BAROTROPIC WAVES

- Another possible simplification of the governing eqs. of GFD is to LINEARIZE the eqs. → restrictions must be imposed on the flows (for Ek $\ll \mathbf{1}$)
- Coriolis terms are linear → no need to simplify
- Advection terms are non-linear → need to be simplified ($\mathrm{Ro} = \frac{U}{\Omega L} \ll \mathbf{1}$)
  - → relatively weak flows (small U)
  - → relatively large scales (large L)
  - → in LAB: fast rotation rates (large Ω)
- Local time rate of velocity change is linear → no need to simplify ($\mathrm{Ro}_T = \frac{1}{\Omega T} \sim \mathbf{1}$)
- → consider slow flow fields under rotation that evolve relatively fast = rapidly moving disturbances do not require large velocities = information (or energy) may travel faster then material particles → the flow is a WAVE FIELD !
- → WAVES supported by <u>inviscid</u>, <u>homogeneous</u> fluid <u>in rotation</u>
- Velocity scale: "celerity" $\mathrm{C} = \frac{\textit{distance covered by the signal}}{\textit{nominal evolution time}} = \frac{L}{T} \sim L\Omega \gg U$

BAROTROPIC WAVES

- slow flow fields under rotation that evolve relatively fast = rapidly moving disturbances do not require large velocities = information (or energy) may travel faster then material particles

- *NOTE: look Appendix B of Cushman to review wave dynamics (already done in the first part of the course)*

BAROTROPIC WAVES

System governing the linear wave dynamics of an inviscid, homogeneous, shallow-water fluid in rotation (for f > 0) => **start from the shallow-water model excluding advection**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

$$h(x,y,t) = \eta(x,y,t) + H(x,y)$$

BAROTROPIC WAVES

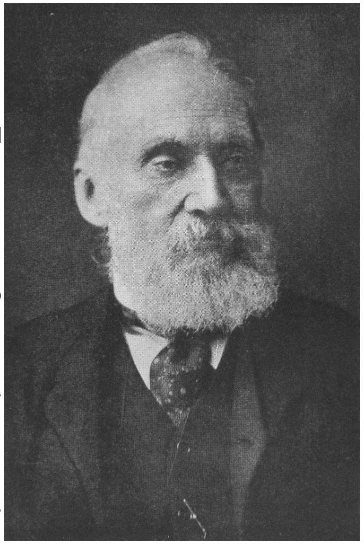
System governing the linear wave dynamics of an inviscid, homogeneous, shallow-water fluid in rotation (for f > 0) => **if $H(x,y) = cost$ [flat bottom] and through the scale analysis we obtain a linearized form of the continuity equation which brings to small amplitude waves [...]**

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

KELVIN WAVE 1



https://it.wikipedia.org/wiki/William_Thomson

A traveling disturbance requiring a lateral boundary layer as a support: u = v = 0 at x = 0 (coastline)

Lord Kelvin's hypothesis was that u = 0 in the whole domain

From the previous equations **[…]** :

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial y^2}$$

$$c = \sqrt{gH}$$

Wave equation => propagation of 1-d non-dispersive waves => speed of surface gravity waves in non-rotating shallow waters

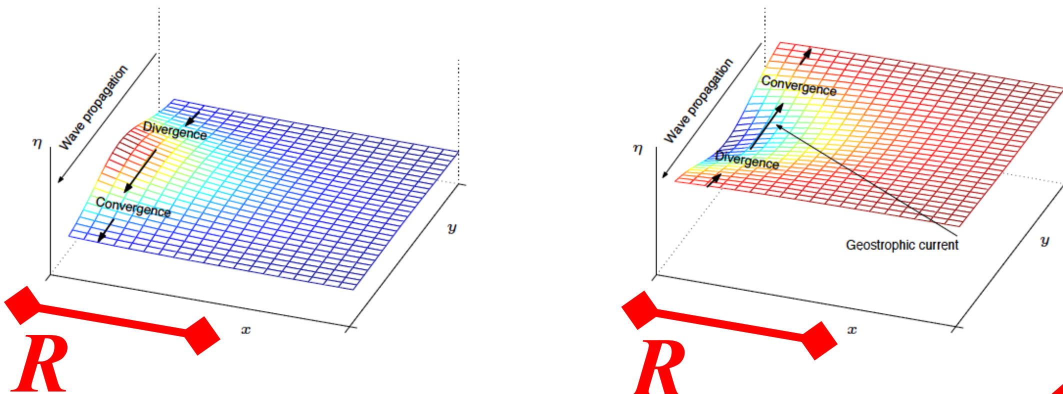
Solution:
[…]

$$\begin{aligned} u &= 0 \\ v &= \sqrt{gH}\, F(y+ct)\, e^{-x/R} \\ \eta &= -H\, F(y+ct)\, e^{-x/R}, \end{aligned}$$

$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}$$

Rossby radius of deformation

KELVIN WAVE 2



$$- fv = -g\frac{\partial \eta}{\partial x}$$

Figure 9-1 Upwelling and downwelling Kelvin waves. In the Northern Hemisphere, both waves travel with the coast on their right, but the accompanying currents differ. Geostrophic equilibrium in the x–momentum equation leads to a velocity v that is maximum at the bulge and directed as the geostrophic equilibrium requires. Because of the different geostrophic velocities at the bulge and further away, convergence and divergence patterns create a lifting or lowering of the surface. The lifting and lowering is such that the wave propagates towards negative y in either case (positive or negative bulge).

Remember:

v => FLOW VELOCITY

c => Surf-Grav WAVE SPEED

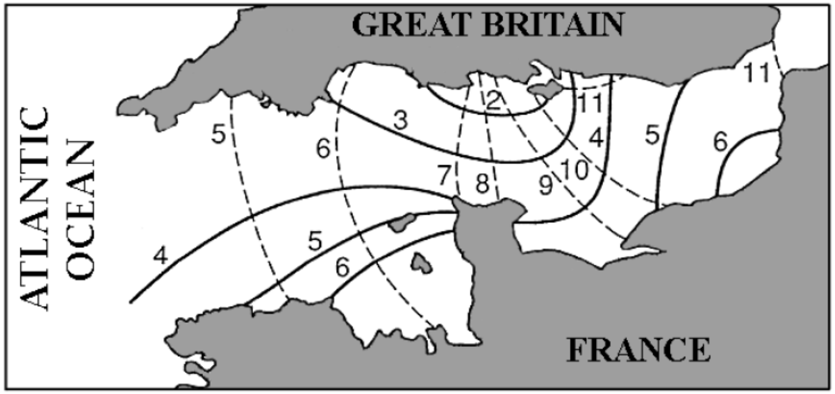
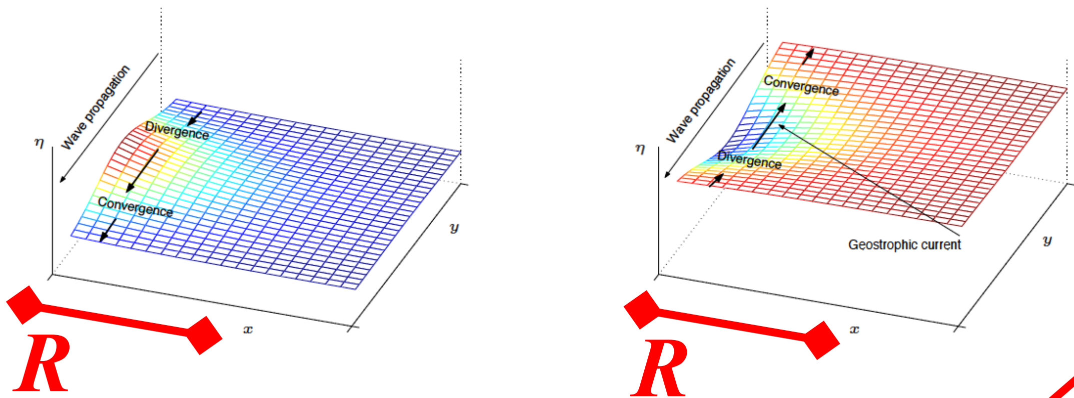


Figure 9-2 Cotidal lines (dashed) with time in lunar hours for the M2 tide in the English Channel showing the eastward progression of the tide from the North Atlantic Ocean. Lines of equal tidal range (solid, with value in meters) reveal larger amplitudes along the French coast, namely to the right of the wave progression in accordance with Kelvin waves. (From Proudman, 1953, as adapted by Gill, 1982.)

KELVIN WAVE 2



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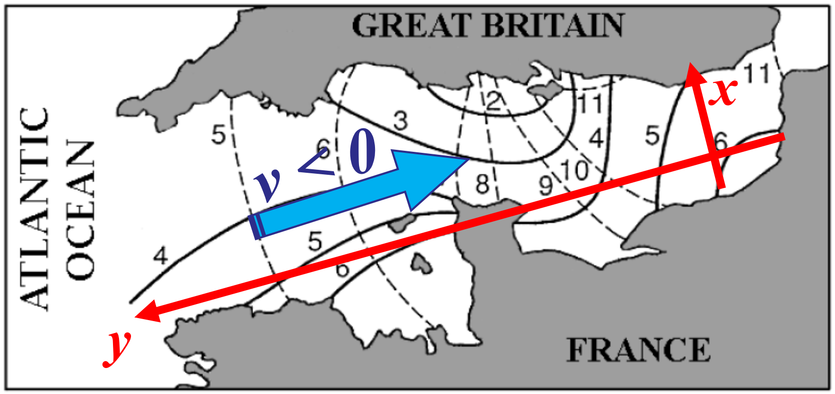

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POINCARE' WAVES 1

https://it.wikipedia.org/wiki/Henri_Poincar%C3%A9#/media/File:JH_Poincare.jpg

Keep u ≠ 0 in the whole domain

The system has to be solved entirely → all coeffs. are constant and a Fourier-like solution can be set:

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re \begin{pmatrix} A \\ U \\ V \end{pmatrix} e^{\mathrm{i}\,(k_x x + k_y y - \omega t)}$$

Dispersion relation **[…]** : $\omega\,[\omega^2 - f^2 - gH\,(k_x^2 + k_y^2)] = 0$

$\omega = 0$ → steady geostrophic flow

solution:

$\omega = \sqrt{f^2 + gH\,k^2}$ → **superinertial travelling waves** (PW)

and cases **[…]**: f = 0, HF, LF with $\dfrac{\omega}{f} = \sqrt{1 + R^2 k^2}$

POINCARE' WAVES 2

$$\boxed{1 \leq \frac{\omega}{f} \leq Rk}$$

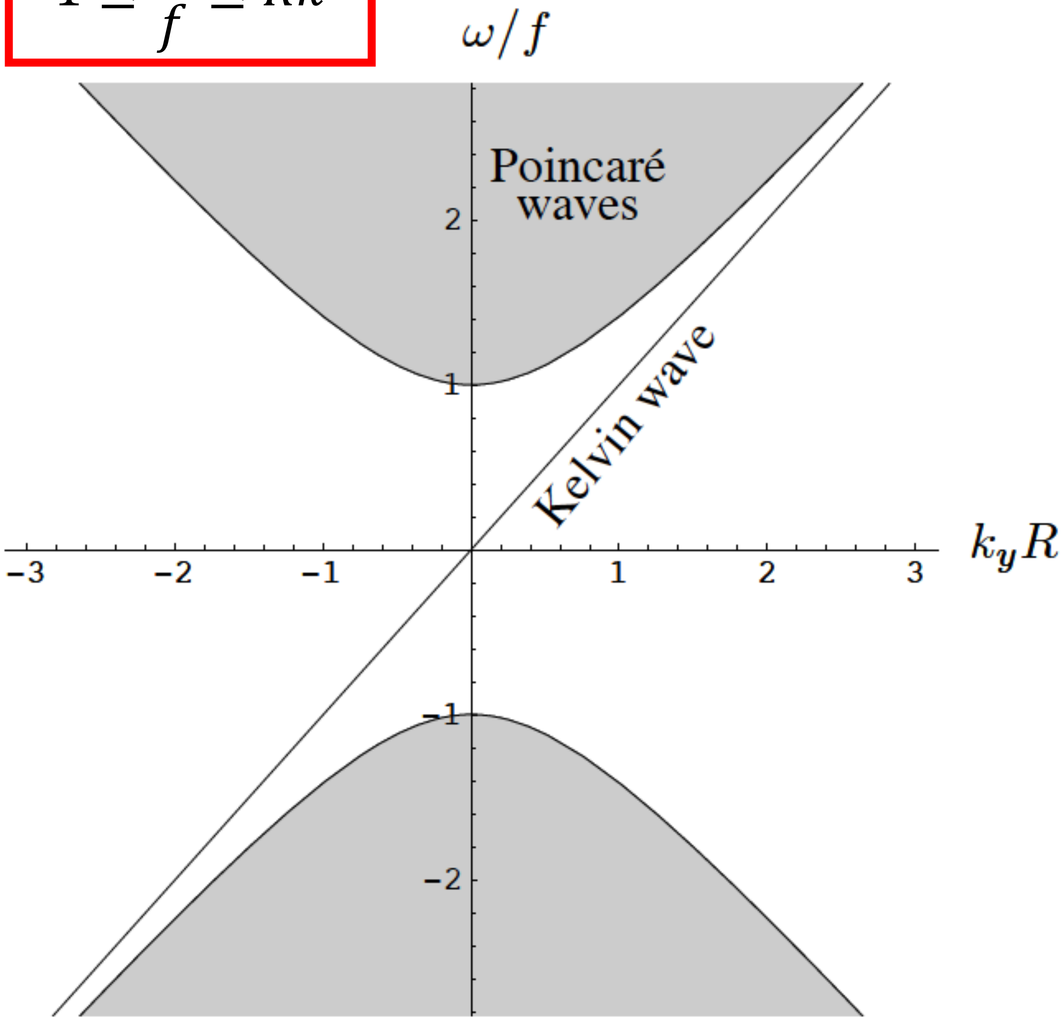
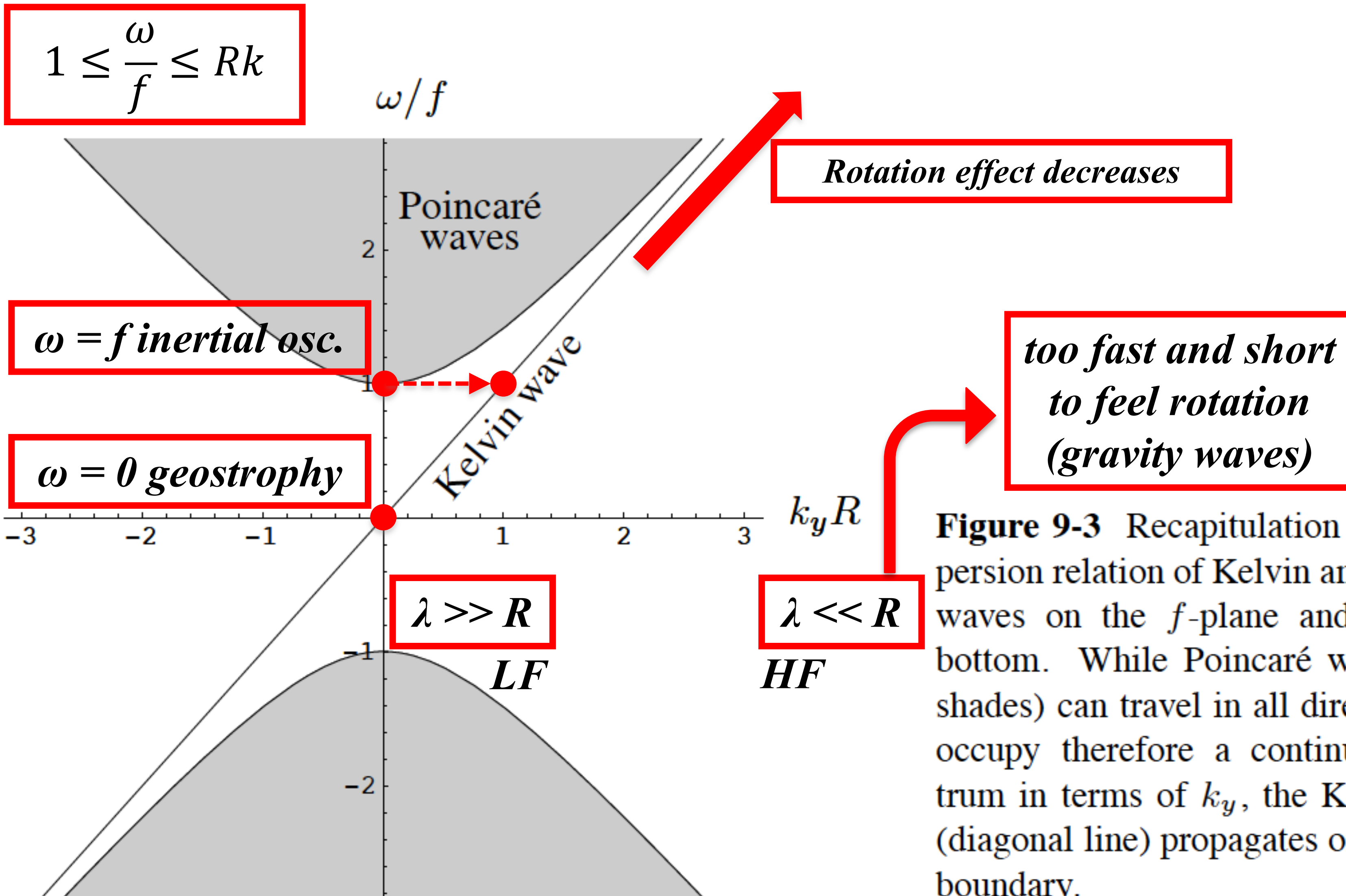


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

The solution of KW as $\omega/f = k_y R$ can be found with Fourier-like solution in the eqs. for KW with $e^{i(ky+\omega t)}$

POINCARE' WAVES 2



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The solution of KW as $\omega/f = k_y R$ can be found with Fourier-like solution in the eqs. for KW with $e^{i(ky+\omega t)}$

PLANETARY or ROSSBY WAVES 1

http://paoc.mit.edu/paoc/education/rossby.htm

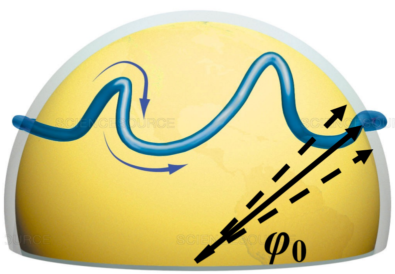
KW and PW are relatively fast waves ($\omega \geq f$): do exist other slower waves ($\omega \ll f$), **associated with evolution of disturbances in the geostrophic flow**?

Coriolis parameter: $f = 2\Omega \sin \varphi$

Taylor expansion around a reference latitude φ_0:

$$f = 2\Omega \sin\varphi_0 + 2\Omega \frac{y}{a} \cos\varphi_0 + \ldots$$

$$f = f_0 + \beta_0 y$$



$$\beta_0 = 2(\Omega/a)\cos\varphi_0$$ BETA PARAMETER

$$\beta = \frac{\beta_0 L}{f_0} \ll 1$$ PLANETARY NUMBER

The system for the β-plane has "large" and "small" terms → **[…]** retaining the large ones we obtain the geostrophic flow (u_g,v_g 1st-approx solution)

PLANETARY or ROSSBY WAVES 2

solving the system with (u_g, v_g) we obtain: velocity = geostr + ageostr

[…] and then including (u,v) in the continuity equation:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0$$

Using a Fourier-like solution for η we have the dispersion relation:

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}$$

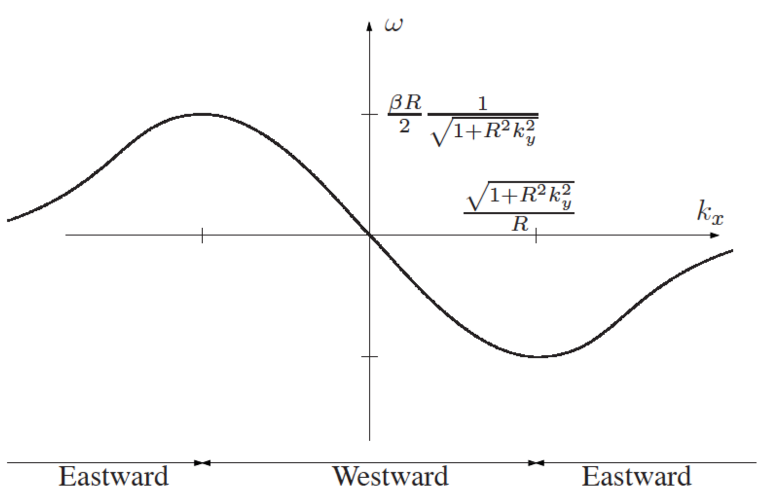


Figure 9-4 Dispersion relation of planetary (Rossby) waves. The frequency ω is plotted against the zonal wavenumber k_x at constant meridional wavenumber k_y. As the slope of the curve reverses, so does the direction of zonal propagation of energy.

- For both cases of LW and SW: $\omega \ll f_0$ subinertial wave
- $c_x = c_x(k)$ dispersive wave
- $c_x = \frac{\omega}{k_x} < 0$ westward propagation
- $c_g = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}\right)$ is westward for LW and eastward for SW

PLANETARY or ROSSBY WAVES 3

- Rewriting the dispersion relation we obtain eq. for circles in (k_x, k_y) at constant ω: $\omega_1 < \omega_2 < \omega_3 < \omega_{max}$ **[...]**

- $\omega_{max} = \frac{\beta_0 R}{2}$ max frequency

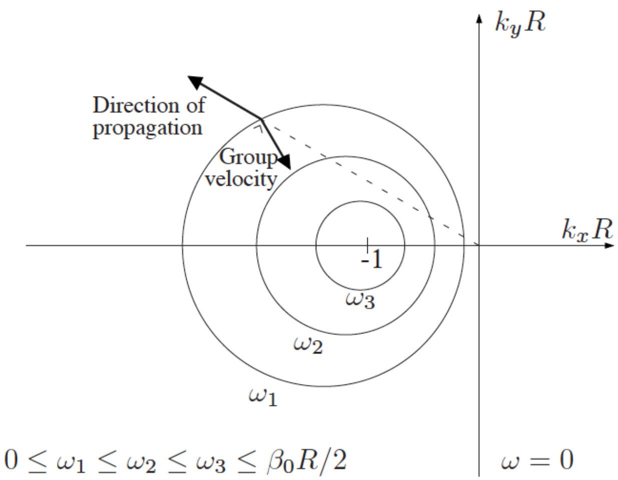
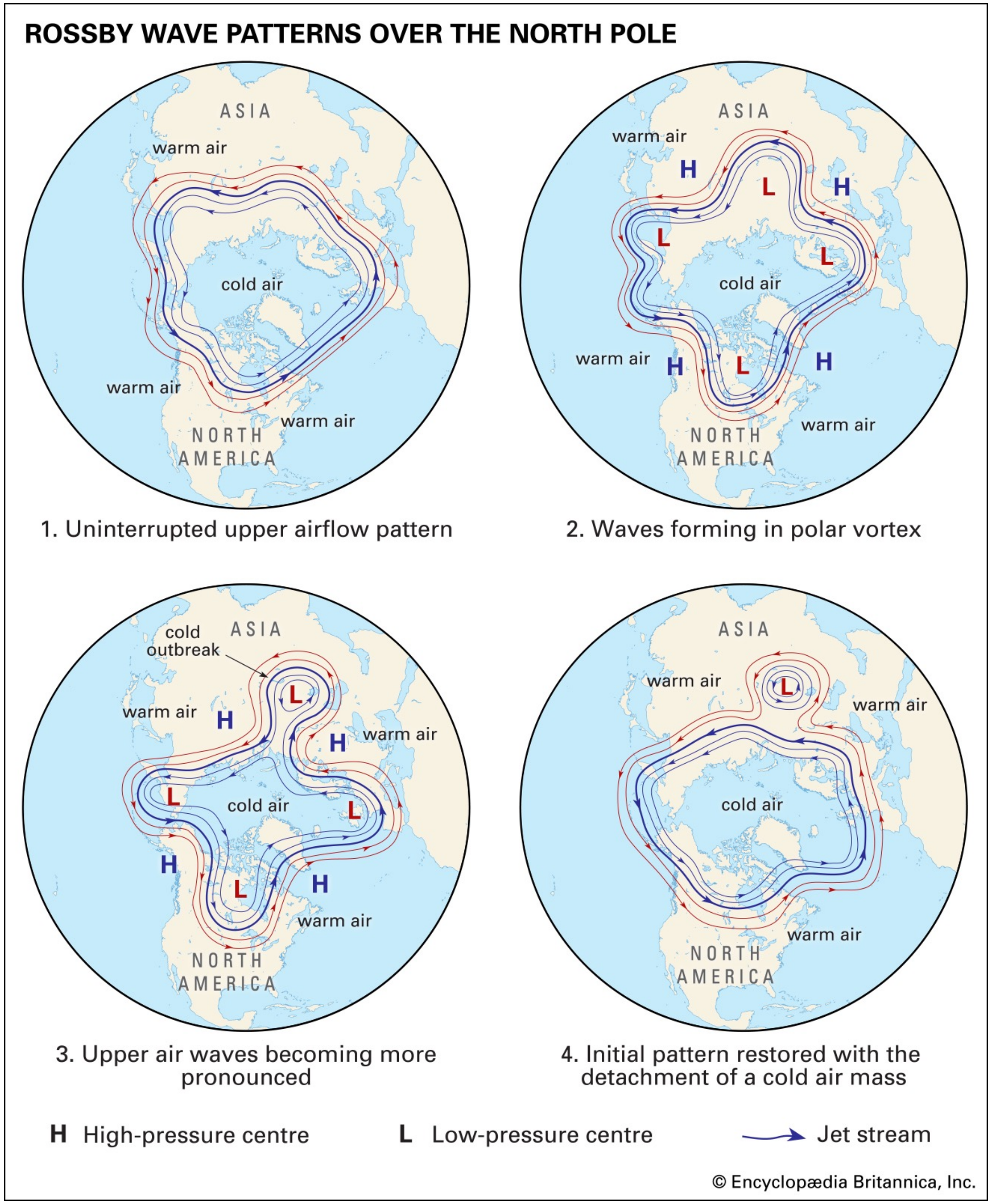


Figure 9-5 Geometric representation of the planetary-wave dispersion relation. Each circle corresponds to a fixed frequency, with frequency increasing with decreasing radius. The group velocity of the (k_x, k_y) wave is a vector perpendicular to the circle at point (k_x, k_y) and directed toward its center.

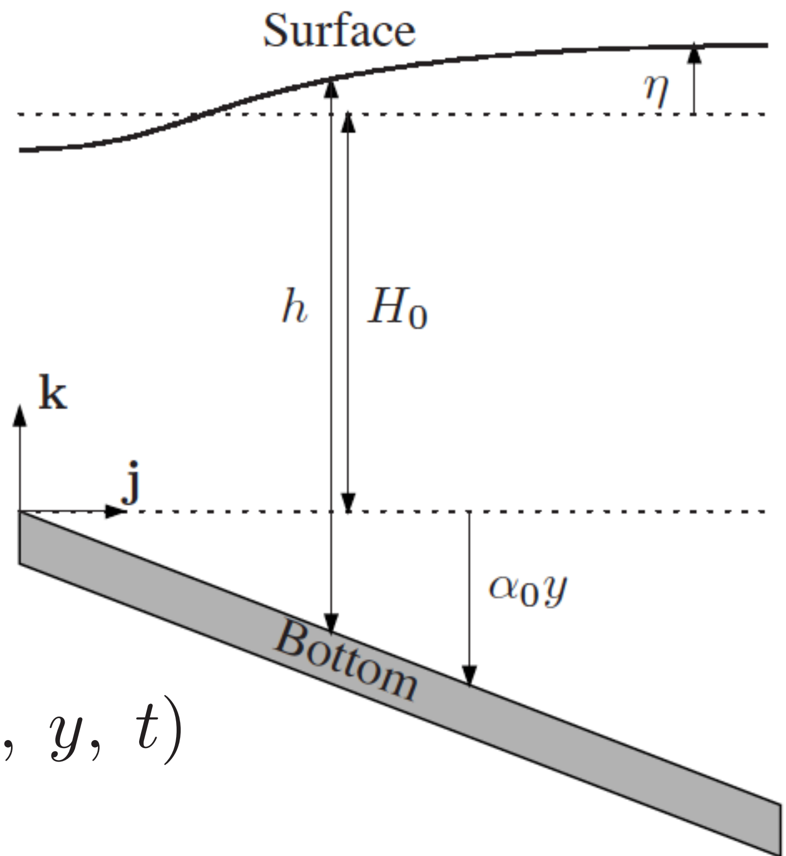


TOPOGRAPHIC WAVES 1

Perturbing effect is small and associated with weak bottom irregularity (not uncommon in the GFD phenomena…)

$$H = H_0 + \alpha_0 y$$

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$



$$h(x,\ y,\ t) = H_0 + \alpha_0 y + \eta(x,\ y,\ t)$$

The system has "large" and "small" terms which scale as Ro_T: retaining the large ones we obtain the geostrophic flow (u_g,v_g 1st-approx solution)

TOPOGRAPHIC WAVES 2

solving the system with (u_g,v_g) we obtain: velocity = geostr + ageostr

[…] and then including (u,v) in the continuity equation:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2\eta + \frac{\alpha_0 g}{f}\frac{\partial \eta}{\partial x} = 0$$

Using a Fourier-like solution for η we have the dispersion relation:

$$\omega = \frac{\alpha_0 g}{f}\frac{k_x}{1 + R^2\left(k_x^2 + k_y^2\right)}$$

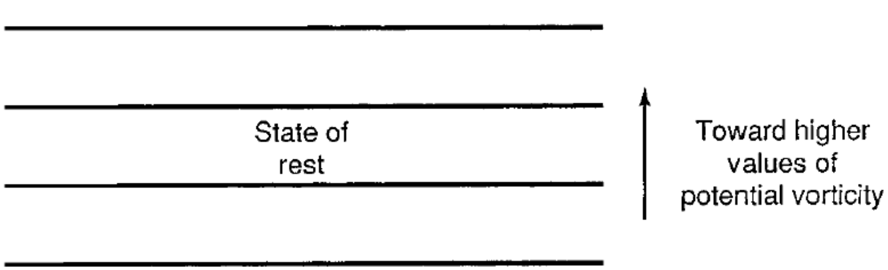
- Phase speed c_x has the same sign as α_0 → TW propagate in the Northern Hemisphere with the shallower side on their right
- For both cases of LW and SW: $\omega \ll f$ subinertial wave
- Since RW always propagate westward = with north on their right, analogy with RW is: "shallow-north" and "deep-south"
- Similar considerations as RW to obtain $\omega_{max} = \frac{\alpha_0 g}{2fR}$ max frequency

ANALOGY BETWEEN PLANETARY AND TOPOGRAPHIC WAVES

Potential vorticity on β-plane and sloping bottom: $$q = \frac{f_0 + \beta_0 y + \partial v/\partial x - \partial u/\partial y}{H_0 + \alpha_0 y + \eta}$$

Taylor expansion: $$q = \frac{1}{H_0}\left(f_0 + \boxed{\beta_0 y} - \boxed{\frac{\alpha_0 f_0}{H_0} y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f_0}{H_0}\eta\right)$$

=> gradient of potential vorticity



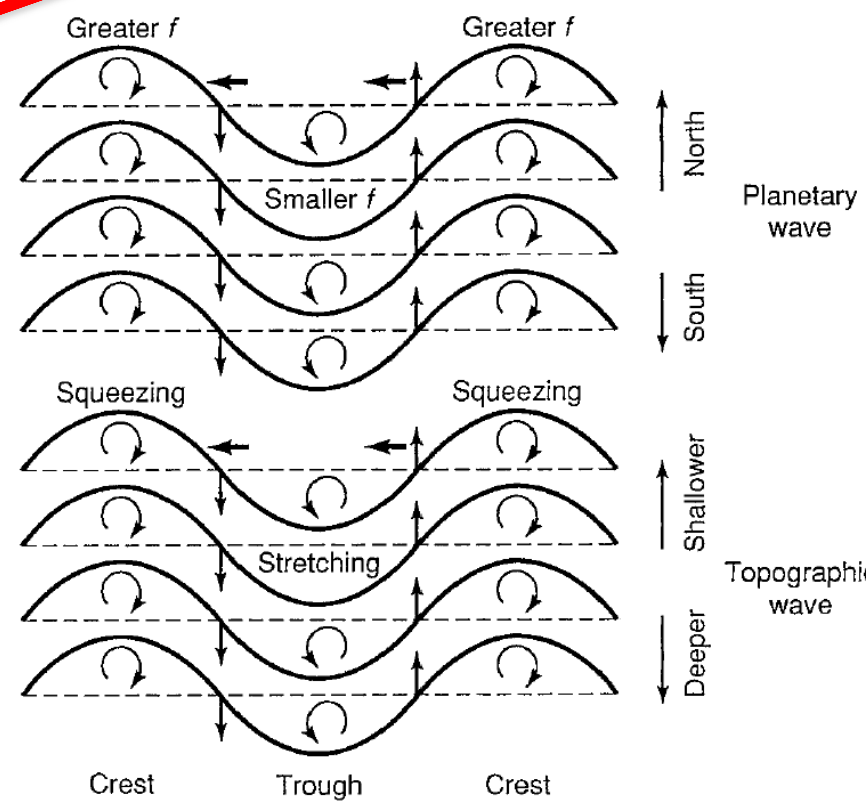


Figure 9-7 Comparison of the physical mechanisms that propel planetary and topographic waves. Displaced fluid parcels react to their new location by developing either clockwise or counterclockwise vorticity. Intermediate parcels are entrained by neighboring vortices, and the wave progresses forward.