DISUGUAGLIANZA TRIANGOLARE IN $\mathbb{C}$ ①

Vogliamo dimostrare che $|z_1+z_2| \leq |z_1|+|z_2|$

Si parte dalla banale osservazione che, se $a, b \in \mathbb{R}$, allora

$$(a-b)^2 \geq 0$$
$$= a^2-2ab+b^2$$

da cui $\quad ab \leq \frac{1}{2}(a^2+b^2) \qquad (*)$

Siano ora (x_1, y_1), (x_2, y_2) due coppie di numeri reali.

da $(*)$ si ha che:

$$\frac{x_1}{(x_1^2+y_1^2)^{1/2}} \cdot \frac{x_2}{(x_2^2+y_2^2)^{1/2}} + \frac{y_1}{(x_1^2+y_1^2)^{1/2}} \cdot \frac{y_2}{(x_2^2+y_2^2)^{1/2}} \leq$$

$$\leq \frac{1}{2}\left(\frac{x_1^2}{x_1^2+y_1^2} + \frac{x_2^2}{x_2^2+y_2^2}\right) + \frac{1}{2}\left(\frac{y_1^2}{x_1^2+y_1^2} + \frac{y_2^2}{x_2^2+y_2^2}\right)$$

$$= 1$$

da cui

$$\boxed{x_1x_2 + y_1y_2 \leq (x_1^2+y_1^2)^{1/2}(x_2^2+y_2^2)^{1/2}} \qquad (**)$$

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ora si ha:

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + (x_1+iy_1)(x_2-iy_2) + (x_1-iy_1)(x_2+iy_2)$$

$$= |z_1|^2 + |z_2|^2 + x_1x_2 - \cancel{ix_1y_2} + \cancel{ix_2y_1} + y_1y_2$$

$$+ x_1x_2 \cancel{+ix_1y_2} \cancel{-ix_2y_1} + y_1y_2$$

$$= |z_1|^2 + |z_2|^2 + 2(x_1x_2+y_1y_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1|+|z_2|)^2$$

per (**)

e quindi $|z_1+z_2| \leq |z_1|+|z_2|$