ES 1

5. $$\{\tilde{p}_h, \tilde{q}_k\} = \left\{ p_h - \epsilon \frac{\partial G}{\partial q_h}, q_k + \epsilon \frac{\partial G}{\partial p_k} \right\} =$$

$$= \delta_{hk} - \epsilon \left\{ \frac{\partial G}{\partial q_h}, q_k \right\} + \epsilon \left\{ p_h, \frac{\partial G}{\partial p_k} \right\} + \cancel{O(\epsilon^2)}$$

$$= \delta_{hk} + \epsilon \left(\frac{\partial^2 G}{\partial p_k \partial q_h} - \frac{\partial G}{\partial q_h \partial p_k} \right) = \delta_{hk}$$

$$\{\tilde{p}_h, \tilde{p}_k\} = \left\{ p_h - \epsilon \frac{\partial G}{\partial q_h}, p_k - \epsilon \frac{\partial G}{\partial q_k} \right\} =$$

$$= -\epsilon \underbrace{\left\{ p_h, \frac{\partial G}{\partial q_k} \right\}}_{= -\frac{\partial^2 G}{\partial q_h \partial q_k}} - \epsilon \left\{ p_k - \frac{\partial G}{\partial q_h} \right\} = 0$$

$$\{q_h, \tilde{q}_k\} = \dots = 0$$

Analoghi passaggi

8. $$H = (p_1 q_2 - p_2 q_1)^2 = M_3^2$$

$$M_3 = p_1 q_2 - p_2 q_1$$

$$\underset{\substack{\uparrow \\ \text{sotto rotaz.} \\ \text{infinitesima}}}{\delta H} \propto \{M_3, H\} = \{M_3, M_3^2\} = 2M_3 \underbrace{\{M_3, M_3\}}_{=0} = 0$$

Alternativam. si fa conto esplicito

$$\{\underbrace{p_1 q_2 - p_2 q_1}_{M_3}, \underbrace{p_1^2 q_2^2 + p_2^2 q_1^2 - 2p_1 p_2 q_1 q_2}_{H}\} =$$

$$= \frac{\partial M_3}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial M_3}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial M_3}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial M_3}{\partial p_2}\frac{\partial H}{\partial q_2} =$$

$$= -p_2\left(\cancel{2p_1 q_2^2} - \cancel{2p_2 q_1 q_2}\right) + p_1\left(\cancel{2p_2 q_1^2} - \cancel{2p_1 q_1 q_2}\right)$$

$$-q_2\left(\cancel{2p_2^2 q_1} - \cancel{2p_1 p_2 q_2}\right) + q_1\left(\cancel{2p_1^2 q_2} - \cancel{2p_1 p_2 q_1}\right) = 0$$

$$\{q_1, p_1\} = \frac{\partial q_1}{\partial r}\frac{\partial p_1}{\partial p_r} + \frac{\partial q_1}{\partial \varphi}\frac{\partial p_1}{\partial p_\varphi} - \frac{\partial q_1}{\partial p_r}\frac{\partial p_1}{\partial r} - \frac{\partial q_1}{\partial p_\varphi}\frac{\partial p_1}{\partial \varphi} =$$

$$= \cos\varphi \cos\varphi - r \operatorname{sen}\varphi \left(-\frac{\operatorname{sen}\varphi}{r}\right) - 0 - 0 = 1$$

$$\{q_2, p_2\} = \operatorname{sen}\varphi \operatorname{sen}\varphi + r\cos\varphi \frac{\cos\varphi}{r} - 0 - 0 = 1$$

$$\{q_1, p_2\} = \cos\varphi \operatorname{sen}\varphi - r\operatorname{sen}\varphi \frac{\cos\varphi}{r} = 0$$

$$\{q_2 p_1\} = \operatorname{sen}\varphi \cos\varphi + r\cos\varphi \left(-\frac{\operatorname{sen}\varphi}{r}\right) = 0$$

$$\{q_1, q_2\} = \cos\varphi \cdot 0 - r\,\mathrm{sen}\,\varphi \cdot 0 - 0\cdot(\ ) - 0\cdot(\ ) = 0$$

$$\{p_1, p_2\} = p_\varphi \cancel{\frac{\mathrm{sen}\,\varphi}{r^2}}\,\mathrm{sen}\,\varphi + \left(-p_r \cancel{\mathrm{sen}\,\varphi} - p_\varphi \cancel{\frac{\cos\varphi}{r}}\right)\frac{\cos\varphi}{r}$$

$$- \cos\varphi\left(\cancel{-p_\varphi \frac{\cos\varphi}{r^2}}\right) + \frac{\mathrm{sen}\,\varphi}{r}\left(\cancel{p_r\cos\varphi} - \cancel{p_\varphi \frac{\mathrm{sen}\,\varphi}{r}}\right) = 0$$

$$K = (p_1 q_2 - p_2 q_1)^2\Big|_{\text{relaz.}} = p_\varphi^2$$

$$p_1 q_2 - p_2 q_1 = \left(\cancel{p_r \cos\varphi} - p_\varphi \frac{\mathrm{sen}\,\varphi}{r}\right) r\,\mathrm{sen}\,\varphi - \left(\cancel{p_r\,\mathrm{sen}\,\varphi} + p_\varphi \frac{\cos\varphi}{r}\right) r\cos\varphi$$

$$= -p_\varphi$$

$$\Rightarrow \quad \dot{p}_r = 0 \rightarrow p_r(t) = p_r^0 \qquad \dot{r} = 0 \rightarrow r(t) = r^0$$

$$\dot{p}_\varphi = 0 \rightarrow p_\varphi(t) = p_\varphi^0 \qquad \dot{\varphi} = 2p_\varphi \rightarrow \varphi(t) = 2p_\varphi^0 t + \varphi^0$$

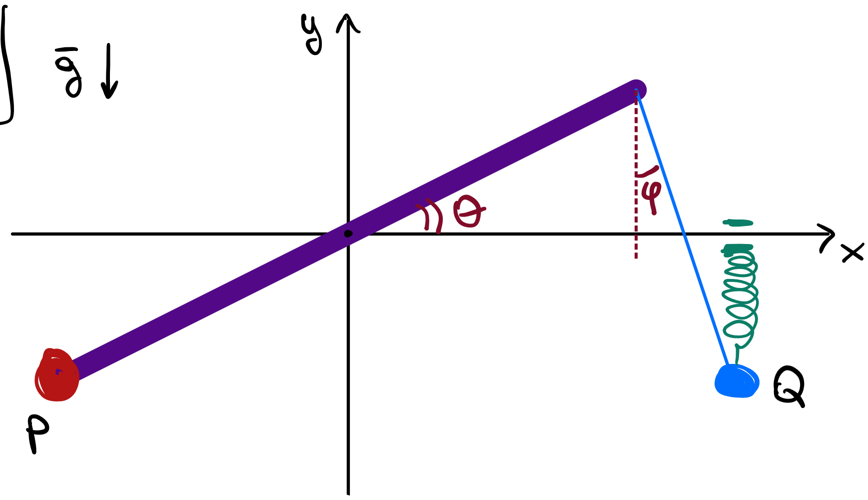
$$\Rightarrow \quad q_1(t) = r^0 \cos(2p_\varphi^0 t + \varphi^0)$$

$$q_2(t) = r^0\,\mathrm{sen}(2p_\varphi^0 t + \varphi^0)$$

$$p_1(t) = p_r^0 \cos(2p_\varphi^0 t + \varphi^0) - p_\varphi^0 \frac{\mathrm{sen}(2p_\varphi^0 t + \varphi^0)}{r^0}$$

$$p_2(t) = p_r^0\,\mathrm{sen}(2p_\varphi^0 t + \varphi^0) + p_\varphi^0 \frac{\cos(2p_\varphi^0 t + \varphi^0)}{r^0}$$

ES2



$$I_{barra} = \frac{M^2 (2\ell)^2}{12} = \frac{M\ell^2}{3} = m\ell^2$$

$M = 3m$

3am

$$x_P = -\ell \cos\theta \qquad \dot{x}_P^2 + \dot{y}_P^2 = \ell^2 \dot{\theta}^2$$

$$y_P = -\ell \,\mathrm{sen}\,\theta$$

$$x_Q = \ell\cos\theta + \ell\,\mathrm{sen}\,\varphi$$

$$y_Q = \ell\,\mathrm{sen}\,\theta - \ell\cos\varphi$$

$$\dot{x}_Q^2 + \dot{y}_Q^2 = \ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2 - 2\ell^2\dot{\theta}\dot{\varphi} \cdot \underbrace{(\mathrm{sen}\,\theta\cos\varphi - \cos\theta\,\mathrm{sen}\,\varphi)}_{\mathrm{sen}(\theta-\varphi)}$$

$$T = \frac{m}{2}\left(\ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2 - 2\ell^2\dot{\theta}\dot{\varphi}\,\mathrm{sen}(\theta-\varphi)\right) + m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$= \frac{m}{2}\left(4\ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2 - 2\ell^2\dot{\theta}\dot{\varphi}\,\mathrm{sen}(\theta-\varphi)\right)$$

$$Q = m\begin{pmatrix} 4\ell^2 & -\ell^2\mathrm{sen}(\theta-\varphi) \\ -\ell^2\mathrm{sen}(\theta-\varphi) & \ell^2 \end{pmatrix}$$

$$V_g = mg\,y_P + mg\,y_Q = -2mg\ell\,\mathrm{sen}\,\theta + mg\ell\,\mathrm{sen}\,\theta - mg\ell\cos\varphi$$

$$= -mg\ell(\mathrm{sen}\,\theta + \cos\varphi)$$

$$V_k = \frac{k}{2}y_Q^2 = \frac{k\ell^2}{2}(\mathrm{sen}\,\theta - \cos\varphi)^2$$

$$+ F\ell\cos\theta$$

1) $$L = \frac{1}{2}\left(4ml^2\dot{\theta}^2 + ml^2\dot{\varphi}^2 - 2ml^2\dot{\theta}\dot{\varphi}\,\mathrm{sen}(\theta-\varphi)\right) + mgl\,(\mathrm{sen}\,\theta + \cos\varphi) - \frac{kl^2}{2}(\mathrm{sen}\,\theta - \cos\varphi)^2$$

2) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt}\left(l^2\dot{\varphi} - l^2\dot{\theta}\,\mathrm{sen}(\theta-\varphi)\right) =$$

$$= l^2\ddot{\varphi} - l^2\ddot{\theta}\,\mathrm{sen}(\theta-\varphi) - l^2\dot{\theta}^2\cos(\theta-\varphi) + l^2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)$$

$$\frac{\partial L}{\partial \varphi} = + l^2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi) - mgl\,\mathrm{sen}\,\varphi - kl^2(\mathrm{sen}\,\theta - \cos\varphi)\,\mathrm{sen}\,\varphi$$

$$\rightarrow \ddot{\varphi} - \ddot{\theta}\,\mathrm{sen}(\theta-\varphi) = \dot{\theta}^2\cos(\theta-\varphi) - \frac{g}{l}\mathrm{sen}\,\varphi - \frac{k}{m}(\mathrm{sen}\,\theta - \cos\varphi)\,\mathrm{sen}\,\varphi$$

3) $$V = -mgl\,(\mathrm{sen}\,\theta + \cos\varphi) + \frac{kl^2}{2}(\mathrm{sen}\,\theta - \cos\varphi)^2$$

$$\frac{\partial V}{\partial \theta} = -mg\cancel{l}\cos\theta + kl^{\cancel{2}}(\mathrm{sen}\,\theta - \cos\varphi)\cos\theta \quad +Fl\,\mathrm{sen}\,\theta = 0 \qquad = Fl\frac{1}{2}$$

$\theta = \frac{\pi}{6}$: $-\frac{\sqrt{3}mg}{2} + + Kl\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\cos\varphi\right)$

$$\frac{\partial V}{\partial \varphi} = mg\cancel{l}\,\mathrm{sen}\,\varphi + kl^{\cancel{2}}(\mathrm{sen}\,\theta - \cos\varphi)\,\mathrm{sen}\,\varphi = 0$$

$$\cos\theta\left((\mathrm{sen}\,\theta - \cos\varphi) - \frac{mg}{kl}\right) = 0$$

$$\mathrm{sen}\,\varphi\left((\mathrm{sen}\,\theta - \cos\varphi) + \frac{mg}{kl}\right) = 0$$

$$\begin{matrix} A_1A_2 = 0 \\ B_1B_2 = 0 \end{matrix} \rightarrow \begin{matrix} A_1 = B_1 = 0 & A_1 = B_2 = 0 \\ A_2 = B_1 = 0 & \cancel{A_2 = B_2} = 0 \end{matrix}$$

← le parentesi tonde non si possono annullare contemporaneamente.

$\cos\theta = 0 \quad \rightarrow \quad \theta = -\frac{\pi}{2}, \frac{\pi}{2}$

$\exists$ se $\frac{mg}{Kl} \leq 2$

$\operatorname{sen}\varphi = 0 \quad \rightarrow \quad \varphi = 0, \pi$

- $\theta = -\frac{\pi}{2} \qquad -1 - \cos\varphi + \frac{mg}{Kl} = 0 \rightarrow \varphi_* = \pm \arccos\left(\frac{mg}{Kl} - 1\right)$

× $\theta = \frac{\pi}{2} \qquad 1 - \cos\varphi + \frac{mg}{Kl} \rightarrow \varphi = \pm \arccos\left(\underbrace{\frac{mg}{Kl} + 1}_{>1}\right)$

NON ESISTE

× $\varphi = 0 \qquad \operatorname{sen}\theta - 1 - \frac{mg}{Kl} \rightarrow \operatorname{sen}\theta = 1 + \frac{mg}{Kl} > 1$

NON ESISTE

- $\varphi = \pi \qquad \operatorname{sen}\theta + 1 - \frac{mg}{Kl} \rightarrow \operatorname{sen}\theta_* = \frac{mg}{Kl} - 1$

$\exists$ se $\frac{mg}{Kl} \leq 2$

- $\varphi = 0 \qquad \theta = -\frac{\pi}{2}$
- $\varphi = \pi \qquad \theta = -\frac{\pi}{2}$
- $\varphi = 0 \qquad \theta = +\frac{\pi}{2}$
- $\varphi = \pi \qquad \theta = \frac{\pi}{2}$

$$\partial^2 V = \begin{pmatrix} mgl\operatorname{sen}\theta + Kl^2(\overbrace{\cos^2\theta - \operatorname{sen}^2\theta}^{1-2\operatorname{sen}^2\theta} + \cos\varphi\operatorname{sen}\theta) & Kl^2\operatorname{sen}\varphi\cos\theta \\ Kl^2\operatorname{sen}\varphi\cos\theta & mgl\cos\varphi + Kl^2\operatorname{sen}\theta\cos\varphi - Kl^2(\underbrace{\cos^2\varphi - \operatorname{sen}^2\varphi}_{2\cos^2\varphi - 1}) \end{pmatrix}$$

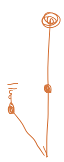
$$\partial^2 V\left(-\frac{\pi}{2}, 0\right) = \begin{pmatrix} -mgl + kl^2(-2) & \\ & \end{pmatrix} \quad \underline{\text{instab.}}$$

$$\partial^2 V\left(-\frac{\pi}{2}, \pi\right) = \begin{pmatrix} -mgl + kl^2(-1+1) & \\ & \end{pmatrix} \quad \underline{\text{instab.}}$$

$$\partial^2 V\left(\frac{\pi}{2}, 0\right) = \begin{pmatrix} mgl + kl^2(-1+1) & 0 \\ 0 & mgl + kl^2(1-1) \end{pmatrix} \text{def. pos} \quad \underline{\text{stab.}}$$

$$\partial^2 V\left(\frac{\pi}{2}, \pi\right) = \begin{pmatrix} mgl + kl^2(-1-1) & 0 \\ 0 & -mgl + kl^2(-1-1) \end{pmatrix} \quad \underline{\text{instab.}}$$

$$\partial^2 V\left(-\frac{\pi}{2}, \varphi^*\right) = \begin{pmatrix} \overbrace{-mgl + kl^2\left(-\cancel{1} - \left(\frac{mg}{kl} - \cancel{1}\right)\right)}^{-2mgl} & 0 \\ 0 & mgl\left(\frac{mg}{kl} - 1\right) + kl^2\left(1 - \frac{mg}{kl} - 2\left(\frac{mg}{kl} - 1\right)^2 + 1\right) \end{pmatrix} \quad \underline{\text{instab.}}$$



$$\partial^2 V(\theta^*, \pi) = \begin{pmatrix} \overbrace{mgl\left(\frac{mg}{kl} - 1\right) + kl^2\left(\cancel{1} - \cancel{2} + \cancel{1} - \frac{mg}{kl}\right)}^{mgl\left(\frac{mg}{kl} - 2\right) \leq 0 \text{ quadrante}} & 0 \\ 0 & \end{pmatrix} \quad \text{instab.}$$

4) $(\theta, \frac{\pi}{2}) = (\frac{\pi}{2}, 0)$

$$B = \begin{pmatrix} mgl & 0 \\ 0 & mgl \end{pmatrix} \qquad A = \begin{pmatrix} 4l^2 m & -l^2 m \\ -l^2 m & l^2 m \end{pmatrix}$$

$$0 = \det(B - \lambda A) = \det \begin{pmatrix} mgl - 4l^2\lambda m & +\lambda l^2 m \\ +\lambda l^2 m & mgl - l^2\lambda m \end{pmatrix}$$

$$= m^2 l^4 \det \begin{pmatrix} \frac{g}{l} - 4\lambda & \lambda \\ \lambda & \frac{g}{l} - \lambda \end{pmatrix} =$$

$$= m^2 l^4 \left[\left(\frac{g}{l}\right)^2 - 5\lambda\left(\frac{g}{l}\right) + 4\lambda^2 - \lambda^2 \right]$$

$$\rightarrow \quad 3\lambda^2 - 5\left(\frac{g}{l}\right)\lambda + \left(\frac{g}{l}\right)^2 = 0$$

$$\lambda_{1,2} = \left(\frac{5}{6} \pm \frac{1}{6}\sqrt{13} \right) \frac{g}{l} \quad \Leftarrow \omega_1^2, \omega_2^2$$

$25 - 12 = 13$

5) $V = -mgl(\operatorname{sen}\theta + \cos\varphi) + \frac{Kl^2}{2}(\operatorname{sen}\theta - \cos\varphi)^2$

Attorno $\left(\theta, \frac{\pi}{2}\right) = \left(\frac{\pi}{2}, 0\right) \rightarrow \theta = \frac{\pi}{2} + \delta\theta \quad \varphi = \delta\varphi$

$$\cos(\delta\varphi) \simeq 1 - \frac{(\delta\varphi)^2}{2}$$

$$\operatorname{sen}\left(\frac{\pi}{2} + \delta\theta\right) = \cos(\delta\theta) \simeq 1 - \frac{(\delta\theta)^2}{2}$$

K appare nel termine

$$\frac{Kl^2}{2}\left(\operatorname{sen}\left(\frac{\pi}{2}-\delta\theta\right)-\cos(\delta\varphi)\right)^2 \simeq \left(\frac{(\delta\varphi)^2-(\delta\theta)^2}{2}\right)^2\frac{Kl^2}{2}$$

sono tutti termini quartici che vengono scartati all'ordine quadratico.

6) $\bar{F} = -F\bar{e}_x \rightsquigarrow \delta V = F x_p = -Fl\cos\theta$

$$\Rightarrow V_{new} = -mgl(\operatorname{sen}\theta + \cos\varphi) + \frac{Kl^2}{2}(\operatorname{sen}\theta - \cos\varphi)^2 - Fl\cos\theta$$

a) $L_{new} = L - \delta V = L + Fl\cos\theta$

b)

$$\frac{\partial V}{\partial \theta} = -mg\cancel{l}\cos\theta + Kl^{\cancel{2}}(\operatorname{sen}\theta-\cos\varphi)\cos\theta + F\cancel{l}\operatorname{sen}\theta = 0$$

$$\frac{\partial V}{\partial \varphi} = mg\cancel{l}\operatorname{sen}\varphi + Kl^{\cancel{2}}(\operatorname{sen}\theta-\cos\varphi)\operatorname{sen}\varphi = 0$$

Voglio che $\theta = \frac{\pi}{6}$ sia pto equil $\Rightarrow$

$$-mg\frac{\sqrt{3}}{2} + Kl\left(\frac{1}{2}-\cos\varphi\right) + \frac{F}{2} = 0$$

$$mg\operatorname{sen}\varphi + Kl\left(\frac{1}{2}-\cos\varphi\right) = 0$$

ES 3

1) $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \longrightarrow E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{\ell^2}$

$$\frac{d^2}{dx^2}\begin{matrix}\cos(\alpha x)\\ \text{sen}(\alpha x)\end{matrix} = \begin{matrix}-\alpha^2\cos(\alpha x)\\ -\alpha^2 \text{sen}(\alpha x)\end{matrix}$$

2)

1° liv.	ψ_1	$m=0$	in	$2m+1$	$\rightarrow n=1$
2° liv.	ψ_2	$m=1$	in	$2m$	$\rightarrow n=2$
3° liv.	ψ_3	$m=1$	in	$2m+1$	$\rightarrow n=3$

$$E_3 = \frac{9\hbar^2\pi^2}{2m\ell^2}$$

$$\langle H\rangle_{\psi_3} = (\psi_3, \hat{H}\psi_3) = (\psi_3, E_3\psi_3) = E_3(\psi_3, \psi_3) = E_3$$

$$= \frac{9\hbar^2\pi^2}{2m\ell^2}$$

3) ψ_n sono a parità definita, ψ_n' ha parità opposta a ψ_n

$$\langle P\rangle_{\psi_{10}} = \int_{-\ell/2}^{\ell/2} dx \underbrace{\psi_{10}^*(x)\frac{\hbar}{i}\psi_{10}'(x)}_{\text{disp.}} = 0$$

4) $\psi_1(x) = \sqrt{\frac{2}{\ell}} \cos\left(\frac{\pi x}{\ell}\right)$

$$\text{Prob.}\left[x \in \left[\frac{\ell}{4}, \frac{\ell}{2}\right]\right] = \int_{\ell/4}^{\ell/2} dx \ \frac{2}{\ell} \cos^2\left(\frac{\pi x}{\ell}\right) =$$

$$= \frac{2}{\ell} \int_{\ell/4}^{\ell/2} dx \ \frac{1 + \cos \frac{2\pi x}{\ell}}{2} =$$

$$= \frac{1}{\ell}\left(\int_{\ell/4}^{\ell/2} dx + \int_{\ell/4}^{\ell/2} \cos\left(\frac{2\pi x}{\ell}\right) dx \right) =$$

$$= \frac{1}{\ell}\left(\frac{\ell}{2} - \frac{\ell}{4} + \frac{\ell}{2\pi} \operatorname{sen}\left(\frac{2\pi x}{\ell}\right)\Big|_{\ell/4}^{\ell/2} \right) =$$

$$= \frac{1}{4} + \frac{1}{2\pi}\left(0 - 1\right) = \frac{1}{4} - \frac{1}{2\pi} .$$