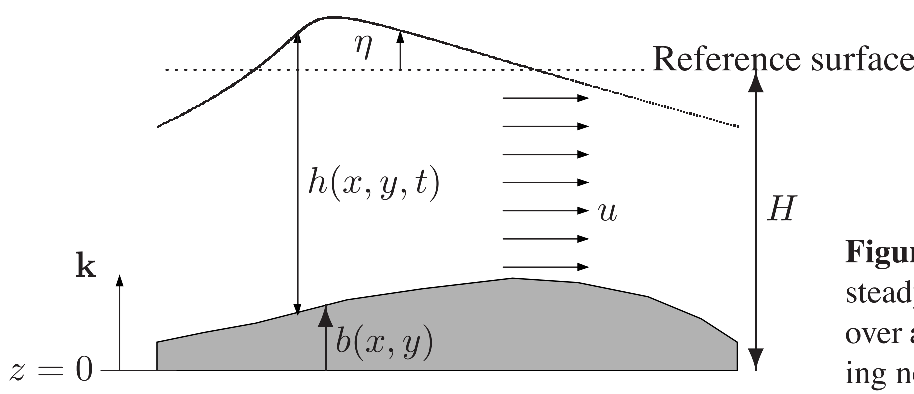


Figure 7-5 Schematic diagram of unsteady flow of a homogeneous fluid over an irregular bottom and the attending notation.

➢ **NEW CONTINUITY EQUATION:**

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

with $\eta = b + h - H$ and $\partial_t \eta = \partial_t h$

Shallow-water model

in case of flat bottom *b(x,y)=0*

3 unknowns in 3 equations

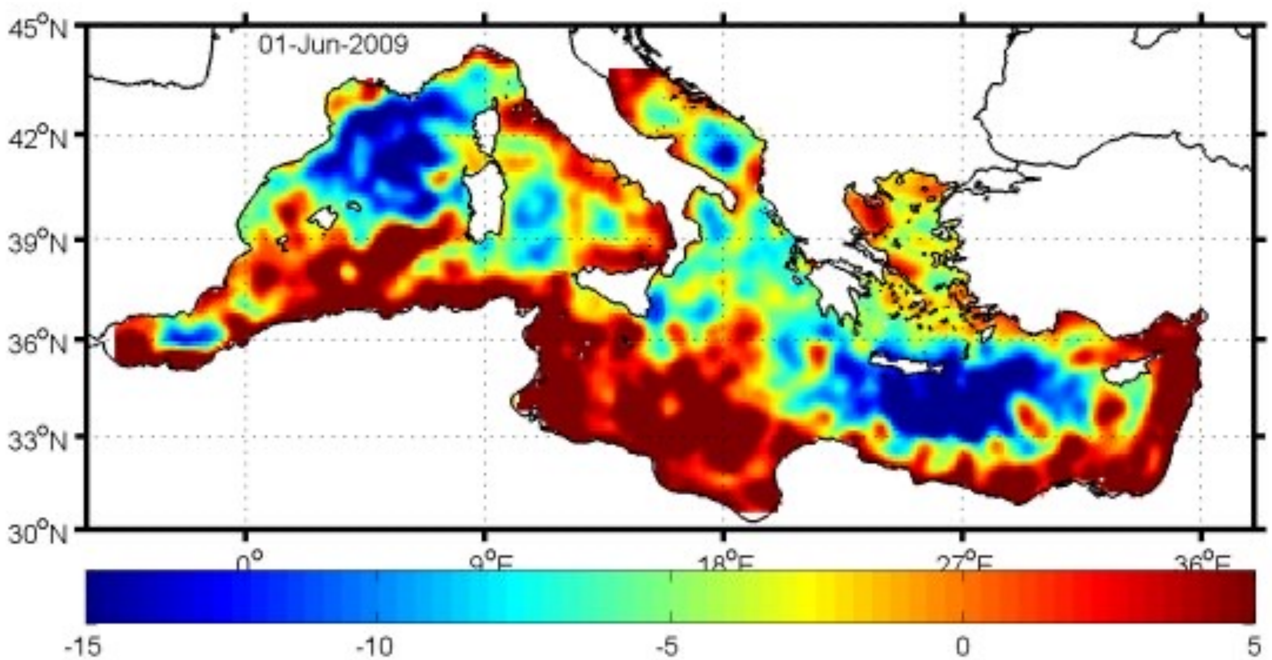
$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - fv &= -g\,\frac{\partial h}{\partial x}\\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + fu &= -g\,\frac{\partial h}{\partial y}\\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

Geostrophy and altimetry

Variation of η with *x,y* measured from satellite gives info on geostrophic currents

$$-\,fv = -g\,\frac{\partial\eta}{\partial x}$$

$$+\,fu = -g\,\frac{\partial\eta}{\partial y}$$



Example of absolute dynamic topography (in cm) of the Mediterranean Sea on 1 June 2009 using the Rio et al. (2007) synthetic mean dynamic topography. http://www.goceitaly.asi.it/GoceIT/index.php?Itemid=94

VORTICITY DYNAMICS

- **Geostrophic flows are non-divergent on the f-plane, with 2d-divergence equal to zero: let's investigate the role of the horizontal divergence in barotropic flows**
- **Subtract y-derivative of x-mom.eq from x-derivative of y-mom.eq of barotropic flow system (or the shallow-water model) […]**

- def. ambient vorticity f
- def. relative vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (the vertical component of $\nabla \times \bar{u}$)
- def. total vorticity $f + \zeta$

- $\frac{d}{dt}(f+\zeta) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \Rightarrow total\ vorticity\ ruled\ by\ horiz.div.$
- $\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \Rightarrow fluid\ column\ heigth\ ruled\ by\ horiz.div.$
- $\frac{d}{dt}(h \cdot dS) = 0 \Rightarrow parcel's\ volume\ is\ conserved\ in\ time$
- Combining the above equations... […]

VORTICITY DYNAMICS

- **Kelvin's theorem for 2-d rotating flows**: *"in barotropic flows without friction the circulation is conserved"*
- **This conservation principle has the same meaning of that of the angular momentum for an isolated system**

$$\frac{d}{dt}\left[(f + \zeta)\, ds\right] = 0$$

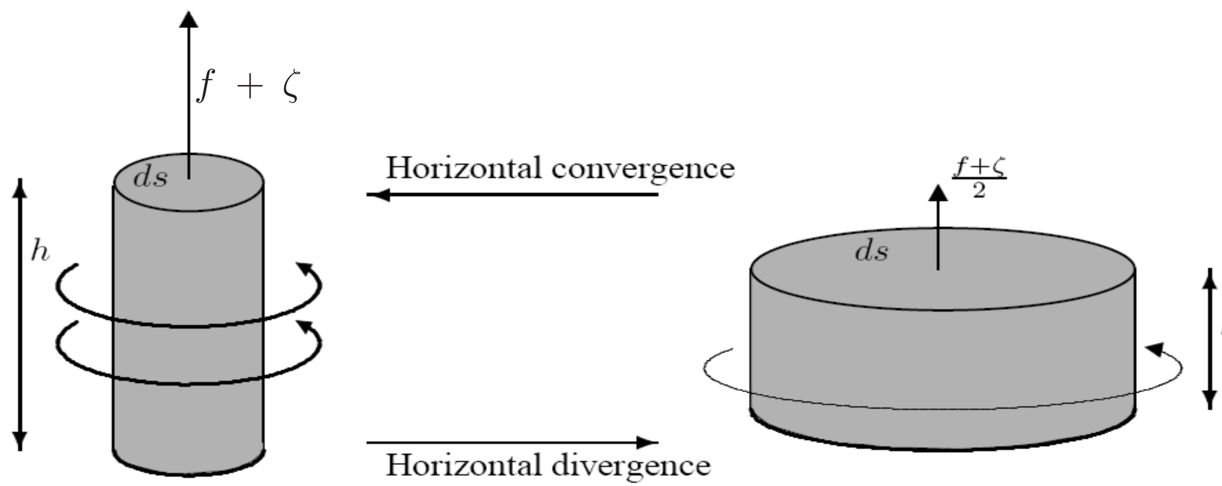
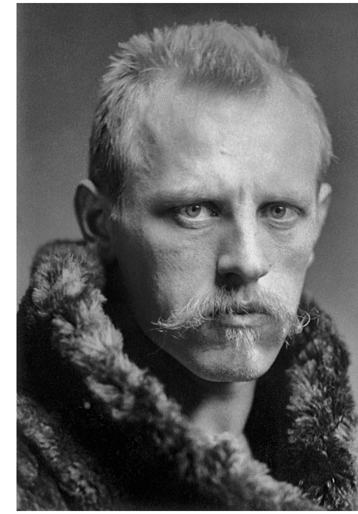


Figure 7-6 Conservation of volume and circulation of a fluid parcel undergoing vertical squeezing or stretching. The products $h\,ds$ and $(f + \zeta)\,ds$ are conserved during the transformation. As a corollary, the ratio $(f + \zeta)/h$, called the potential vorticity, is also conserved.

VORTICITY DYNAMICS

- **Kelvin's theorem for 2-d rotating flows**: *"in barotropic flows without friction the circulation is conserved"*
- **This conservation principle has the same meaning of that of the angular momentum for an isolated system**
- **IF both circulation and volume are conserved, so is their ratio, allowing to eliminate dependency on cross-section**
- $\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$ **where** $q = \frac{f+\zeta}{h} = \frac{f + \partial v/\partial x - \partial u/\partial y}{h}$
- **and q is called POTENTIAL VORTICITY, or "circulation per volume", thus obtaining the conservation of potential vorticity**
- **<u>For rapidly rotating flows</u>:** $Ro = \frac{U}{\Omega L} \ll 1 \Rightarrow f + \zeta \sim \Omega + \frac{U}{L} \sim \Omega \Rightarrow q = \frac{f}{h}$
- **and, IF $f = cost$ each fluid column must conserve its height h: and in particular, if the upper boundary is flat, fluid parcels must follow the isobaths => <u>barotropic flows become geostrophic</u>**

The Ekman models



Fridtjof Nansen (1861 –1930)

Norwegian scientist, explorer, diplomat.

Nobel Peace Prize 1922

https://en.wikipedia.org/wiki/Fridtjof_Nansen

Vagn Walfrid Ekman (1874 –1954)

Swedish oceanographer.

http://en.wikipedia.org/wiki/Vagn_Walfrid_Ekman

- Prandtl hypothesis on Boundary Layers
- $\boldsymbol{Ek} \to \mathbf{1}$ close to the wall - $\boldsymbol{Ek} \ll \mathbf{1}$ far from the wall
- study of iceberg's motion (Nansen/Fram → Bjerknes → Ekman)

The Ekman models

Fridtjof Nansen (1861 –1930)

Norwegian scientist, explorer, diplomat.

Nobel Peace Prize 1922

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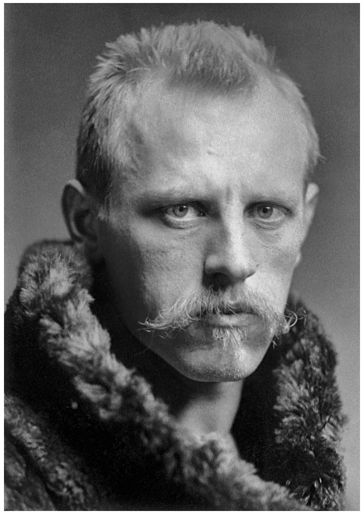
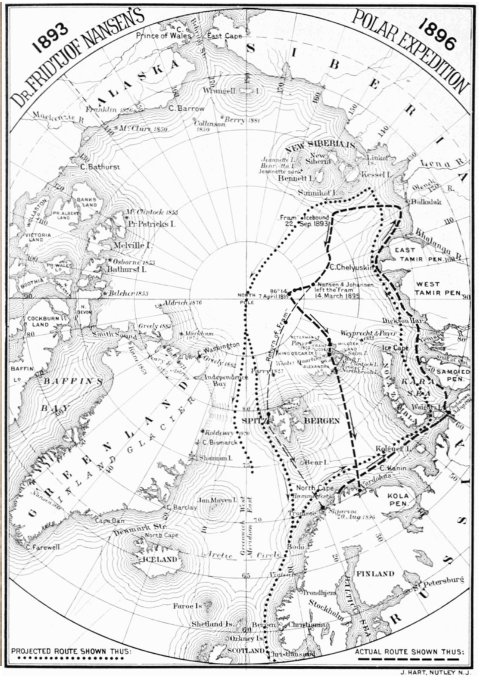




https://frammuseum.no/polar-history/vessels/the-polar-ship-fram/ (VIDEO)

https://en.wikipedia.org/wiki/Nansen%27s_Fram_expedition
https://www.youtube.com/watch?v=pLS5SJzwaas

EKMAN LAYER

- **As seen from the scale analysis of the primitive eqs. vertical friction has a very minor role in the balance of forces ($Ek \ll 1$) and may be omitted**
- **But we lost something, since the frictional terms have the highest derivative order => when $Ek \ll 1$ not all the BCs can be applied, the result is that SLIPPING ON THE BOUNDARY is allowed**
- **Prandtl hypothesis**: *the fluid has 2 distinct behaviors:*
  - *far from the boundary (INTERIOR, vertical scale* ***H****), friction can be neglected (*$Ek \ll 1$*):* $Ek = \frac{\nu_T}{\Omega H^2} \sim \frac{10^{-2} m^2/s}{10^{-4} s^{-1} \cdot (10^3 m)^2} \sim 10^{-4}$
  - *across a short distance near the boundary (BOUNDARY LAYER, vertical scale* ***d****), friction acts to bring the finite interior velocity to zero at the wall (*$Ek \sim 1$*):* $Ek = \frac{\nu_T}{\Omega d^2} \sim 1 \Rightarrow d = \sqrt{\frac{\nu_T}{\Omega}} \sim 10\, m \Rightarrow d \ll H$
- **Because of the Coriolis effect, the frictional layer of the geophysical flows, called EKMAN LAYER, greatly differs from the BL in non-rotating flows (δ), which does not have a thickness and grows downstream ($\delta \propto \sqrt{x}$)**

THE BOTTOM EKMAN LAYER

- **The bottom exerts a frictional stress against the flow bringing its interior velocity gradually to zero within a thin layer above the wall $d \ll H$**
- **Hypotheses:**
  - **Interior flow is uniform and geostrophic: $Ro_T \ll 1$ *and* $Ro \ll 1$**
  - **Homogeneous fluid: $\rho_0 = cost$ *and* $\rho' = 0$**
  - **Flat bottom**
- **Primitive equations:**

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_E \frac{\partial^2 u}{\partial z^2} \\
+fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_E \frac{\partial^2 v}{\partial z^2} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z},
\end{aligned}
$$

- **Boundary conditions:**

$$
\begin{aligned}
&\text{Bottom } (z=0): && u = 0, \quad v = 0, \\
&\text{Toward the interior } (z \gg d): && u = \bar{u}, \quad v = 0, \quad p = \bar{p}(x,y)
\end{aligned}
$$

- **Interior flow is uniform, no horizontal gradient**
- **[…]**

THE BOTTOM EKMAN LAYER

- **Solutions:**

$$u = \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right)$$
$$v = \bar{u}\, e^{-z/d}\sin\frac{z}{d}\,.$$

$$d = \sqrt{\frac{2\nu_E}{f}}$$ **Ekman depth**

- **Considerations:**
  - **As expected, the Ekman depth corresponds to $Ek \sim 1$**
  - **Although the driving interior flow is along x, we have a transversal velocity (along y) which is not negligible**
  - **Close to wall $z \to 0$ or $\frac{z}{d} \ll 1 \Rightarrow u \sim v \sim \bar{u}z/d$ …the velocity near the bottom is at 45 degree to the left of the interior velocity (with f>0) […]**
  - **Where u reaches its maximum at $z = \frac{3\pi}{4}d$ the velocity is $u = 1.07\bar{u}$ that is, larger than its interior value**
  - **The net transport of fluid transverse to the main flow is $V = \int_0^\infty v\,dz = \bar{u}d/2$ while $U = -\bar{u}d/2$**

THE BOTTOM EKMAN LAYER

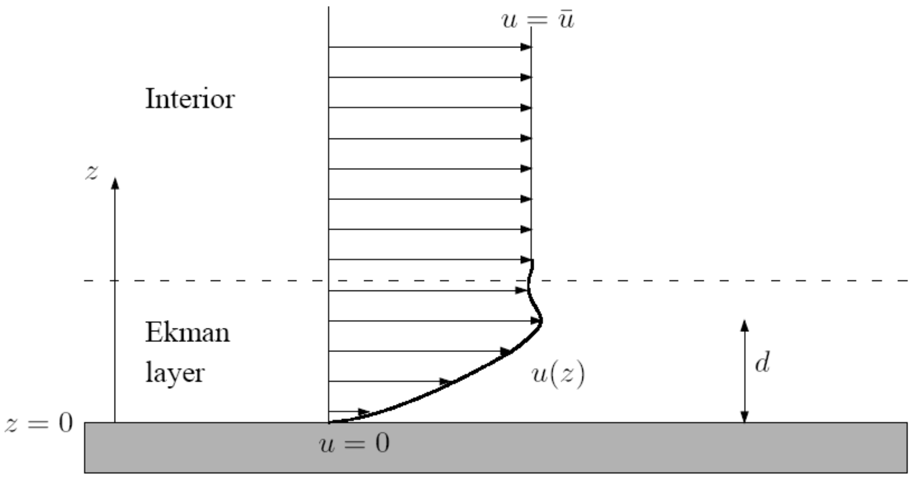


Figure 8-3 Frictional influence of a flat bottom on a uniform flow in a rotating framework.

$$u = \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right)$$
$$v = \bar{u}\, e^{-z/d}\sin\frac{z}{d}.$$

$$d = \sqrt{\frac{2\nu_E}{f}}$$

$$f > 0$$

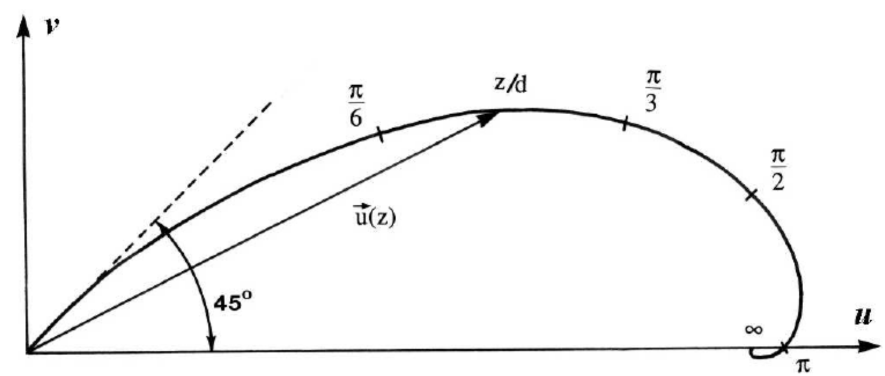


Figure 8-4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.

THE BOTTOM EKMAN LAYER - GENERALIZED

- **Interior geostrophic flow varying on a scale sufficiently large to be in geostrophic equilibrium:**

$$-f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \qquad f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$

- **for a constant Coriolis parameter (on f-plane) the flow is non-divergent:** $\partial\bar{u}/\partial x + \partial\bar{v}/\partial y = 0$
- **The BL equations:**

$$\begin{aligned} -f(v-\bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\ f(u-\bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

- **…with BCs** $\boldsymbol{u \to \bar{u}\ and\ v \to \bar{v}\ for\ z \to \infty\ and\ u(z=0) = v(z=0) = 0}$
- **…and solutions:**

$$\begin{aligned} u &= \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right) - \bar{v}\, e^{-z/d}\sin\frac{z}{d} \\ v &= \bar{u}\, e^{-z/d}\sin\frac{z}{d} + \bar{v}\left(1 - e^{-z/d}\cos\frac{z}{d}\right). \end{aligned}$$

THE BOTTOM EKMAN LAYER - GENERALIZED

- **We can compute the transport related to the Ekman bottom layer:**

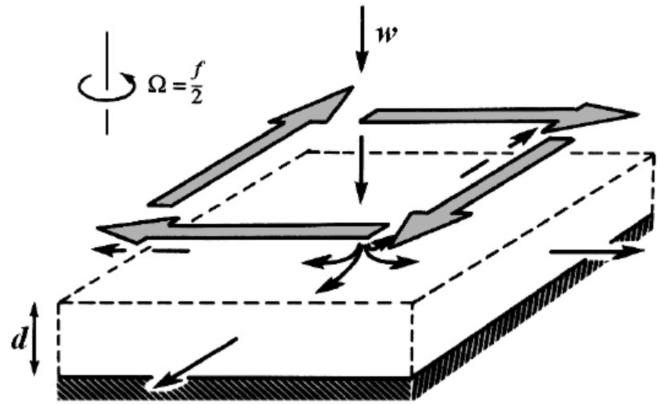
$$\begin{aligned} \mathrm{U} &= \int_0^{\infty} (u - \bar{u})\, dz = -\frac{d}{2} (\bar{u} + \bar{v}) \\ \mathrm{V} &= \int_0^{\infty} (v - \bar{v})\, dz = \frac{d}{2} (\bar{u} - \bar{v})\,. \end{aligned}$$

- **this transport is not necessarily parallel to the interior geostrophic flow and may be divergent [...]:**

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_0^{\infty} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = -\frac{d}{2} \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right) = -\frac{d}{2\rho_0 f} \nabla^2 \bar{p}.$$

- **The flow in the BL converges/diverges if interior has a relative vorticity $\bar{\zeta} = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \neq 0$ (pos/neg):**
  - **Divergence in BEL and compensating downwelling in the interior + ACyc gyre**
  - **Convergence in BEL and compensating upwelling in the interior + Cyc gyre**
- **Due to the solid bottom, the only possibility to provide convergence / divergence which supports upwelling / downwelling is a vertical velocity $\bar{w}$ from the interior**

THE BOTTOM EKMAN LAYER - GENERALIZED



$f > 0$

Figure 8-5 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such a situation arises in the presence of an anticyclonic gyre in the interior, as depicted by the large horizontal arrows. Similarly, interior cyclonic motion causes convergence in the Ekman layer and upwelling in the interior.

- **Interior is geostrophic and on f-plane $\partial_z \bar{w} = 0 \Rightarrow \bar{w} = cost \Rightarrow$ the vertical velocity must occur throughout the depth of the fluid**
- **Since divergence is $\propto d \ll H \Rightarrow$ the vertical velocity is very weak […]**
- **def. EKMAN PUMPING: $\bar{w} = \frac{d}{2}\bar{\zeta} = \frac{d}{2\rho_0 f}\nabla^2\bar{p} = -\nabla\cdot(U,V) = -\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)$**
- **The larger the vorticity from the interior, the greater the upwelling / downwelling, with an effect increasing toward the equator ($f \rightarrow 0$)**

THE BOTTOM EKMAN LAYER OVER UNEVEN TERRAIN

- Irregular topography has an effect over the structure of BEL
- Terrain with elevation $z = b(x, y)$ above a horizontal reference level
- Since GFD flows are almost 2D: $\nabla b(x, y) = (\partial_x b, \partial_y b) \ll 1$
- Interior geostrophic flow not uniform
- The BL equations:

$$
\begin{aligned}
-f(v-\bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\
f(u-\bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2}
\end{aligned}
$$

- …with BCs $u \rightarrow \bar{u}$ *and* $v \rightarrow \bar{v}$ *for* $z \rightarrow \infty$ *and* $u(z=b) = v(z=b) = 0$
- …and solutions are the same as previous case with $z \rightarrow z - b$:

$$
\begin{aligned}
u &= \bar{u} - e^{(b-z)/d} \left(\bar{u} \cos \frac{z-b}{d} + \bar{v} \sin \frac{z-b}{d} \right) \\
v &= \bar{v} + e^{(b-z)/d} \left(\bar{u} \sin \frac{z-b}{d} - \bar{v} \cos \frac{z-b}{d} \right)
\end{aligned}
$$

THE BOTTOM EKMAN LAYER OVER UNEVEN TERRAIN

- **Computing vertical velocity from continuity eq.:**

$$\begin{aligned}
\frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\
&= e^{(b-z)/d} \left\{ \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right) \sin \frac{z-b}{d} \right. \\
&\quad + \frac{1}{d} \frac{\partial b}{\partial x} \left[(\bar{u} - \bar{v}) \cos \frac{z-b}{d} + (\bar{u} + \bar{v}) \sin \frac{z-b}{d} \right] \\
&\quad \left. + \frac{1}{d} \frac{\partial b}{\partial y} \left[(\bar{u} + \bar{v}) \cos \frac{z-b}{d} - (\bar{u} - \bar{v}) \sin \frac{z-b}{d} \right] \right\}
\end{aligned}$$

- **…then we can integrate from $z = b, w = 0$ to $z \to \infty, w = \bar{w}$:**

$$\bar{w} = \left(\bar{u} \frac{\partial b}{\partial x} + \bar{v} \frac{\partial b}{\partial y} \right) + \frac{d}{2} \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right)$$

- **The first component was found during the analysis of geostrophic flow over irregular bottom, and it ensures no normal flow to the bottom; the second is the Ekman pumping as in the flat bottom case, which is not affected by the bottom slope**

THE SURFACE EKMAN LAYER

- **The frictional stress against the flow is exerted by the WIND STRESS (historically, the 1st problem investigated by Ekman)**
- **Hypotheses:**
  - **Interior flow is uniform and geostrophic:** $Ro_T \ll 1$ *and* $Ro \ll 1$
  - **Homogeneous fluid:** $\rho_0 = cost$ *and* $\rho' = 0$
  - **Presence of wind stress:** $\vec{\tau} = (\tau_x, \tau_y)$
- **Primitive equations + BCs:**

$$
\begin{aligned}
-\ f\ (v - \bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\
+\ f\ (u - \bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \\
\text{Surface } (z = 0): &\quad \rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x, \quad \rho_0 \nu_E \frac{\partial v}{\partial z} = \tau^y \\
\text{Toward interior } (z \to -\infty): &\quad u = \bar{u}, \quad v = \bar{v}.
\end{aligned}
$$

- **Solutions:**

$$
\begin{aligned}
u &= \bar{u} + \frac{\sqrt{2}}{\rho_0 f d}\, e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right)\right] \\
v &= \bar{v} + \frac{\sqrt{2}}{\rho_0 f d}\, e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]
\end{aligned}
$$

THE SURFACE EKMAN LAYER

- **The solution has a wind-driven component fully related to the wind stress $\vec{\tau}$, independent by the interior flow but dependent on $1/d \Rightarrow$ the wind-driven component can be very large if d is very small (for example with almost inviscid flow with ν_E very small or near the Equator), and even a moderate wind stress may generate a large wind-driven component**

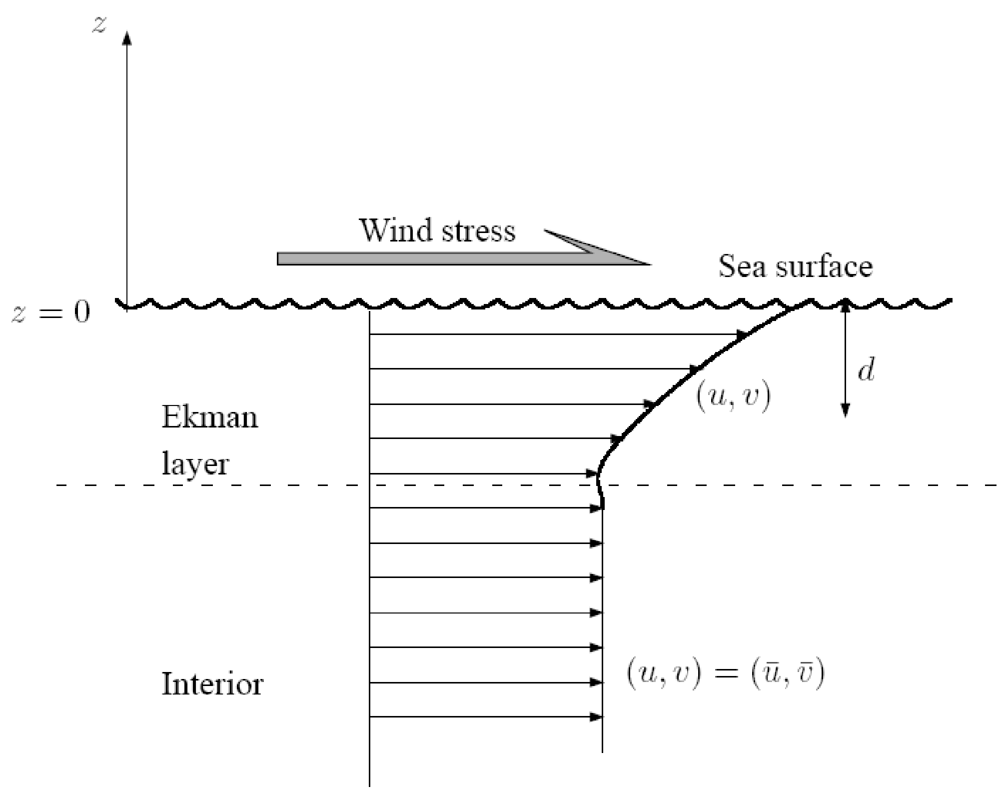


Figure 8-6 The surface Ekman layer generated by a wind stress on the ocean.

THE SURFACE EKMAN LAYER

- The **wind-driven (Ekman) transport** in the SEL has components [...]:

$$U = \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f}\, \tau^y$$

$$V = \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f}\, \tau^x.$$

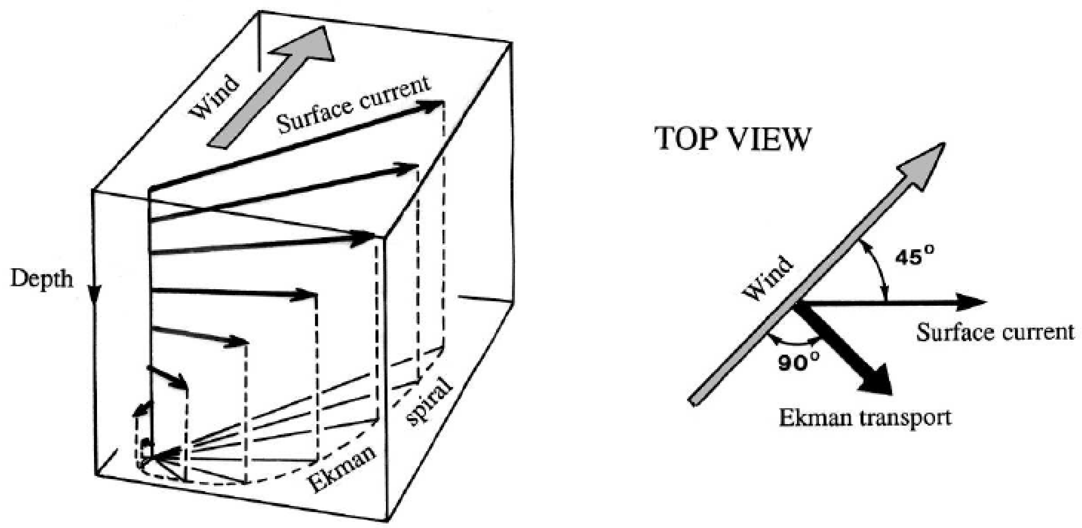


Figure 8-7 Structure of the surface Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the right of the surface stress. The reverse holds for the Southern Hemisphere.

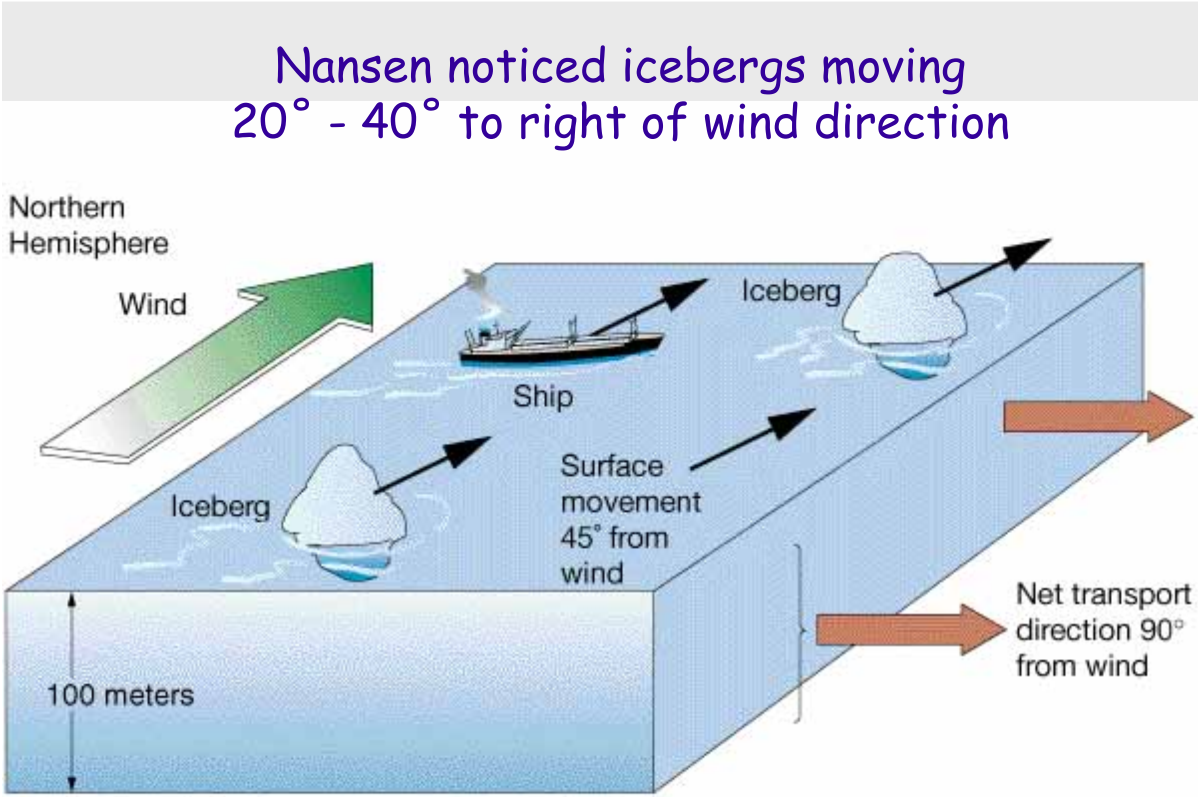
$$f > 0$$

THE SURFACE EKMAN LAYER

- **The Ekman transport is perpendicular to the wind stress, to the right in the N. Hemisphere, to the left in the S. Hemisphere, explaining why icebergs, mostly floating underwater, drift to the right of the wind as observed by Fridtjof Nansen**
- **The surface velocity $\overrightarrow{u_0} = \vec{u}(z = 0)$ has an angle of $45°$ with $\vec{\tau}$ […]**

https://people.ucsc.edu/~mdmccar/migrated/ocea1/01_Public/lectures/lect_notes_2/14_SURF_Ocean_Circ_12Fall.pdf

$$f > 0$$



THE SURFACE EKMAN LAYER

- **Compute the divergence of the Ekman transport (as done for BEL):**

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$

- **The divergence is totally independent by $\boldsymbol{v_E}$ and is entirely dependent on $\vec{\tau}$: $\nabla \cdot \vec{U} \propto \nabla \times \vec{\tau}|_z \rightarrow$ *wind − stress curl***

- **On f-plane: $\nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z$**

- **IF $\nabla \times \vec{\tau}|_z \neq 0$ the divergence of the Ekman transport must be provided by a vertical velocity throughout the interior (as in BEL) [...] :**

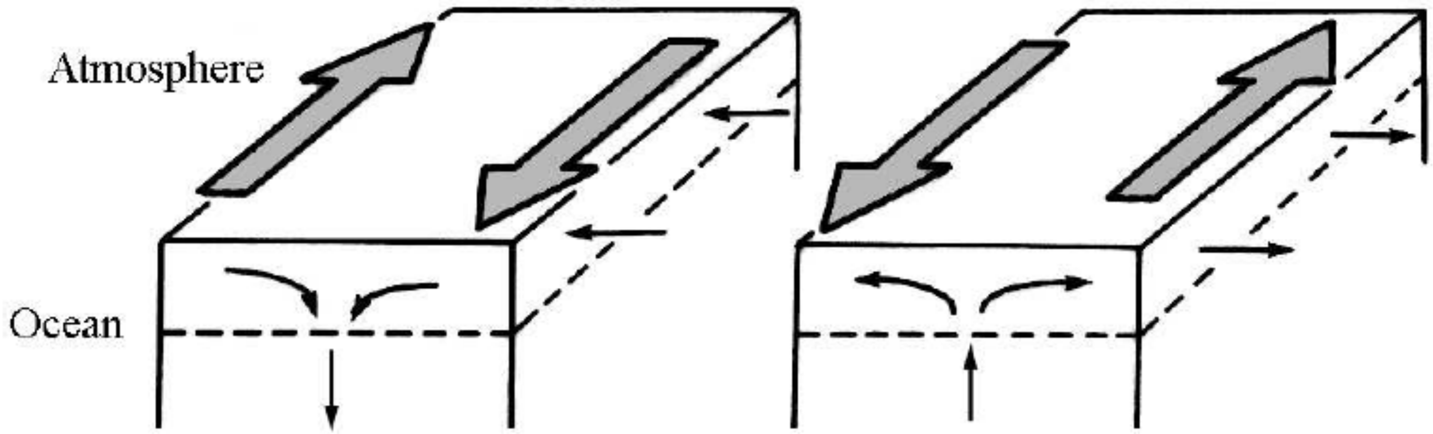
$$\bar{w} = + \int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right] = w_{\mathrm{Ek}}$$

- **def. EKMAN PUMPING: $\bar{w} = w_{Ek} = \frac{1}{\rho_0} \nabla \times \frac{\vec{\tau}}{f}|_z$**

- **Ekman pumping on f-plane: $w_{Ek} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z$**

THE SURFACE EKMAN LAYER

- Ekman pumping on f-plane: $w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z \lesseqgtr 0$
- Ekman pumping: a very effective mechanism to drive subsurface ocean currents through the action of winds



$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z < 0$$

Clockwise wind pattern

Convergence in SEL

Downwelling

$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z > 0$$

Anticlockwise wind pattern

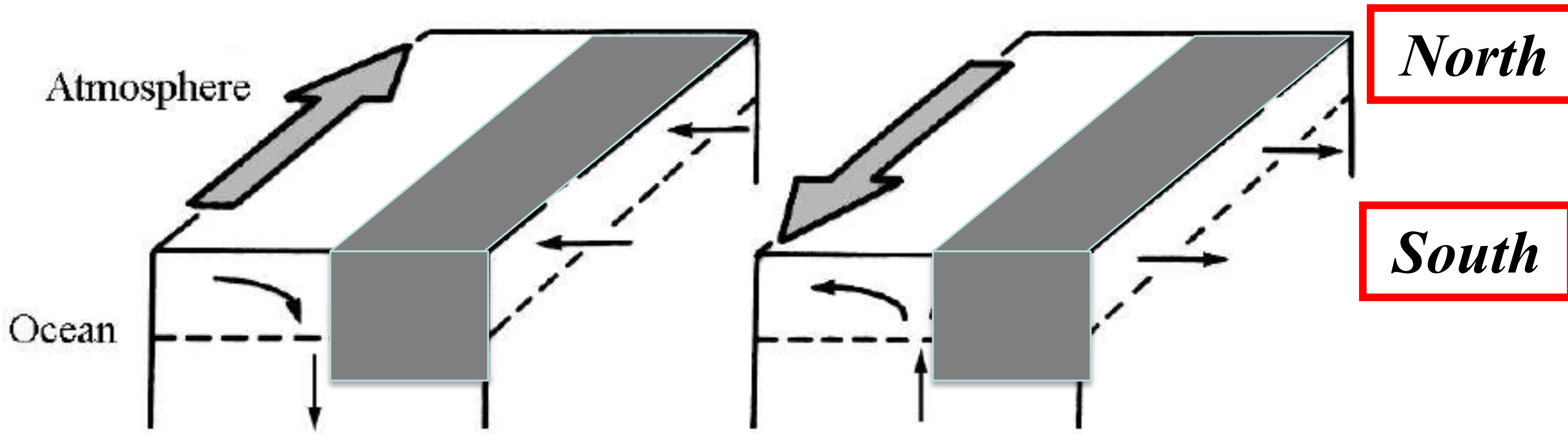
Divergence in SEL

Upwelling

$f > 0$

THE SURFACE EKMAN LAYER

- **Ekman pumping on f-plane:** $w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z \lessgtr 0$
- **Ekman pumping: a very effective mechanism to drive subsurface ocean currents through the action of winds => near the coast**



$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z < 0$$

Clockwise wind pattern

Convergence in SEL

Downwelling

$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z > 0$$

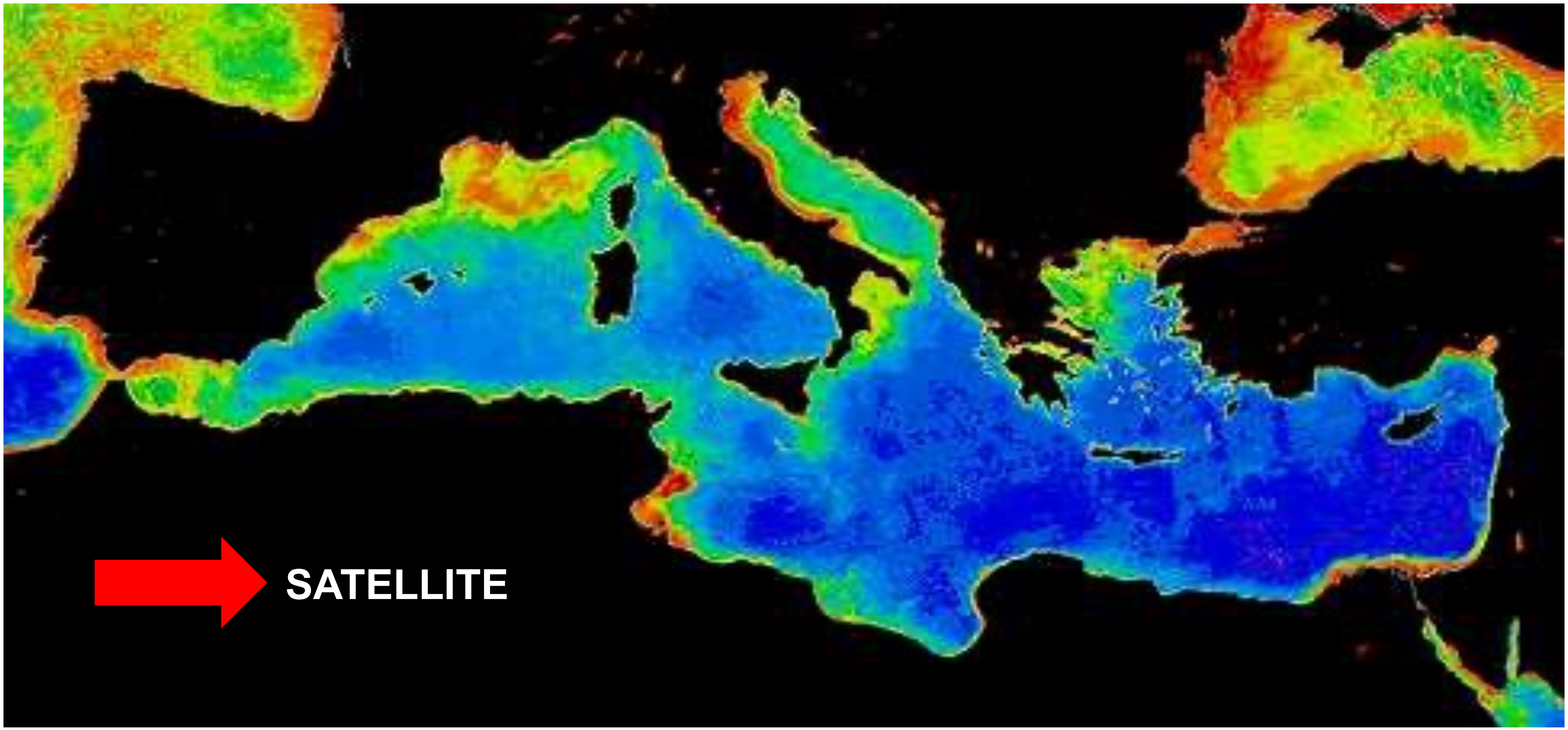
Anticlockwise wind pattern

Divergence in SEL

Upwelling

$$f > 0$$

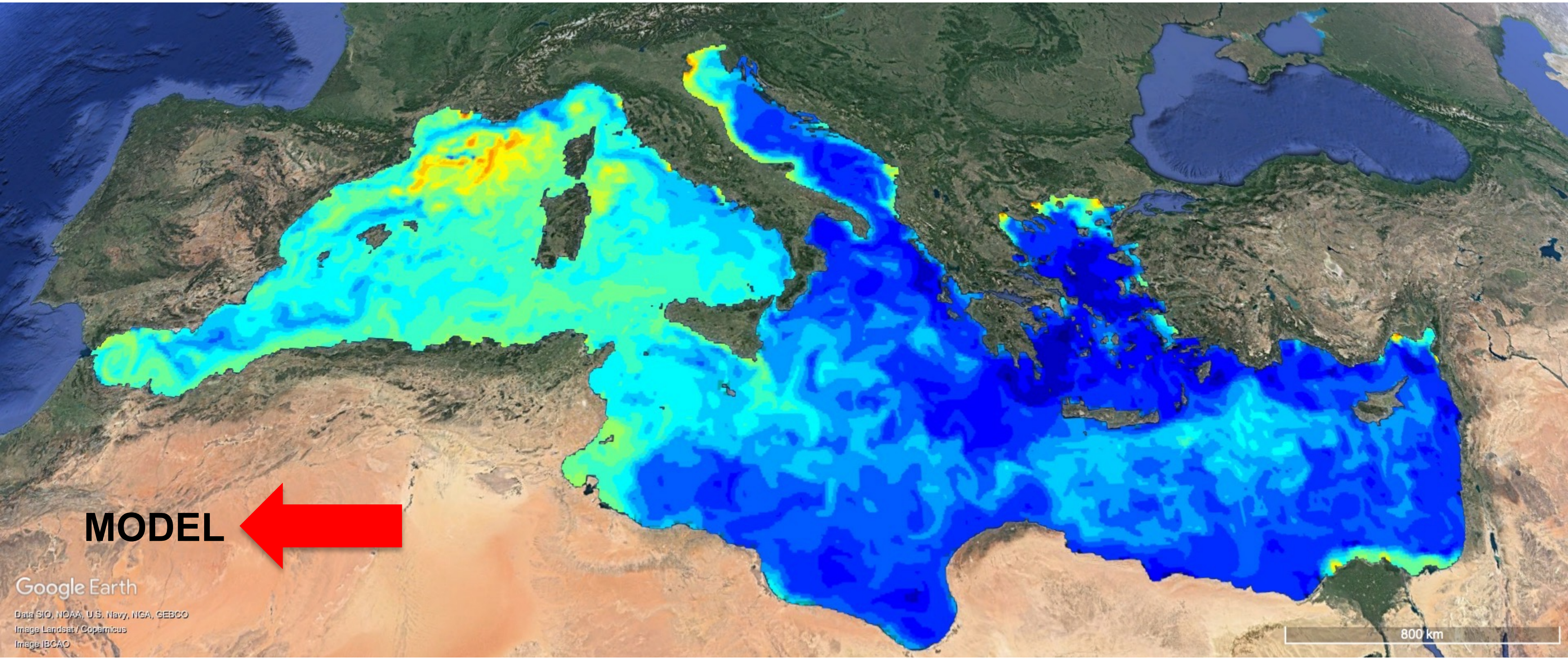
Effect of upwelling on biogochemistry in the Mediterranean Sea



North-Western Med Sea is an area of upwelling and high productivity due to Ekman pumping

https://www.youtube.com/watch?v=KBKmKI3tl4Q

Effect of upwelling on biogochemistry in the Mediterranean Sea



North-Western Med Sea is an area of upwelling and high productivity due to Ekman pumping

https://www.youtube.com/watch?v=KBKmKI3tl4Q

THE EKMAN LAYER IN REAL GEOPHYSICAL FLOWS

- Real geophysical flows are characterized by turbulence and stratification $\Rightarrow$ observations cannot match the highly idealized models of BEL and SEL
- GFD flows have $Re \gg 1 \Rightarrow$ we can replace υ with υ_E to account for the enhanced momentum transfer in a turbulent flow
- Ekman layers are SHEAR FLOWS and turbulence is not homogenous, increasing with the shear and suppressed close to the boundary where size of turbulent eddies is limited $\Rightarrow$ a general theory of turbulence does not exist, as a minimum $\upsilon_E = \upsilon_E(z)$ but observations do not agree with simple models
- The angle between near-boundary velocity / surface velocity and interior in BEL / SEL is $< 45°$ ranging $5° \div 20°$
- Eddy viscosity υ_E scales with friction velocity $u^* = \sqrt{|\vec{\tau}|/\rho_0}$ and d as mixing length ($\sim$ size of the most turbulent eddies): $\upsilon_E \sim u^* d$
- Ekman depth scales with $d \sim \sqrt{\upsilon_E/f} \sim \sqrt{u^* d\ /f} \Rightarrow d \sim u^*/f$
- Empirically: $d = 0.4\, u^*/f$

THE EKMAN LAYER IN REAL GEOPHYSICAL FLOWS

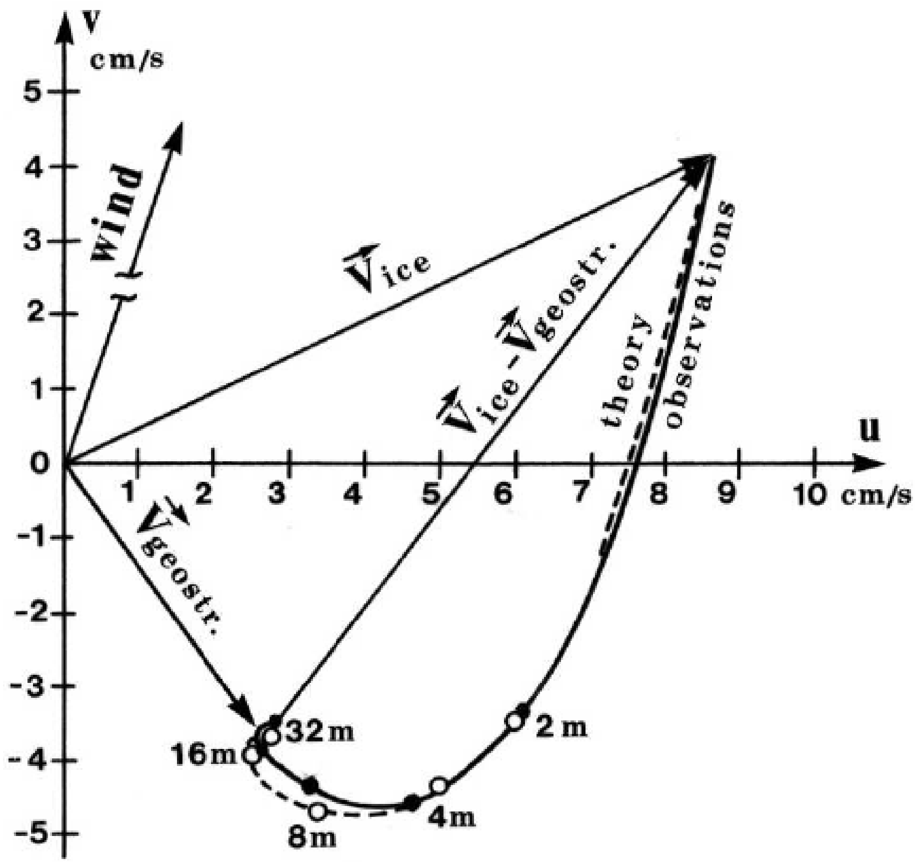
- **Real geophysical flows are characterized by vertical density stratification $\rho = \rho(z)$: the gradual change of density with z hinders vertical movements ⇒ reduction of vertical mixing of momentum by turbulence and decoupling motions at separate levels**

- **Stratification reduces the Ekman depth and increases the veering of the velocity vector with z**

- **Surface atmospheric layer during daytime over land and above warm currents at sea is frequently in a state of CONVECTION due to the heating from below: the Ekman dynamics is related to convective motions, driven both by the geostrophic flow aloft and by the intensity of the surface heat flux ⇒ Atmospheric Boundary Layer (ABL)**

- **Ekman depth scales with $d = \dfrac{1.3\, u^*}{f\left(1+\frac{N^2}{f^2}\right)^{1/4}}$**

One of the few cases when obs → theory



=> *surface current*

Figure 8-9 Comparison between observed currents below a drifting ice floe at 84.3°N and theoretical predictions based on an eddy viscosity $\nu_E = 2.4 \times 10^{-3}$ m^2/s. (Reprinted from *Deep-Sea Research*, 13, Kenneth Hunkins, Ekman drift currents in the Arctic Ocean, p. 614, ©1966, with kind permission from Pergamon Press Ltd, Headington Hill Hall, Oxford 0X3 0BW, UK)

observations: angle u_0 and u_{INT} < 45º

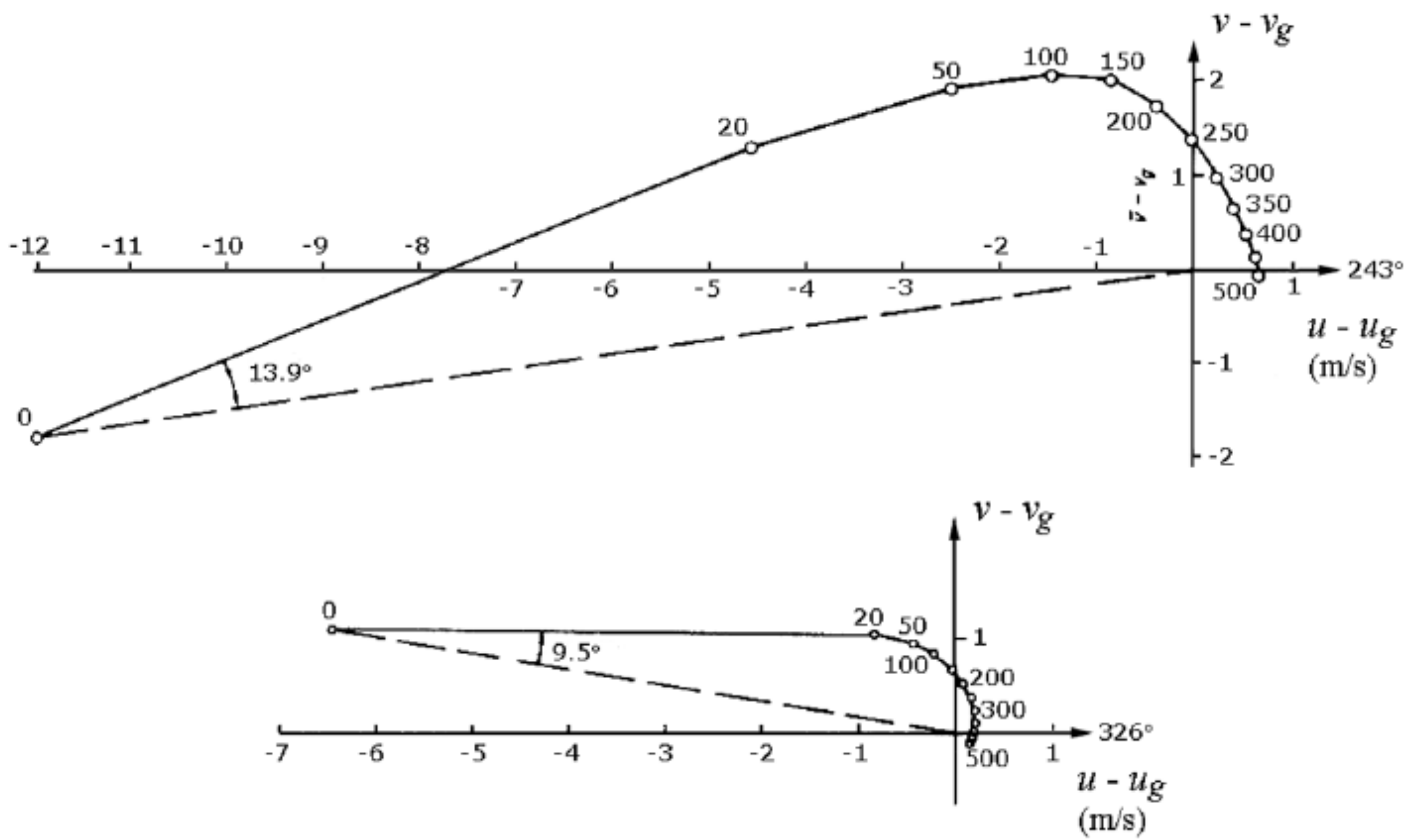


Figure 8-10 Wind vectors minus geostrophic wind as a function of height (in meters) in the maritime friction layer near the Scilly Isles. *Top diagram*: Case of warm air over cold water. *Bottom diagram*: Case of cold air over warm water. (Adapted from Roll, 1965)