

ES 1

$$5. \{q, p\} = \{ \tilde{p}^{1/3} e^{\gamma \tilde{q}}, \tilde{p}^\alpha e^{-\beta \tilde{q}} \} =$$

$$= (\gamma e^{\gamma \tilde{q}} \tilde{p}^{1/3}) (\alpha \tilde{p}^{\alpha-1} e^{-\beta \tilde{q}}) - (\frac{1}{3} \tilde{p}^{-2/3} e^{\gamma \tilde{q}}) (-\beta \tilde{p}^\alpha e^{-\beta \tilde{q}})$$

$$= e^{\gamma \tilde{q}} e^{-\beta \tilde{q}} \tilde{p}^{\alpha - \frac{2}{3}} (\alpha \gamma + \frac{\beta}{3})$$

$$\gamma = \beta \quad \alpha = \frac{2}{3} \quad \frac{2}{3}\beta + \frac{\beta}{3} = \beta$$

$$\beta = 1 \quad \gamma = 1 \quad \alpha = \frac{2}{3}$$

$$\leadsto q = \tilde{p}^{1/3} e^{\tilde{q}} \quad p = \tilde{p}^{2/3} e^{-\tilde{q}}$$

$$6. H(p, q) = \frac{1}{2} p^2 q^2 \quad \xrightarrow[\substack{c=1 \\ k_0=0}]{K} H(p(\tilde{p}, \tilde{q}), q(\tilde{p}, \tilde{q})) = \frac{1}{2} \tilde{p}^{2/3} e^{2\tilde{q}} \tilde{p}^{4/3} e^{-2\tilde{q}} = \frac{1}{2} \tilde{p}^2$$

$$\rightarrow \begin{aligned} \dot{\tilde{p}} &= 0 \\ \dot{\tilde{q}} &= \tilde{p} \end{aligned}$$

$$\begin{aligned} \tilde{p}(t) &= \tilde{p}_0 \\ \tilde{q}(t) &= \tilde{p}_0 t + \tilde{q}_0 \end{aligned}$$

$$p(t) = \tilde{p}_0^{2/3} e^{-\tilde{q}_0} e^{-\tilde{p}_0 t}$$

$$q(t) = \tilde{p}_0^{1/3} e^{\tilde{q}_0} e^{\tilde{p}_0 t}$$

$$7. p(0) = \tilde{p}_0^{2/3} e^{-\tilde{q}_0} \equiv p_0 \quad \rightarrow \quad \tilde{p}_0 = p_0 q_0$$

$$q(0) = \tilde{p}_0^{1/3} e^{\tilde{q}_0} \equiv q_0 \quad \rightarrow \quad \tilde{q}_0 = \log \frac{q_0^{2/3}}{p_0^{1/3}}$$

$$\Phi_t(p, q) = \begin{pmatrix} p e^{-pq t} \\ q e^{pq t} \end{pmatrix}$$

$$t \ll 1 \Rightarrow \Phi_t(p, q) = \begin{pmatrix} p - p^2 q t \\ q + p q^2 t \end{pmatrix}$$

$$\delta p = t \left\{ p, \frac{1}{2} p^2 q^2 \right\} = -p^2 q t$$

$$\delta q = t \left\{ q, \frac{1}{2} p^2 q^2 \right\} = p q^2 t$$

ES2

$$x_p = R \sin \theta$$

$$y_p = -R \cos \theta$$

1)

$$T = \frac{M}{2} \dot{s}^2 + \frac{m}{2} R^2 \dot{\theta}^2 \quad a = \begin{pmatrix} M & 0 \\ 0 & mR^2 \end{pmatrix}$$

$$V = -mgR \cos \theta + \frac{1}{2} k \left(\underbrace{(R \sin \theta - s)^2 + R^2 \cos^2 \theta}_{R^2 - 2Rs \sin \theta + s^2} \right)$$

$$L = \frac{M}{2} \dot{s}^2 + \frac{m}{2} R^2 \dot{\theta}^2 + mgR \cos \theta - \frac{1}{2} k s^2 + kRs \sin \theta - \frac{kR^2}{2}$$

const. imple. in eq. Lag.

$$2) \left. \begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} &= M \dot{s} \\ \frac{\partial L}{\partial s} &= -ks + kR \sin \theta \end{aligned} \right\} \ddot{s} = -\frac{k}{M} s + \frac{k}{M} R \sin \theta$$

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= mR^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mgR \sin \theta + kRs \cos \theta \end{aligned} \right\} \ddot{\theta} = -\frac{g}{R} \sin \theta + \frac{k}{mR} \frac{s}{R} \cos \theta$$

3) $s \rightarrow -s \quad \theta \rightarrow -\theta$

4) $V = -mgR \cos \theta + \frac{k}{2} s^2 - kRs \sin \theta$

$$\frac{\partial V}{\partial s} = ks - kR \sin \theta = 0 \rightarrow s = R \sin \theta$$

$$\frac{\partial V}{\partial \theta} = mgR \sin \theta - kRs \cos \theta = 0 \rightarrow s = \frac{mg}{k} \frac{\sin \theta}{\cos \theta}$$

$$\rightarrow R \frac{\sin \theta}{\cos \theta} \left(\cos \theta - \frac{mg}{KR} \right) = 0$$

$$\theta = 0 \quad \leadsto \quad s = 0$$

($\theta = \pi$ non fa parte delle config.)

$$\exists \text{ se } \frac{mg}{KR} \leq 1 \quad \theta_{\pm}^* = \pm \arccos \frac{mg}{KR} \quad \leadsto \quad s_{\pm}^* = \pm R \sqrt{1 - \frac{mg}{KR}} = R \sin \theta_{\pm}^*$$

$$\partial^2 V = \begin{pmatrix} K & -KR \cos \theta \\ -KR \cos \theta & mgR \cos \theta + KR s \sin \theta \end{pmatrix}$$

$$\partial^2 V_{(0,0)} = \begin{pmatrix} K & -KR \\ -KR & mgR \end{pmatrix} \rightarrow \det = mgKR - (KR)^2 = (KR)^2 \left(\frac{mg}{KR} - 1 \right)$$

STAB se $\frac{mg}{KR} > 1$

INSTAB se $\frac{mg}{KR} < 1$

$$\partial^2 V_{(s_{\pm}^*, \theta_{\pm}^*)} = \begin{pmatrix} K & -mg \\ -mg & \cancel{\left(\frac{mg}{KR}\right)^2} \frac{1}{K} + KR^2 \left(1 - \cancel{\left(\frac{mg}{KR}\right)^2} \right) \end{pmatrix} =$$

$$= \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix} \rightarrow \det = (KR)^2 - (mg)^2 = (KR)^2 \left(1 + \frac{mg}{KR} \right) \left(1 - \frac{mg}{KR} \right)$$

STAB quod est, cioè $\frac{mg}{KR} < 1$

$$5) \quad M = m \quad \frac{\omega g}{kR} < \frac{1}{10} < 1$$

$$\text{pt: stab.} \rightarrow (s, \theta) = (s_{\pm}^*, \theta_{\pm}^*)$$

$$A = \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \quad B = \begin{pmatrix} k & -\omega g \\ -\omega g & kR^2 \end{pmatrix}$$

$$\det(B - \lambda A) = (k - \lambda m)R^2(k - \lambda m) - (\omega g)^2 = 0 \rightarrow \frac{1}{m^2 R^2}$$

$$\rightarrow \left(\lambda - \frac{k}{m}\right)^2 - \left(\frac{\omega g}{R}\right)^2 = 0$$

$$\lambda_{1,2} = \frac{k}{m} \pm \frac{\omega g}{R}$$

$$s = \pm s_* + \delta s$$

$$\theta = \pm \theta_* + \delta \theta$$

$$\begin{aligned} L_{\text{lin}} &= \frac{1}{2} (\delta s, \delta \dot{\theta}) \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \begin{pmatrix} \delta \dot{s} \\ \delta \ddot{\theta} \end{pmatrix} - \frac{1}{2} (\delta s, \delta \theta) \begin{pmatrix} k & -\omega g \\ -\omega g & kR^2 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \theta \end{pmatrix} \\ &= \frac{m}{2} \delta \dot{s}^2 + \frac{\omega R^2}{2} \delta \dot{\theta}^2 - \frac{k}{2} \delta s^2 + \omega g \delta s \delta \theta - \frac{kR^2}{2} \delta \theta^2 \end{aligned}$$

$$\begin{pmatrix} \delta \dot{s} \\ \delta \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1/m & \\ & 1/mR^2 \end{pmatrix} \begin{pmatrix} k & -\omega g \\ -\omega g & kR^2 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \theta \end{pmatrix} =$$

$$= \begin{pmatrix} k/m & -g \\ -g/R^2 & k/m \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \theta \end{pmatrix} = \begin{pmatrix} \frac{k}{m} \delta s - g \delta \theta \\ -\frac{g}{R^2} \delta s + \frac{k}{m} \delta \theta \end{pmatrix}$$

$$6) \quad k = \frac{mg}{R} \quad \vec{F} = F \vec{e}_x$$

$$a) \quad \delta V(s) \text{ t.c.} \quad -\delta V'(s) = F \Rightarrow \delta V(s) = -Fs$$

$$L = L_{\text{old}} + Fs$$

$$V = V_{\text{old}} - Fs \quad k = \frac{mg}{R}$$

$$\frac{\partial V}{\partial s} = \frac{\partial V_{\text{old}}}{\partial s} - F = \frac{mg}{R} s - mg \sin \theta - F$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V_{\text{old}}}{\partial \theta} = mgR \sin \theta - mg s \cos \theta$$

Vogliamo che $s=R$ sia pt. di equil. \rightarrow

$$\frac{\partial V}{\partial s} \Big|_{s=R} = mg(1 - \sin \theta) - F = 0 \quad \leadsto \quad (\#)$$

$$\frac{\partial V}{\partial \theta} \Big|_{s=R} = mgR \sin \theta - mgR \cos \theta = 0$$

$$\hookrightarrow \tan \theta = 1 \rightarrow \theta = \pi/4$$

$$(\#): \quad mg \left(1 - \frac{\sqrt{2}}{2} \right) = F$$

$$\partial^2 V = \partial^2 V_{\text{old}} = \begin{pmatrix} \frac{mg}{R} & -mg \cos \theta \\ -mg \cos \theta & mgR \cos \theta + mg s \sin \theta \end{pmatrix}$$

$$\downarrow \Big|_{\substack{s=R \\ \theta=\pi/4}} = \begin{pmatrix} \frac{mg}{R} & -mg \frac{\sqrt{2}}{2} \\ -mg \frac{\sqrt{2}}{2} & mgR \sqrt{2} \end{pmatrix} \rightarrow \det = (mg)^2 \left(\sqrt{2} - \frac{1}{2} \right) > 0$$

STAB.

7) Troviamo solut. di $Bu = \lambda Au$

$$\frac{1}{2} (\delta s, \delta \dot{\theta}) \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \begin{pmatrix} \delta \dot{s} \\ \delta \dot{\theta} \end{pmatrix} - \frac{1}{2} (\delta s, \delta \dot{\theta}) \begin{pmatrix} k & -mg \\ -mg & kR^2 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \dot{\theta} \end{pmatrix}$$

\uparrow A \uparrow B

$$\lambda_{\pm} = \frac{k}{m} \pm \frac{g}{R}$$

$$\lambda = \lambda_{\pm} : (B - \lambda_{\pm} A)u = \begin{pmatrix} \mp mg/R & -mg \\ -mg & \mp mgR \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$= -mgR \begin{pmatrix} \pm 1 & R \\ R & \pm R^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha \begin{pmatrix} R \\ \mp 1 \end{pmatrix}$$

$$\Rightarrow U = \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \delta s \\ \delta \dot{\theta} \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix}$$

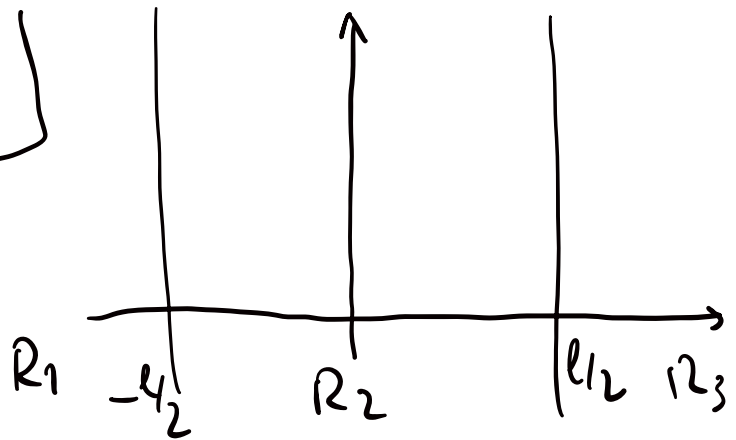
$$\begin{pmatrix} R & 1 \\ R & -1 \end{pmatrix} \begin{pmatrix} k & -mg \\ -mg & kR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} kR - mg & -mgR + kR^2 \\ kR + mg & -mgR - kR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2kR^2 - 2mgR & 0 \\ 0 & 2kR^2 + 2mgR \end{pmatrix} = 2mR^2 \begin{pmatrix} \frac{k}{m} - \frac{g}{R} & 0 \\ 0 & \frac{k}{m} + \frac{g}{R} \end{pmatrix}$$

$$\begin{pmatrix} R & 1 \\ R & -1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} mR & mR^2 \\ mR & -mR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} =$$

$$= 2mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow L = 2mR^2 \left(\frac{1}{2} (\dot{x} \dot{y}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \frac{1}{2} (xy) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \right)$$

ES3



1)

Condiz. raccordi $\psi_2(-\frac{l}{2}) = 0$ $\psi_2(\frac{l}{2}) = 0$

$$\psi_2(x) = A e^{ikx} + B e^{-ikx}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

vedi note

$$\boxed{n \neq 0} \Rightarrow m \neq 0$$

$$n=2m \quad \psi_{2m}(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{2m\pi x}{l}\right)$$

$$n=2m+1 \quad \psi_{2m+1}(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{(2m+1)\pi x}{l}\right)$$

funzioni d'onda normalizzate

$$2) E_m = \frac{\pi^2 \hbar^2}{2ml^2} m^2 \quad m = 1, 2, \dots \quad (m \neq 0!)$$

$$3) \frac{(\psi_2, \hat{H} \psi_2)}{\|\psi_2\|^2} = E_2 \frac{(\psi_2, \psi_2)}{\|\psi_2\|^2} = E_2 = \frac{2\pi^2 \hbar^2}{ml^2}$$

$$4) \psi_3(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{3\pi x}{l}\right) \leftarrow \text{funz. pari in } x \rightarrow -x$$

$$\Rightarrow \psi_3'(x) = -\sqrt{\frac{2}{l}} \frac{3\pi}{l} \sin\left(\frac{3\pi x}{l}\right) \leftarrow \text{funz. dispari in } x \rightarrow -x$$

$$\frac{(\psi_3, \hat{P} \psi_3)}{\|\psi_3\|^2} = \int_{-l/2}^{l/2} \psi_3 \frac{\hbar}{i} \psi_3' dx = 0$$

funz. dispari

$$5) \psi_1(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{\pi x}{l}\right) \leftarrow \text{funz. normalizzate.}$$

$$\text{Prob. } (x \in [0, \frac{l}{4}]) = \int_0^{\frac{l}{4}} |\psi_1(x)|^2 dx =$$

$$= \int_0^{\frac{l}{4}} \left(\frac{2}{l}\right) \cos^2\left(\frac{\pi x}{l}\right) dx = \frac{1}{l} \int_0^{\frac{l}{4}} (1 + \cos\left(\frac{2\pi x}{l}\right)) dx =$$

$$= \frac{1}{l} \left(x + \frac{l}{2\pi} \sin\left(\frac{2\pi x}{l}\right) \right) \Big|_0^{\frac{l}{4}} =$$

$$= \frac{1}{4} + \frac{1}{2\pi} \sin\left(\frac{\pi}{2}\right) = \frac{1}{4} + \frac{1}{2\pi} \quad (\text{che } e^- < 1/2).$$