# **STRATIFICATION**





**STRATIFICATION** induces a certain degree of decoupling between the various fluid masses which have different densities (vertical layering) => stratified systems contain more degrees of freedom than homogeneous systems => stratified systems exhibit additional types of motion

### **Static stability** 11.2

Let us first consider a fluid in static equilibrium. Lack of motion can occur only in the absence of horizontal forces and thus in the presence of horizontal homogeneity. Stratification is then purely vertical (Figure 11-1).



Figure 11-1 When an incompressible fluid parcel of density  $\rho(z)$  is vertically displaced from level z to level  $z + h$ in a stratified environment, a buoyancy force appears because of the density difference  $\rho(z) - \rho(z+h)$  between the particle and the ambient fluid.

### 11.2 **Static stability**

 $t_{\text{min}}$  I eak of mation aan again only in the charge. brium. Lack of motion can occur only in the absence presence of horizontal homogeneity. Stratification is then purely vertical (Figure 11-1). changes, and at the subject to a new level is subject to its own weight to its



Figure 11-1 When an incompressible force appears because of the density<br>difference  $\rho(z) - \rho(z+h)$  between the particle and the ambient fluid.

With **incompressible fluids**, the displaced with **incompressible fluids**, the displaced parcel retains its former density and at the new level is subject to a *net downward force equal to its own weight minus the weight of the displaced fluid* (Archimede's principle): with **incompressible fluids**, the displaced  $\frac{y_1}{z}$  to level  $z + n$  force equal to us own weight minus the Led environment, a buoyancy weight of the uispluced fund retains its former density and at the  $\text{density } \rho(z) \text{ is vertically}$  from some line density variations, the density variations, also trom level z to level  $z + n$  force equal to us own weight minus the

$$
\rho(z) - \rho(z+h)
$$
 between the  
d the ambient fluid.  $\rho(z)$   $V \frac{d^2 h}{dt^2} = g [\rho(z+h) - \rho(z)] V$ 

**Example 12.5.** All using the **Boussinesq approx**\*

$$
\rho(z+h) - \rho(z) \simeq \frac{d\rho}{dz} h
$$

$$
\frac{d^2h}{dt^2} - \frac{g}{\rho_0} \frac{d\rho}{dz} h = 0
$$

 $\Rightarrow$  2 cases: STABLE and UNSTABLE =>  $\frac{d\rho}{dz}$  < 0 and  $\frac{d\rho}{dz}$  > 0 [...]  $\alpha$  $0$  [..  $\frac{d\rho}{dt}$  and  $\frac{d\rho}{dt}$  or  $\frac{1}{\rho}$  $\frac{dz}{dz}$  b and  $\frac{dz}{dz}$ 

> \*not like an air balloon in seawater!  $\mathcal{P}$  , equation by V , equation (1) reduces to  $\mathcal{P}$  , equation (1) reduces to  $\mathcal{P}$ ke an air balloon  $\overline{\phantom{a}}$

### 11.2 **Static stability**

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Figure 11-1 When an incompressible 11.3. ATMOSPHERIC STRATIFICATION 321 force appears because of the density

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(Archimede's principle):  
\n
$$
\rho(z) - \rho(z+h)
$$
 between the  
\nd the ambient fluid. 
$$
\rho(z) V \frac{d^2 h}{dt^2} = g [\rho(z+h) - \rho(z)] V
$$

 $\begin{bmatrix} \dots \end{bmatrix}$  using the **Boussinesq approx** negative. If it is positive (depending to a fluid with the greater density  $\lfloor \ldots \rfloor$  using the boursinesq

$$
\rho(z+h) - \rho(z) \simeq \frac{d\rho}{dz} h
$$

$$
N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz} \qquad \frac{d^2h}{dt^2} - \frac{g}{\rho_0} \frac{d\rho}{dz} h = 0
$$

=> 2 cases: STABLE and UNSTABLE =>  $\frac{d\rho}{dz}$  < 0 and  $\frac{d\rho}{dz}$  > 0 [...]  $\alpha$  $0$  [..  $\frac{d\rho}{d\phi}$  is  $\frac{d\rho}{d\phi}$  and  $\frac{d\rho}{d\phi}$  or  $\frac{1}{\phi}$ 

#### **BUOYANCY FORCE**  $\boldsymbol{O}$ YANCY FORCE  $\Rightarrow$   $\frac{d\mathbf{r}}{dz}<0$  and  $\frac{d\mathbf{r}}{dz}$  $\frac{1}{2}$  cases. Shipper and its surrounding that, the parameter than its surrounding surface that its surface  $\frac{d\alpha}{dt}$ reaching its original level the particle's inertia causes it to go further downward and to become surrounded by heavier fluid. The parcel, now buoyant, is recalled upward, and oscillations

#### **We will restrict to STABLY STRATIFIED FLUIDS => N2** D FLUII dρ will restrict to STABLY STRATIFIED FLUIDS =>  $N^2$ provides the frequency of the oscillation and can thus be called the *stratification frequency*.

It goes more commonly, however, by the name of  $B$  and  $\bar{B}$  and  $\bar{C}$  are contribution of  $B$ 

# **Can we found an adimensional number with similar role as Ro ?**



**Figure 11-5** Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

**Stratification will act to restrict or minimize the vertical displacement, forcing the flow to pass around rather the over the obstacle: the greater the restriction, the greater the importance of stratification**

## Can we found an adimensional number with similar role as Ro?



Figure 11-5 Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

Stratification will act to restrict or minimize the vertical displacement, forcing the flow to pass around rather the over the obstacle: the greater the restriction, the greater the importance of stratification

$$
[\ldots] \frac{\text{vert.comv.}}{\text{horiz.div.}} = \frac{\partial_z w}{\partial_x u + \partial_y v} = \frac{\Delta z}{H} = \frac{W/H}{U/L} = \frac{U^2}{N^2 H^2} = Fr^2 \implies \text{Froude n.} \implies Fr \lesssim 1
$$



**Alps act as an obstacle: in case of high stratification vertical displacements are strongly suppressed and air quality gets worse**



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**Alps act as an obstacle: in case of high stratification vertical displacements are strongly suppressed and air quality gets worse**

**Also here [28-1-2024]**

*WHAT controls the vertical motions?*

*Rotation?*

*Stratification?*

*Both?*







Figure 14-1 Mixing of a two-layer stratified fluid with velocity shear. Rising of dense fluid and lowering of light fluid both require work against buoyancy forces and thus lead to an increase in potential energy. Concomitantly, the kinetic energy of the system decreases during mixing. Only when the kinetic-energy drop exceeds the potential-energy rise can mixing proceed spontaneously.

**= center of gravity**

### **2-layer stratified fluid + shear.**  $\mathbf{u}$  $\overline{\phantom{a}}$

Figure 14-1 Mixing of a two-layer stratified fluid with velocity sl **Example 14-1** Figure 14-1 Figure of a two-layer stratified fluid with<br>ering of light fluid both require work against buoyancy energy. Concomitantly, the kinetic energy of the system decreases during mixing. Only when the kinetic-energy drop exceeds the potential-energy rise can mixing proceed spontaneously.

$$
PE \text{ gain } = \int_0^H \rho_{\text{final}} \, gz \, dz - \int_0^H \rho_{\text{initial}} \, gz \, dz
$$
  
=  $\frac{1}{2} \rho g H^2 - \left[ \frac{1}{2} \rho_2 g \, \frac{H^2}{4} + \frac{1}{2} \rho_1 g \, \frac{3H^2}{4} \right]$   
=  $\frac{1}{8} (\rho_2 - \rho_1) g H^2.$ 

ering of light fluid both require work against buoyance and thus lead to an increase in potential to an increase in

0

0

$$
KE \text{ loss} = \int_0^H \frac{1}{2} \rho_0 u_{\text{initial}}^2 dz - \int_0^H \frac{1}{2} \rho_0 u_{\text{final}}^2 dz
$$
  
= 
$$
\frac{1}{2} \rho_0 U_2^2 \frac{H}{2} + \frac{1}{2} \rho_0 U_1^2 \frac{H}{2} - \frac{1}{2} \rho_0 U^2 H
$$
  
= 
$$
\frac{1}{8} \rho_0 (U_1 - U_2)^2 H.
$$

#### **2-layer stratified fluid + shear.** ering of light fluid both require work against buoyance and thus lead to an increase in potential to an increase in  $\mathbf{u}$ allıtu liulu<br>—————————  $\overline{P}$ initial d $\overline{P}$

Figure 14-1 Mixing of a two-layer stratified fluid with velocity sh **Eight fluid both require work against buoyancy forces and**<br>ering of light fluid both require work against buoyancy forces and energy. Concomitantly, the kinetic energy of the system decreases during mixing. C nitantly, the kinetic energy of the system decrea:<br>rop exceeds the potential-energy rise can mixing p d thus le<br>2ses du 2  $\frac{1}{2}$ d to a<br>מי מי  $\ddot{\phantom{0}}$ rincrea  $\overline{1}$ proceed spontaneously.

<span id="page-15-0"></span>0

$$
PE \text{ gain } = \int_0^H \rho_{\text{final}} \, gz \, dz - \int_0^H \rho_{\text{initial}} \, gz \, dz \qquad \text{COMPLETE VERTICAL MIX naturally possible when } KE_{loss}
$$
  
=  $\frac{1}{2} \rho g H^2 - \left[ \frac{1}{2} \rho_2 g \frac{H^2}{4} + \frac{1}{2} \rho_1 g \frac{3H^2}{4} \right]$   
=  $\frac{1}{8} (\rho_2 - \rho_1) g H^2.$   $\frac{(\rho_2 - \rho_1) g H}{\rho_0 (U_1 - U_2)^2} <$ 

0

$$
KE \text{ loss } = \int_0^H \frac{1}{2} \rho_0 u_{\text{initial}}^2 dz - \int_0^H \frac{1}{2} \rho_0 u_{\text{final}}^2 dz \quad \text{ is not too strong or the shear is} \n= \frac{1}{2} \rho_0 U_2^2 \frac{H}{2} + \frac{1}{2} \rho_0 U_1^2 \frac{H}{2} - \frac{1}{2} \rho_0 U^2 H \quad \text{overcome the stratification } (\dots)
$$
\n
$$
= \frac{1}{8} \rho_0 (U_1 - U_2)^2 H.
$$
\nIn the opposite case  $(KE_{loss} <$ 

 $\frac{J_0}{H^2}$  aturally possible when  $KE_{loss}$ 

$$
\frac{(\rho_2 - \rho_1)gH}{\rho_0(U_1 - U_2)^2} <
$$

arrier barrier barrier in the initial density variation be sufficiently weak in  $\ldots$  meaning that the density variation

have LOCALIZED MIXING o In the opposite case ( $KE_{loss}$  < only near the initial interface, n to the entire system…

### and let us explore interfacial waves of infinitesimal amplitudes. Mathematical derivations,  $\alpha$  $\bf LOCALIZED$  **MIXING**

We know (Kundu, 1990) that a sinusoidal perturbation of wavenumber  $k$  is unstab

$$
(\rho_2^2 - \rho_1^2)g < \rho_1 \rho_2 k \left( U_1 - U_2 \right)^2,
$$

or for a Boussinesq fluid ( $\rho_1 \simeq \rho_2 \simeq \rho_0$ ),

 $2(\rho_2 - \rho_1)g < \rho_0 k (U_1 - U_2)^2$ .



In a stability analysis, waves of all wavenumbers must be considered  $\rightarrow \forall k \exists$  at least a  $\tilde{k}$  large enough to cause instability shear flow is always unstable. This is known as the *Kelvin–Helmholtz instability*. Among  $\Rightarrow$  ∀k ∃ at least a  $\tilde{k}$  large enough to cause instability



**Kelvin-Helmoltz instability video**

*is always unstable to interface accorder interface according to the interface of the interface of the order of the interface of the interface of the interface of the ord short waves*

*(localized mixing)*

Kelvin-Helmholtz in-Figure 14-2 stability: (a) initial perturbation of wavenumber  $k$ , (b) temporal evolution of an unstable perturbation. The system is always unstable to short waves, which steepen, overturn and ultimately cause mixing. As waves overturn, their vertical and lateral dimensions are comparable.

### ${\bf LOCALIZED~MINIG}$ and let us explore interfacial waves of infinitesimal amplitudes. Mathematical derivations,  $\alpha$

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Kelvin-Helmholtz in-Figure 14-2 stability: (a) initial perturbation of wavenumber  $k$ , (b) temporal evolution of an unstable perturbation. The system is always unstable to short waves, which steepen, overturn and ultimately cause mixing. As waves overturn, their vertical and lateral dimensions are comparable.

**Kelvin-Helmoltz instability video**

Interfacial unstable waves grow and form ROLLS of height comparable to their width

# $\Delta H =$  mixing zone

 $\Delta H \propto \lambda_{max} = \frac{2\pi}{k_{min}} \sim \frac{\rho_0 (U_1 - U_2)^2}{2(\rho_2 - \rho_1)g}$ 

... link with H from  $KE_{loss} < PE_{gain}$ 

 $H > \Delta H$ : localized mixing  $H < \Delta H$ : complete mixing



Figure 14-3 Development of a Kelvin-Helmholtz instability in the laboratory. Here, two layers flowing from left to right join downstream of a thin plate (visible on the left of the top photograph). The upper and faster moving layer is slightly less dense than the lower layer. Downstream distance (from left to right on each photograph and from top to bottom panel) plays the role of time. At first, waves form and overturn in a two-dimensional fashion (in the vertical plane of the photo) but, eventually, threedimensional motions appear that lead to turbulence and complete the mixing. (Courtesy of Greg A. Lawrence. For more details on the laboratory experiment, see Lawrence et al., 1991.)

# $ROLLING + BREAKING = TURBULERT MIXING$



Figure 14-4 Kelvin-Helmholtz instability generated in a laboratory with fluids of two different densities and colours. (Adapted from GFD-online, Satoshi Sakai, Isawo Iizawa, Eiji Aramaki)



Figure 14-6 Kelvin-Helmholtz instability of the Sahara desert. (Photo by the second author)

# **Kelvin-Helmoltz instability**



Figure 14-5 Kelvin-Helmholtz instability in the. Algerian sky. (Photo by the second author)



# **Instability of a stratified shear flow**

*Q: For a given density stratification* ( $N^2$ ), what is the *critical velocity shear for the instability*  $\rightarrow$  *mixing?* 

# **Instability of a stratified shear flow**

*Q: For a given density stratification* ( $N^2$ ), what is the *critical velocity shear for the instability*  $\rightarrow$  *mixing?* 

*Consider a 2-d (x,z) inviscid, non-rotating, non-diffusive fluid with velocity* (*u,w*), dynamic pressure p and density anomaly  $\rho$ 

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}
$$
  

$$
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0}
$$
  

$$
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
$$
  

$$
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0.
$$

# *Basic state + small Perturbation and Linearization*

Our basic state consists of a steady, sheared horizontal flow  $[u = \bar{u}(z), w = 0]$  in a vertical density stratification  $[\rho = \bar{\rho}(z)]$ . The accompanying pressure field  $\bar{p}(z)$  obeys  $d\bar{p}/dz =$  $-g\bar{\rho}(z)$ . The addition of an infinitesimally small perturbation ( $u = \bar{u} + u'$ ,  $w = w'$ ,  $p =$  $\bar{p} + p', \rho = \bar{\rho} + \rho'$  and a subsequent linearization of the equations yield:  $\bf{O(2)} \rightarrow 0$ 

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1) 
$$
\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w' \frac{d\bar{u}}{dz} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}
$$
  
\n2) 
$$
\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}
$$
  
\n3) 
$$
\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0
$$
  
\n4) 
$$
\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + w' \frac{d\bar{\rho}}{dz} = 0.
$$

**def. Perturbation streamfunction**  $\psi : u' = +\partial \psi / \partial z$ ,  $w' = -\partial \psi$ 

#### *Basic state + small Perturbation and Linearization* Hypotheses: ∂u′  $te + s$ Id  $P$ ertu t<mark>urbation</mark> a ρ0 ∂p′ l Q  $\ln \text{cofl} \cdot N^2 = -\frac{g}{\rho} \frac{d\overline{\rho}}{d\overline{\rho}}$  $=$   $\cos t$

- a linear density vertical profile:  $N^2 = -\frac{g}{g}$  $\boldsymbol{\rho}_\mathbf{0}$  $d\overline{\rho}$  $\boldsymbol{dz}$  $= cost$ ∂≀•™•∙
- the horizontal and temporal evolution of the perturbations  $u'w' \rho' \psi'$  are formally **harmonic functions** in  $(x, t)$ , propagating with  $\lambda_x = 2\pi/k$  and having a Fourier like wave structure  $\approx e^{ik(x-ct)}$  **RIT**  $f(z) = 2$ having a Fourier-like wave structure  $\sim e^{ik(x-ct)}$  ...  $BUTf(z) = ?$  $\overline{v}$ µ¤ of the nerturbations  $u'w'$  o' $u'$  are
- substituting in Eq. 4 we obtain the **density perturbation**  $\rho' = \frac{-N^2 \rho_0}{\sigma(\overline{n} \rho)}$  $g(\bar{u}-c)\psi$ **Introducion** substituting in Eq. 4 we obtain the **density perturbation**  $\rho' = \frac{-N^2 \rho_0}{a(\overline{u}-c)w}$  $g(u-c)\psi$ <br>Ancy frequency is a fact a form of the Fourier Goldstein squation soverning the
- [*after some maths*] we obtain the Taylor-Goldstein equation governing the vertical structure of a perturbation  $\psi' = \psi(z)e^{il(x-ct)}$  in a stratified shear flow:  $i$  [*after some maths*] we obtain the Taylor-Goldstein equation governing the

$$
(\bar{u}-c)\,\left(\frac{d^2\psi}{dz^2}\,-\,k^2\psi\right)\,\,+\,\,\left(\frac{N^2}{\bar{u}-c}\,-\,\frac{d^2\bar{u}}{dz^2}\right)\psi\,\,=\,\,0
$$

• with the boundary conditions  $w'(0) = w'(H) = 0 \Rightarrow \psi(0) = \psi(H) = 0$ in a domain vertically bounded by two horizontal planes at  $z = 0$ , H we obtain an eigenvalue problem which in general may have complex eigenvalues  $c = c_r + ic_i$  and  $c^* = c_r - ic_i$  $T_{\text{th}}$  is called the *Taylor–Goldstein equation*  $G_{\text{th}}$  (Taylor). It governs  $T_{\text{th}}$  (Taylor). It governs  $G_{\text{th}}$ with the boundary conditions w  $(v) = w(11) = v \Rightarrow \varphi(v) = \varphi(11) = v$ In a domain vertically bounded by two horizontal planes at  $z = 0, \pi$  we obtain an eigenvalue problem which in general may have complex

# *Basic state + small Perturbation and Linearization*

- from  $c = c_r + ic_i$  and  $c^* = c_r ic_i \Rightarrow \psi' \sim e^{ik[x (c_r \pm ic_i)t]} \sim e^{ikx} e^{-ikc_r t} e^{\mp kc_i t}$
- real exponential: the presence of  $c_i \neq 0 \Rightarrow \exists$  at least one unstable mode
- the flow is stable  $\Leftrightarrow$   $c_i = 0$

#### **Basic state + small Perturbation and Linearization** speeds c are purely real.  $B = B \cup B$  is interesting to solve problem (14.10) and (14.11) in the general case of and (14.11) in the general case of and (14.11) in the general case of an interesting of an interesting of an interesting of an interestin + small Pertunbation and Lineanzation speeds c are purely real. Basic state + small Perturbation arbitrary shear flow use will limit our shear flow use  $\alpha$  in Section 10.2, to deriving integrals in Section 10.2, to derive integrals in Section 10.2, to derive integrals in Section 10.2, to derive in Section 10.2, to d  $\bm \mu$  small Perturhation and Linearizatio  $\ddot{\phantom{a}}$  original vitation  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$ tate + small Perturbation and Li neariza tic  $\overline{\mathbf{a}}$

- from  $c = c_r + ic_i$  and  $c^* = c_r ic_i \Rightarrow \psi' \sim e^{ik[x (c_r \pm ic_i)t]} \sim e^{ikx} e^{-ikc_r t} e^{\mp kc_i t}$ arbitrary shear flow units in  $ik[x-(c+ic)]$  integrated integrals in Section 10.2, to  $\pm kct$  $\alpha$  is  $c_r - c_r - c_i \rightarrow \psi \sim e$  is the most powerful one is the most powerful one is the most powerful one is  $\psi$  $B = \frac{1}{2} \int_{0}^{1} \frac{1}{2} e^{-\frac{(x-1)(x-1)}{x}} dx$  $\lim_{\epsilon \to 0} c - c_r + i c_i$  and  $c - c_r - i c_i \to \psi \sim e^{-c}$ constraints. A variety of such constraints can be established, but the most powerful one is and  $c^* = c_s - ic_s \Rightarrow u/v \sim u$ From  $c = c + ic$  and  $c^* = c - ic$   $\Rightarrow$   $u^* \sim e^{ik|x-1|}$  $\sum_{i=1}^n$ • from  $c = c_r + ic_i$  and  $c^* = c_r - ic_i \Rightarrow \psi' \sim e^{i k [x - (c_r \pm ic_i)t]} \sim e^{i k x} e^{-i k c_r t}$  $\psi \rightarrow \psi' \sim e^{i k [x - (c_r \pm i c_i)t]}$  $\overline{1}$ u¯ − c  $x_{\ell}$  $, -ikcrt$ <sub> $e^{\mp k}$ </sub>  $c_i t$ 
	- real exponential: the presence of  $c_i \neq 0 \Rightarrow \exists$  at least one unstable mode construction of  $\alpha$  variety of  $\alpha$  x  $\alpha$  and  $\alpha$  is the most powerful one is  $\alpha$ real exponential: the presence of  $c_i \neq 0 \Rightarrow \exists$  at least one unstable mode ponemual, the presence of  $c_i \neq 0 \Rightarrow z$  at lea  $\sum_{n=1}^{\infty}$ +  $\vdots$  0  $\Rightarrow$  3 at least one unstable mode
- the flow is stable  $\Leftrightarrow c_i = 0$  $\mathbf{r} = \mathbf{0}$ 
	- using integral constraints we can analyze the T.-G. eq.: with  $\psi \;=\; \sqrt{\bar{u} - c}\;\phi$  (  $= 0$ using integral constraints we can analyze the  $T.-G$ .  $: T.-G. eq.:$  $\rm{vit}$  $\bullet$  dsing integral constraints we can analyze the 1.-O. eq., with  $\in$  T.-G. eq.: with  $\psi = \sqrt{\overline{u}} -$ If constraints we can analyze the 1.-O. eq., with  $\psi =$ of the domain, and utilizing conditions (14.14), we obtain:

$$
\frac{d}{dz}\left[(\bar{u}-c)\frac{d\phi}{dz}\right] - \left[k^2(\bar{u}-c) + \frac{1}{2}\frac{d^2\bar{u}}{dz^2} + \frac{1}{\bar{u}-c}\left(\frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 - N^2\right)\right]\phi = 0
$$
  

$$
\phi(0) = \phi(H) = 0.
$$

$$
\phi(0) \quad = \quad \phi(H) \ = \ 0.
$$

• [*after some math and using the BCs*] we obtain a **complex equation**: J we obtain a **comp c** equation: • *Lafter some math and using the BCs* we obtain a **complex**  $M_{\rm H}$  , integration (14.13) by the complex complex complex conjugate  $p_{\rm eff}$  $\mathbf{u}$ , and utilizing conditions (14.14), we obtain • [after some math and using the BCs] we obtain a complex r<br>!  $\mathbf{q}$  $\mathbf{r}$ using the  $\overline{B}$  $Cs$ ] we obtain a complex equation we obtain a

$$
\int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)^2 \right] \frac{|\phi|^2}{\bar{u} - c} dz = \int_0^H \left( \bar{u} - c \right) \left( \left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) dz + \frac{1}{2} \int_0^H \frac{d^2 \bar{u}}{dz^2} |\phi|^2 dz
$$

• whose imaginary part is  ${\bf t} \ \mathop{\bf{is}}\nolimits \quad c_i \ \int_0^H \ \left[ N^2 \ - \ \frac{1}{4} \left( \frac{d \bar{u}}{d z} \right)^2 \right] \ \frac{|\phi|^2}{|\bar{u} - c|^2} \ dz \ = \ - \ c_i \int_0^H \ \left( \left| \frac{d \phi}{d z} \right|^2 \ + \ k^2 |\phi|^2 \right) \ .$  $c_i \; \int_0^H$  $N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)$  $\frac{1}{4}$  $\langle dz \rangle$  $c_i \left[ \begin{array}{c} N^2 - \frac{1}{4} \left( \frac{\alpha \alpha}{dz} \right) \end{array} \right]$  $\int_0^H$  $\begin{bmatrix} N & - \end{bmatrix}$  $\frac{1}{4}$  $\frac{du}{dt}$ dz \* \* \* \*  $\mathbf{C}$  $\sqrt{|\bar{u}-c|^2}$  $\overline{a}$  $\alpha$  $\overline{1}$  $\int_0^H$  $\overline{a}$  $\int \frac{a}{2}$  $\left(\left|\frac{1}{dz}\right| + k^2|\phi|^2\right) dz,$  $\overline{\phantom{a}}$ rgina<br>  $\mathop{\text{L}}\nolimits$  $\frac{1}{2}$  $\int \mathbf{S} \quad c_i$  $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$  $\frac{1}{2}$   $\left(\frac{u}{2}\right)$   $\left| \frac{u}{2} \right|$  $\bullet$  −  $\bullet$  $v \}$  $\boldsymbol{\mathsf{O}}$  $\overline{\mathbf{S}}$ imagi  $\mathbf{r}$  $\mathcal{E}$  $\overline{\phantom{0}}$ **att is**  $c_i \int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)^2 \right] \frac{|\phi|^2}{|\bar{u} - c|^2} dz = - c_i \int_0^H \left( \left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right)$ art is  $c_i \int_0$  $\int$ <sup>H</sup> 0  $\bigg[ N^2 \ - \ \frac{1}{4}$  $\left(\frac{d\bar{u}}{dz}\right)^2$   $\frac{|\phi|^2}{|\bar{u}-\alpha|}$ |<br>|<br>|  $|\bar{u}-c|$  $\frac{1}{2}$  dz = -  $c_i$  $\int$ <sup>H</sup> 0  $\left(\rule{0pt}{10pt}\right.$  $d\phi$ dz  $\begin{array}{|c|c|} \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{|c|}{4} \multicolumn{1}{|c|}{5} \multicolumn{1}{|c|}{6} \multicolumn{1}{|c|$ 2  $+ k^2 |\phi|^2$  $dz$ ,

#### **Basic state + small Perturbation and Linearization** speeds c are purely real.  $B = B \cup B$  is interesting to solve problem (14.10) and (14.11) in the general case of and (14.11) in the general case of and (14.11) in the general case of an interesting of an interesting of an interesting of an interestin + small Pertunbation and Lineanzation speeds c are purely real. Basic state + small Perturbation arbitrary shear flow use will limit our shear flow use  $\alpha$  in Section 10.2, to deriving integrals in Section 10.2, to derive integrals in Section 10.2, to derive integrals in Section 10.2, to derive in Section 10.2, to d  $\bm \mu$  small Perturhation and Linearizatio  $\ddot{\phantom{a}}$  original vitation  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$  =  $\mathcal{L}_{\mathcal{M}}$ tate + small Perturbation and Li neariza tic  $\overline{\mathbf{a}}$

- from  $c = c_r + ic_i$  and  $c^* = c_r ic_i \Rightarrow \psi' \sim e^{ik[x (c_r \pm ic_i)t]} \sim e^{ikx} e^{-ikc_r t} e^{\mp kc_i t}$ arbitrary shear flow units in  $ik[x-(c+ic)]$  integrated integrals in Section 10.2, to  $\pm kct$  $\alpha$  is  $c_r - c_r - c_i \rightarrow \psi \sim e$  is the most powerful one is the most powerful one is the most powerful one is  $\psi$  $B = \frac{1}{2} \int_{0}^{1} \frac{1}{2} e^{-\frac{(x-1)(x-1)}{x}} dx$  $\lim_{\epsilon \to 0} c - c_r + i c_i$  and  $c - c_r - i c_i \to \psi \sim e^{-c}$ constraints. A variety of such constraints can be established, but the most powerful one is and  $c^* = c_s - ic_s \Rightarrow u/v \sim u$ From  $c = c + ic$  and  $c^* = c - ic$   $\Rightarrow$   $u^* \sim e^{ik|x-1|}$  $\sum_{i=1}^n$ • from  $c = c_r + ic_i$  and  $c^* = c_r - ic_i \Rightarrow \psi' \sim e^{i k [x - (c_r \pm ic_i)t]} \sim e^{i k x} e^{-i k c_r t}$  $\psi \rightarrow \psi' \sim e^{i k | \Omega|}$  $\overline{f}$  $\overline{1}$ u¯ − c  $x_{\ell}$  $, -ikcrt$ <sub> $e^{\mp k}$ </sub>  $c_i t$ 
	- real exponential: the presence of  $c_i \neq 0 \Rightarrow \exists$  at least one unstable mode construction of  $\alpha$  variety of  $\alpha$  x  $\alpha$  and  $\alpha$  is the most powerful one is  $\alpha$ real exponential: the presence of  $c_i \neq 0 \Rightarrow \exists$  at least one unstable mode ponemual, the presence of  $c_i \neq 0 \Rightarrow z$  at lea  $\sum_{n=1}^{\infty}$ +  $\theta \Rightarrow \exists$  at least one unstable mode
- the flow is stable  $\Leftrightarrow c_i = 0$  $\mathbf{r} = \mathbf{0}$ 
	- using integral constraints we can analyze the T.-G. eq.: with  $\psi \;=\; \sqrt{\bar{u} - c}\;\phi$  (  $= 0$ using integral constraints we can analyze the  $T.-G$ .  $: T.-G. eq.:$  $\rm{vit}$  $\bullet$  dsing integral constraints we can analyze the 1.-O. eq., with  $\in$  T.-G. eq.: with  $\psi = \sqrt{\overline{u}} -$ If constraints we can analyze the 1.-O. eq., with  $\psi =$ of the domain, and utilizing conditions (14.14), we obtain:

$$
\frac{d}{dz}\left[(\bar{u}-c)\frac{d\phi}{dz}\right] - \left[k^2(\bar{u}-c) + \frac{1}{2}\frac{d^2\bar{u}}{dz^2} + \frac{1}{\bar{u}-c}\left(\frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 - N^2\right)\right]\phi = 0
$$
  

$$
\phi(0) = \phi(H) = 0.
$$

$$
\phi(0) \quad = \quad \phi(H) \ = \ 0.
$$

• [*after some math and using the BCs*] we obtain a **complex equation**: J we obtain a **comp c** equation: • *Lafter some math and using the BCs* we obtain a **complex**  $M_{\rm H}$  , integration (14.13) by the complex complex complex conjugate  $p_{\rm eff}$  $\mathbf{u}$ , and utilizing conditions (14.14), we obtain • [after some math and using the BCs] we obtain a complex r<br>!  $\mathbf{q}$  $\mathbf{r}$ using the  $\overline{B}$  $Cs$ ] we obtain a complex equation we obtain a

$$
\int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)^2 \right] \frac{|\phi|^2}{\bar{u} - c} dz = \int_0^H \left( \bar{u} - c \right) \left( \left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) dz + \frac{1}{2} \int_0^H \frac{d^2 \bar{u}}{dz^2} |\phi|^2 dz
$$

- whose imaginary part is  $\int_0^H \left[ \int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{d \bar u}{dz} \right)^2 \right] \frac{|\phi|^2}{|\bar u - c|^2} dz = - c_i \int_0^H \left( \left| \frac{d \phi}{dz} \right|^2 + k^2 |\phi|^2 \right) \right]$  $c_i \; \int_0^H \; \Bigg|$  $N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)$  $\frac{1}{4}$  $\langle dz \rangle$  $c_i \left[ \begin{array}{c} N^2 - \frac{1}{4} \left( \frac{\alpha \alpha}{dz} \right) \end{array} \right]$  $\int_0^H$  $\begin{bmatrix} N & - \end{bmatrix}$  $\frac{1}{4}$  $\frac{du}{dt}$  $\frac{az}{2}$ \* \* \* \*  $\mathbf{C}$  $\sqrt{|\bar{u}-c|^2}$  $\overline{a}$  $\alpha$  $\overline{1}$  $\int_0^H$  $\overline{a}$  $\int \frac{a}{2}$  $\left(\left|\frac{1}{dz}\right| + k^2|\phi|^2\right) dz,$  $\overline{\phantom{a}}$ rgina<br>  $\mathop{\text{L}}\nolimits$  $\frac{1}{2}$  $\int \mathbf{S} \quad c_i$  $\left\| \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \right\|$  $\frac{1}{2}$   $\begin{bmatrix} a & b \end{bmatrix}$  $\bullet$  −  $\bullet$  $v \}$  $\boldsymbol{\mathsf{O}}$  $\overline{\mathbf{S}}$ imagi  $\mathbf{r}$  $\mathcal{E}$  $\overline{\phantom{0}}$ art is  $c_i \int_0^H \left\| N^2 - \frac{1}{4} \left( \frac{d\bar{u}}{dz} \right)^2 \right\| \frac{|\phi|^2}{|\bar{u} - c|^2} dz = - c_i \int_0^H \left( \left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right)$ art is  $c_i \int_0$  $\int$ <sup>H</sup> 0  $\bigg[\!N^2 \; - \; \frac{1}{4}\! \bigg]$  $\left(\frac{d\bar{u}}{dz}\right)^2\left|\frac{|\phi|^2}{|\bar{u}-\alpha|}\right|$ ׀<br>׀  $|\bar{u}-c|$  $\frac{1}{2}$  dz = -  $c_i$ #\*  $\int$ <sup>H</sup> 0  $\left(\rule{0pt}{10pt}\right.$  $d\phi$ dz  $\begin{array}{|c|c|} \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{|c|}{4} \multicolumn{1}{|c|}{5} \multicolumn{1}{|c|}{6} \multicolumn{1}{|c|$ 2  $+ k^2 |\phi|^2$  $\setminus$  $dz$ ,
- IF  $N^2 > \frac{1}{4}$  $\overline{4}$  $d\bar{u}$  $\overline{dz}$  $\overline{2}$  $\Rightarrow c_i \cdot ( > 0 ) = -c_i \cdot ( > 0 ) \Rightarrow c_i = 0 \Rightarrow \text{STABLE}$  $\left(d\overline{u}\right)^2$  $\frac{1}{4}$  absolute value of complex quantities. The imaginary part of complex quantities. The imaginary part of complex  $\frac{1}{4}$  $t_{\text{max}} = \frac{1}{d\bar{u}} \sqrt{2}$  $\bullet$  IF  $\frac{1}{4}$   $\frac{1}{4}$  $\frac{4}{d z}$   $\frac{1}{d \bar{u}} \frac{d^2}{2}$ ',  $\left(\begin{array}{cc} 0 & \lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{array}\right)$  $=$   $\Omega$  $\sim$   $\sigma$ <sub>2</sub>  $\sim$   $\sigma$ <sub>1</sub>  $\sim$  0  $\sim$ 
	- IF  $N^2 < \frac{1}{4}$  $\overline{4}$  $d\bar{u}$  $\overline{dz}$  $\overline{2}$  $\Rightarrow c_i \cdot (< 0) = -c_i \cdot (> 0) \Rightarrow \forall c_i \Rightarrow$  STABLE or UNSTABLE ID  $M^2$ ,  $\frac{1}{d\bar{u}}\Big)^2$ , and  $\Big(\leq 0\Big)$ , we obtain into the set intervals in i.  $(> 0) =$  $\Rightarrow \forall c_i \Rightarrow S$ > ST  $\overline{AB}$  $\leq$  $\mathbf{1}$ ٦  $\left(\frac{d\overline{u}}{dz}\right)^2 \Rightarrow$  $c_i \cdot (< 0)$ '<br>'  $\mathcal{L}$  $\equiv$  :  $\frac{12}{4}$ \* 1  $Z$ ) &  $\ddot{\phantom{1}}$

# <span id="page-28-0"></span>*Q: For a given density stratification (N<sup>2</sup>), what critical velocity shear for the instability*  $\rightarrow$  *mix*

- [from  $c = c_r + ic_i$  and  $c^* = c_r ic_i \Rightarrow \psi' \sim e^{ik[x (c_r \pm ic_i)t]} \sim e^{ikx}e^{-\frac{c_r}{c}t}$
- def. Richardson number  $Ri =$  $N^2$  $d\bar{u}$  $d\mathbf{z}$  $\frac{1}{2}$   $\rightarrow$   $\begin{cases} \frac{1}{2} & \text{if } Ri > \frac{1}{4} \\ \frac{1}{2} & \text{if } Ri > \frac{1}{4} \end{cases}$  $\overline{4}$  $\Rightarrow$  STABLE IF Ri  $\lt \frac{1}{4}$  $\overline{4}$  $\Rightarrow$  STABLE or UN.
- *sufficient condition* for stability is  $N^2 > \frac{1}{4}$  $\overline{4}$  $d\bar{u}$  $\overline{dz}$ <sup>2</sup>:  $Ri > \frac{1}{4}$  ... ( $\Rightarrow$  STABLI
- *necessary condition* for instability is  $N^2 < \frac{1}{4}$  $\overline{4}$  $d\bar{u}$  $\overline{dz}$  $\frac{2}{3}$ :  $Ri < \frac{1}{4}$ ... ( $\Leftarrow$  UNS
- But measurements in atmosphere / ocean / laboratory indicate that Ri *reliable condition of instability* ⇒  $d\bar{u}$  $\overline{dz}$  $\overline{2}$  $> 4N^2$
- if the shear flow has linear variations of velocity and density we may re 2-layer flow case, finding a similarity with  $KE_{loss}$  >  $PE_{gain}$  (complete  $\overline{\text{mixing}}$   $\Rightarrow$  Ri is the ratio between PE and KE !!!
- $Ri = \frac{potential$  energy barrier that mixing if occurring must overcome

#### Q: can we say something more about the properties of the growing perturbation? parts of the wave velocity c can be derived by inspection integrals. This analysis, the certain integrals. This analysis,  $\mathbf{r}$  $Q$ : can we say something more about the  $\beta$ and growing pertandation: perturbation, defined by ∂a ∂a shear flow. To begin, we introduce the vertical displacement a caused by the small wave , can we sa rc  $\overline{\phantom{a}}$ n  $\overline{y}$

- if we introduce the vertical displacement  $\alpha$  caused by the small wave perturbation:  $\partial a$   $\partial a$  which corresponds to  $rac{\partial u}{\partial t} + \bar{u}$ ∂a  $rac{\partial x}{\partial x} = w$ or  $(\bar{u} - c) a = - \psi.$  $\int a \sim a(z) e^{i\kappa(x-t)}$  and  $w' = -\partial \psi / \partial x$  $= w \left( \frac{\text{wmin}}{\sigma} \frac{\text{vmin}}{\sigma} \right)$  $\partial t$   $\partial x$   $\partial x$   $\partial t$   $\partial t$ 
	- we can rewrite the T.-G. eq. obtaining an eigenvalue problem for  $a$ :

$$
\frac{d}{dz}\left[ (\bar{u} - c)^2 \frac{da}{dz} \right] + [N^2 - k^2 (\bar{u} - c)^2] a = 0
$$
  

$$
a(0) = a(H) = 0.
$$

#### Q: can we say something more about the properties of the growing perturbation? parts of the wave velocity c can be derived by inspection integrals. This analysis, the certain integrals. This analysis,  $\mathbf{r}$  $Q$ : can we say something more about the  $\beta$ and growing pertandation: perturbation, defined by ∂a ∂a shear flow. To begin, we introduce the vertical displacement a caused by the small wave : can we say s rc  $\overline{\phantom{a}}$ n ng pertunpation? <sup>∂</sup><sup>x</sup> <sup>=</sup> <sup>w</sup> parts of the wave velocity c can be derived by inspection of certain integrals. This analysis, the certain integrals. This analysis,  $\alpha$ ducan we say something me  $\alpha$  is summarize surjession in the context of stratified in the context of strategy  $\alpha$  $\mathbf{f}_{\mathsf{h}}$  flow. The vertical displacement and  $\mathbf{f}_{\mathsf{h}}$  $\Omega$  con we cause problem (14.10) and  $\theta$  $\boldsymbol{\mathsf{Q}}$ .  $\frac{1}{2}$

- if we introduce the vertical displacement  $\alpha$  caused by the small wave perturbation:  $\partial a$   $\partial a$  which corresponds to  $rac{\partial u}{\partial t} + \bar{u}$ ∂a  $rac{\partial x}{\partial x} = w$  $(\bar{u} - c) a = - \psi.$  $\int a \sim a(z) e^{i\kappa(x-t)}$  and  $w' = -\partial \psi / \partial x$ if we introduce the vertical displacement  $\boldsymbol{a}$  caused by the small wave  $= w \left( \frac{\text{wmin}}{\sigma} \frac{\text{vmin}}{\sigma} \right)$ We then eliminate ψ from (14.10) and (14.11) and obtain an equivalent problem for the variable a:  $\frac{100}{2}$   $\frac{1$  $\partial t$   $\partial x$   $\partial \alpha$  $\text{perturbation:}\quad{\partial a}\quad \text{and}\quad \text{which correspond}$ and  $\overline{\partial t}$  the boundary conditions yields y  $a \sim a(z) e^{ik(x-ct)}$ 
	- we can rewrite the T.-G. eq. obtaining an eigenvalue problem for  $a$ : + # <sup>N</sup><sup>2</sup> <sup>−</sup> <sup>k</sup>2(¯<sup>u</sup> <sup>−</sup> <sup>c</sup>) We then eliminate ψ from (14.10) and (14.11) and obtain an equivalent problem for the ite the T.-G.  $\epsilon$ . obtaining an eigenvalue

$$
\frac{d}{dz}\left[ (\bar{u} - c)^2 \frac{da}{dz} \right] + [N^2 - k^2 (\bar{u} - c)^2] a = 0
$$
  

$$
a(0) = a(H) = 0.
$$

- [*after some math and using the BCs*]: l*after* some z man<br>'  $J_0$  and  $J_0$  $\begin{array}{cc} \textbf{a} & \textbf{b} & \textbf{c} \\ \textbf{b} & \textbf{c} & \textbf{c} \end{array}$  for  $\begin{array}{cc} (u - c)^2 P \, dz = \int_0^{\infty} N^2 |a|^2 \, dz \end{array}$ and use of the boundary conditions  $\mathcal{L}_{\mathcal{A}}$ Lattor some math and using the RCs]  $\int^H$  (= 0.2 p integration over the domain over the domain over the domain ager some man and using the  $\int$ <sup>H</sup> 0  $(\bar{u} - c)^2 P \, dz =$  $\int$ <sup>H</sup> 0  $N^2|a|^2\;dz$  $\int_{0}^{\infty} \int_{0}^{\infty} f(x) dx$  $\overline{a}$  $\mathcal{C}^H$ • atter some math and using the BCs  $\left| \cdot \right\rangle$  if  $\bar{u} = c$  $r_{\rm 0}$  and the local coupling between the  $\int_{0}$  is precisely what the flow is pre
- through the analysis of the real and the imaginary parts of the integral and requiring the instability condition  $c_i \neq 0$ : requiring the instability condition  $c_i \neq 0$ : of the real and the imaginary parts of the integral and of the real and the imaginary parts of the integral  $2 \times 2$  is a non-zero positive quantity. The imaginary part of the imaginary part of the imaginary part of this contribution through the analysis of the real and the imaginary parts of the integral and where  $\bullet$  $\frac{1}{2}$  is instability (circumstability instability  $\frac{1}{2}$ ), contains the minimum and minimum an maxim regular  $\frac{1}{\sqrt{2}}$ the instability cond  $\ddot{\ }$  . ion  $c_i \neq 0$ :

 $\triangleright U_{\text{min}} < c_r < U_{\text{max}}$  the growing perturbation travels with the flow at some intermediate speed ring perturbation travels with the flow at some intermediate speed  $\blacktriangleright$   $U_{\min} < c_r < U_{\max}$  the growing perturbation travels with the flow at some inter- $\triangleright U_{\text{min}} < c_r < U_{\text{max}}$  the growing perturbation travels with the flow at some intermediate speed

$$
\triangleright \left( c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 \le \left( \frac{U_{\max} - U_{\min}}{2} \right)^2 \text{ in the complex plane, } c = c_r + ic_i \text{ must lie within the circle with the range of } \bar{u} \text{ as the diameter on the real axis}
$$

#### Q: can we say something more about the properties of the growing perturbation? <sub>2</sub><br>2004 = co 7 y  $\frac{1}{2}$  $\Omega$ <sub>th</sub>ere we cause on this at more che **Respect to the local flow in the flow in the flow is precisely wave and the flow is precisely what is precise** allows the wave to extract energy from the flow and to grow at its expense.

• instability condition  $c_i \neq 0$ : equation is instability (ci  $\frac{1}{2}$ ), correction the minimum and minimum and minimum and minimum and minimum and minimum and correction  $\alpha$ • instability condition  $c_i \neq 0$ :

 $\triangleright U_{\text{min}} < c_r < U_{\text{max}}$  the growing perturbation travels with the flow at some intermediate speed

$$
\triangleright \left( c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 \le \left( \frac{U_{\max} - U_{\min}}{2} \right)^2 \text{ in the complex plane } (c_r, c_i): c = c_r + ic_i \text{ must } \\ \text{lie within the circle with the range of } \bar{u} \text{ as the diameter on the real axis}
$$

• since instability requires  $c_i > 0 \Rightarrow \psi' \sim e^{ikx} e^{-ikt} e^{kc_i t}$  we are interested in the upper part of the circle  $\Rightarrow$  Howard semicircle theorem nce instability re uires  $c_i > 0 \Rightarrow \psi' \sim \epsilon$ 

•  $c_i \leq \frac{U_{max} - U_{min}}{2} \Rightarrow kc_i \leq \frac{k}{2}(U_{max} - U_{min}) \Rightarrow$  the perturbation does not grow to infinite



https://www.dartmouth.edu/~cushman/books/EFM/chap5.pdf







https://www.youtube.com/watch?v=aLvk2cbsvzM https://www.youtube.com/watch?v=0YatiDf9A8A



# **THANKS**