ES1)

4) $P = \sum_k \frac{\partial \varphi_k}{\partial \alpha}(0)\, p_k$ $\quad p_k = \frac{\partial L}{\partial \dot{q}_k}$

q_ℓ ciclica: $\varphi_k(q, \alpha) = q_k + \alpha\, \delta_{k\ell}$

$\Rightarrow \frac{\partial \varphi_k}{\partial \alpha} = \delta_{k\ell}$

$P = \sum_k \delta_{k\ell}\, p_k = p_\ell$ → mom. coniugato alla coord. ciclica

5) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\sqrt{x^2+y^2+z^2})$

$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(\sqrt{x^2+y^2+z^2})$

6) $[H, M_i] = 0$ perché H è invariante per rotazioni

$\Rightarrow M_1$ è cost. del moto (indip. da tempo esplicitam.)

7) $H, \; M_z, \; \bar{M}^2 = M_x^2 + M_y^2 + M_z^2$

$$[H, M_z] = 0 \qquad [H, \bar{M}^2] = \sum_i 2M_i [H, M_i] = 0$$

$$[M_z, \bar{M}^2] = [M_z, M_x^2] + [M_z, M_y^2] =$$

$$= 2M_x(M_y) + 2M_y(-M_x) = 0$$

8) Perchè allora è integrabile e quindi posso portarlo in una forma con H_{new} che dip solo da momenti coniugati

9)
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \hat{p}_x \\ \tilde{p}_y \end{pmatrix}$$

$$\{x, y\} = \{\cos\alpha\, \tilde{x} - \operatorname{sen}\alpha\, \hat{y},\ \operatorname{sen}\alpha\, \tilde{x} + \cos\alpha\, \tilde{y}\} =$$

$$= 0 \quad \text{poiché} \quad \{\hat{x}, \tilde{y}\} = 0$$

$$\{x, p_x\} = \{\cos\alpha\, \tilde{x} - \operatorname{sen}\alpha\, \tilde{y},\ \cos\alpha\, \tilde{p}_x - \operatorname{sen}\alpha\, \hat{p}_y\}$$

$$= \cos^2\alpha + \operatorname{sen}^2\alpha = 1$$

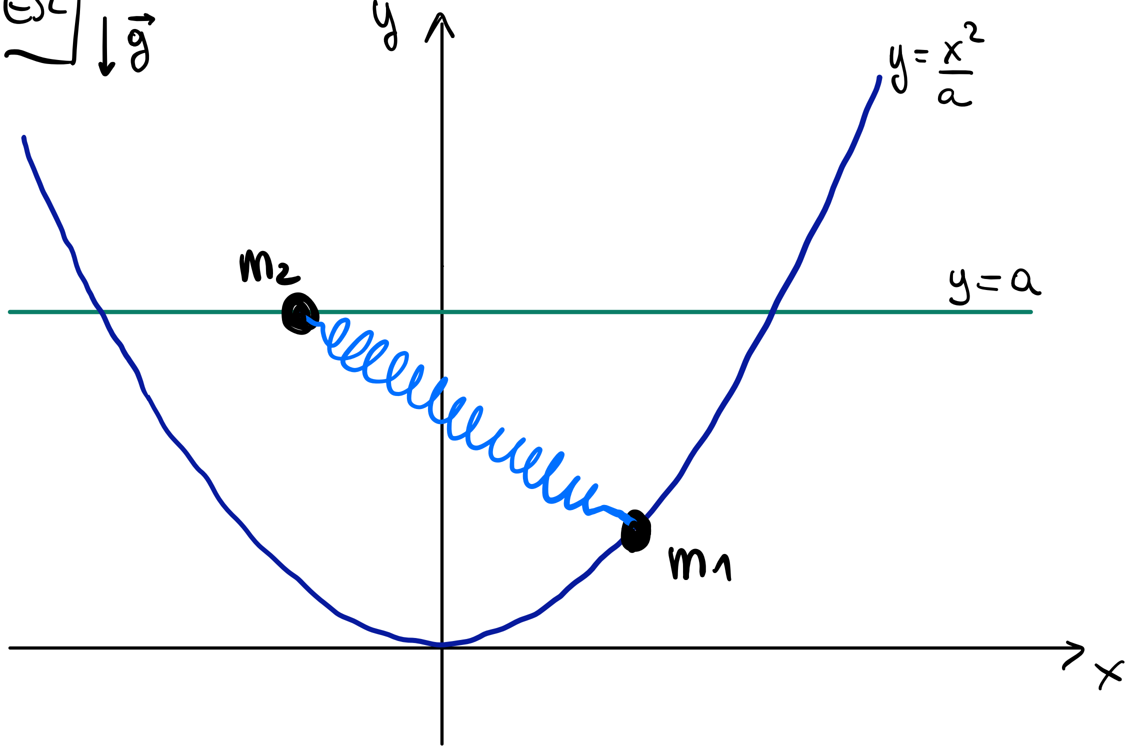
$$\{y, p_y\} = \operatorname{sen}^2\alpha + \cos^2\alpha = 1$$

$$\{x, p_y\} = \{\cos\alpha\, \hat{x} - \operatorname{sen}\alpha\, \tilde{y},\ \operatorname{sen}\alpha\, \tilde{p}_x + \cos\alpha\, \tilde{p}_y\}$$

$$= \cos\alpha \operatorname{sen}\alpha - \operatorname{sen}\alpha \cos\alpha = 0$$

$$\{y, p_x\} = \operatorname{sen}\alpha \cos\alpha - \cos\alpha \operatorname{sen}\alpha = 0$$

ES2 $\downarrow \vec{g}$



$x_1 = s \qquad \dot{x}_1 = \dot{s} \qquad x_2 = z$

$y_1 = \frac{s^2}{a} \qquad \dot{y}_1 = \frac{2s\dot{s}}{a} \qquad y_2 = a$

$$T = \frac{m_1}{2}\dot{s}^2\left(1+\frac{4s^2}{a^2}\right) + \frac{m_2}{2}\dot{z}^2 \qquad \mathcal{a} = \begin{pmatrix} m_1\left(1+\frac{4s^2}{a^2}\right) & 0 \\ 0 & m_2 \end{pmatrix}$$

$$V = m_1 g \frac{s^2}{a} + \frac{k}{2}\left((s-z)^2 + \left(\frac{s^2}{a} - a\right)^2\right) + \cancel{m_2 g a}$$

$$= m_1 g \frac{s^2}{a} + \frac{k}{2}\left(s^2 - 2sz + z^2 + \frac{s^4}{a^2} - 2s^2 + a^2\right)$$

$$= \frac{k}{2}z^2 - ksz + \frac{ks^4}{2a^2} + \left(m_1\frac{g}{a} - \frac{k}{2}\right)s^2 + \cancel{\frac{ka^2}{2}}$$

1)

$$L = T - V = \frac{m_1}{2}\dot{s}^2\left(1+\frac{4s^2}{a^2}\right) + \frac{m_2}{2}\dot{z}^2$$

$$- \frac{k}{2}z^2 + ksz - \frac{ks^4}{2a^2} - \left(m_1\frac{g}{a} - \frac{k}{2}\right)s^2$$

2) $\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{d}{dt}\left(m_1 \dot{s}\left(1+\frac{4s^2}{a^2}\right)\right) = m_1 \ddot{s}\left(1+\frac{4s^2}{a^2}\right) + 8 m_1 \dot{s}^2 \frac{s}{a^2}$

$$\frac{\partial L}{\partial s} = 4 m_1 \dot{s}^2 \frac{s}{a^2} + kz - 2\frac{ks^3}{a^2} - \left(\frac{2m_1 g}{a} - k\right)s$$

$$\ddot{s}\left(1+\frac{4s^2}{a^2}\right) = -4\frac{\dot{s}^2 s}{a^2} + \frac{k}{m_1}z - 2\frac{ks^3}{m_1 a^2} - \left(2\frac{g}{a} - \frac{k}{m_1}\right)s$$

3) $V = \frac{k}{2}z^2 - ksz + \frac{ks^4}{2a^2} + \left(m_1\frac{g}{a} - \frac{k}{2}\right)s^2$

$$\frac{\partial V}{\partial s} = -kz + 2\frac{ks^3}{a^2} + \left(2m_1\frac{g}{a} - k\right)s = 0$$

$$\frac{\partial V}{\partial z} = k(z-s) = 0 \rightarrow z = s$$

$$\frac{ks^3}{a^2} + \left(m_1\frac{g}{a} - k\right)s = 0$$

$$\frac{ks}{a^2}\left(s^2 + \left(\frac{m_1 g}{ka} - 1\right)a^2\right) = 0$$

Ho soluz.

$s = 0$

$s = \pm a\sqrt{1 - \frac{m_1 g}{ka}}$ se $\frac{m_1 g}{ka} \leq 1$

$\equiv s^*_\pm$

$P_1: (s, z) = (0, 0)$

$P_{2,3}: (s, z) = (s^*_\pm, s^*_\pm)$

$$\frac{\partial V}{\partial s} = -kz + 2\frac{ks^3}{a^2} + \left(2\omega_1 \frac{g}{a} - k\right)s$$

$$\frac{\partial V}{\partial z} = k(z - s)$$

$$\partial^2 V = \begin{pmatrix} 6\frac{ks^2}{a^2} + 2\omega_1 \frac{g}{a} - k & -k \\ -k & k \end{pmatrix}$$

$$\partial^2 V(0,0) = \begin{pmatrix} 2\omega_1 \frac{g}{a} - k & -k \\ -k & k \end{pmatrix}$$

$$\det = \left(2\omega_1 \frac{g}{a} - k\right)k - k^2 = 2k\left[\frac{\omega_1 g}{ka} - 1\right]$$

$$\mathrm{tr} = 2\omega_1 \frac{g}{a} > 0$$

Stab. se $\frac{\omega_1 g}{ka} \geq 1$

Instab. altrim.

$$\partial^2 V(s^*_\pm, s^*_\pm) = \begin{pmatrix} \frac{6k}{a^2} a^2\left(1 - \frac{\omega_1 g}{ka}\right) + 2\omega_1 \frac{g}{a} - k & -k \\ -k & k \end{pmatrix}$$

$$= \begin{pmatrix} 6k\left(1 - \frac{m_1 g}{k a}\right) + k\left(\frac{2 m_1 g}{k a} - 1\right) & -k \\ -k & k \end{pmatrix}$$

$$\det = 6k^2\underbrace{\left(1 - \frac{m_1 g}{k a}\right)}_{\text{Qndo esistono è } > 1} + 2k^2\left(1 - \frac{m_1 g}{k a}\right) > 0 \quad \searrow \text{ STAB. qndo esiste}$$

$$tr = 6k\left(1 - \frac{m_1 g}{k a}\right) + 2 m_1 \frac{g}{a} > 0 \quad \nearrow$$

4) $m_1 = m_2 = m$

$\frac{m g}{k a} = \frac{7}{4} > 1 \implies s = z = 0$ è STABILE $\qquad \frac{k}{m} = \frac{4}{7}\frac{g}{a}$

$$A = a(s = z = 0) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$B = \begin{pmatrix} 2m\frac{g}{a} - k & -k \\ -k & k \end{pmatrix} = k\begin{pmatrix} \frac{5}{2} & -1 \\ -1 & 1 \end{pmatrix}$$

$$\det(B - \lambda A) = \det\left(m\begin{pmatrix} \frac{5}{2}\frac{k}{m} - \lambda & -k/m \\ -k/m & \frac{k}{m} - \lambda \end{pmatrix}\right) =$$

$$= m^2\left[\lambda^2 - \frac{7}{2}\frac{k}{m}\lambda + \frac{3}{2}\left(\frac{k}{m}\right)^2\right] = 0$$

$$\lambda_{1/2} = \frac{k}{m}\left[\frac{7}{4} \pm \sqrt{\frac{49}{16} - \frac{3}{2}}\right] =$$

$$= \frac{k}{m}\left[\frac{7}{4} \pm \sqrt{\frac{25}{16}}\right] = \frac{k}{m}\left[\frac{7}{4} \pm \frac{5}{4}\right]$$

$$= \begin{cases} 3\dfrac{k}{m} = \dfrac{12}{7}\dfrac{g}{a} \\[2ex] \dfrac{k}{2m} = \dfrac{2}{7}\dfrac{g}{a} \end{cases}$$

$$a = \begin{pmatrix} m_1\left(1 + \frac{4s^2}{a^2}\right) & 0 \\ 0 & m_2 \end{pmatrix}$$

5) $m_1 = m_2 = m$

$\frac{mg}{ka} = \frac{1}{4} < 1 \Rightarrow$ $S_\pm^*$ sono STABILI $\left(S_\pm^*\right)^2 = a^2\left(1 - \frac{mg}{ka}\right)$

$\frac{k}{m} = \frac{4g}{a}$

$$A = \begin{pmatrix} m\left(1 + \frac{4}{\cancel{a^2}}\cancel{a^2}\left(1 - \frac{mg}{ka}\right)\right) & 0 \\ 0 & m \end{pmatrix} =$$

$$= \begin{pmatrix} m\left(5 - \frac{4mg}{ka}\right) & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} 4m & 0 \\ 0 & m \end{pmatrix}$$

$$B = \begin{pmatrix} 4k & -k \\ -k & k \end{pmatrix}$$

$$\det(B - \lambda A) = m^2 \det\begin{pmatrix} 4\frac{k}{m} - 4\lambda & -k/m \\ -k/m & \frac{k}{m} - \lambda \end{pmatrix}$$

$$= m^2\left[4\left(\lambda-\frac{k}{m}\right)^2-\left(\frac{k}{m}\right)^2\right] =$$

$$= m^2\left[\left(2\lambda-2\frac{k}{m}-\frac{k}{m}\right)\left(2\lambda-2\frac{k}{m}+\frac{k}{m}\right)\right]=0$$

$$\lambda=\begin{cases}\frac{3}{2}\frac{k}{m} = 6\frac{g}{a}\\ \frac{1}{2}\frac{k}{m} = 2\frac{g}{a}\end{cases}$$

6) $L = T - V = \frac{m_1}{2}\dot{s}^2\left(1+\frac{4s^2}{a^2}\right)+\frac{m_2}{2}\dot{z}^2$
$-\frac{k}{2}z^2+ksz-\frac{ks^4}{2a^2}-\left(\frac{m_1 g}{a}-\frac{k}{2}\right)s^2$

$\downarrow$ $z=0$ $\dot{z}=0$

$$L=\frac{m_1}{2}\dot{s}^2\left(1+\cancel{\frac{4s^2}{a^2}}\right)-\cancel{\frac{ks^4}{2a^2}}-\left(\frac{m_1 g}{a}-\frac{k}{2}\right)s^2$$

Perchè lin. attorno $\begin{cases}s=0\\ \dot{s}=0\end{cases}$

$$L_{lin}=\frac{m_1}{2}\dot{s}^2-\frac{m_1}{2}\left(\frac{2g}{a}-\frac{k}{m_1}\right)s^2$$

$$\Rightarrow \omega^2=\frac{2g}{a}-\frac{k}{m_1}$$

7) $x_1 = r\cos\omega t$ $\rightarrow$ $\dot{x}_1 = \dot{r}\cos\omega t - r\omega\,\text{sen}\,\omega t$

$x_2 = r\,\text{sen}\,\omega t$ $\qquad \dot{x}_2 = \dot{r}\,\text{sen}\,\omega t + r\omega\cos\omega t$

$y = r^2/a$ $\qquad \dot{y} = 2r\dot{r}/a$

$$T = \frac{m_1}{2}\dot{r}^2\left(1 + \frac{4r^2}{a^2}\right) + \frac{m_1\omega^2 r^2}{2}$$

$$V = \frac{kr^4}{2a^2} + \left(\frac{m_1 g}{a} - \frac{k}{2}\right)r^2$$

$$V_{eff} = \frac{kr^4}{2a^2} + \left(\frac{m_1 g}{a} - \frac{k}{2} - \frac{m_1\omega^2}{2}\right)r^2$$

Dist. a da asse $y \Rightarrow r = a$

$$V_{eff}' = \frac{2kr^3}{a^2} + \left(\frac{2m_1 g}{a} - k - m_1\omega^2\right)r = 0$$

$r=a \searrow$ $\cancel{2}ka + 2m_1 g - \cancel{ka} - m_1\omega^2 a = 0$

$$\Rightarrow \omega^2 = \frac{k}{m_1} + \frac{2g}{a}$$